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Existence of solutions for some nonlinear Volterra integral equations via Petryshyn's fixed point theorem

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Abstract

In this paper, we study the existence of solutions of some nonlinear Volterra integral equations by using the techniques of measures of non compactness and the Petryshyn's fixed point theorem in Banach space. We also present some examples of the integral equation to confirm the efficiency of our results.

Keywords: Nonlinear integral equations, existence of solution, measures of non compactness, Petryshyn's fixed point theorem.

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1. Introduction

Integral equations provide important tools for modeling a wide range of phenomena and processes [14] and have found a wide variety of applications in the various field including, mathematical physics, economics, biology, mechanics and population dynamics [23, 19, 7], (for more applications of integral equations, see also [25]). The concept of measures of non compactness was first devised by Kuratowski[15]. Recently, there have been several successful attempt to apply the concept of measure of non compactness in the study of the existence of solutions of nonlinear integral equations [1, 2, 4, 9, 18, 21].

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In this paper, we present and prove a new existence theorem for solution of nonlinear Volterra integral equations

$$x(t) = q\left(t, x(\alpha(t)), x(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds\right), \quad t \in I_a = [0,a]. \tag{1.1}$$

The basic tools used in these investigations are the techniques of measure of non compactness and Petryshyn's fixed point theorem [22] which is a generalization of Darbo's Fixed Point theorem [5]. The main goal of this study is to investigate existence of solution Eq. (1.1). Numerous authors have carried out some successful efforts to solve many functional-integral equations by applying Darbo condition which is a powerful tool to study these equations [1, 2, 9, 18, 21, 17]. In our consideration, we use the Petryshyn's theorem (instead of Darbo's theorem) to study the solvability of Eq. (1.1). This work focuses on the general form (1.1) which has been resulted from [18, 21, 5, 17, 16] and others. The following statements describe the main reasons why we use Eq. (1.1) and what is the excellence of our work: The first is that the conditions in many papers will be simplified. The second reason is that this paper unifies the related work in this area. The third reason is that bounded condition (H3) of Theorem 3.1, shows that the sublinear condition that has been discussed in several literature (see e.g. (M3) below and [18, 17, 10, 16]) have not an important role.
The paper is organized as five sections including the introduction. In Section 2, we introduce some preliminaries and describes the concept of measures of non compactness. Section 3 is devoted to state and prove existence theorem for equations involving condensing operators using the Petryshyn's fixed point theorem. In Section 4, we offer some examples that verify the application of this kind of nonlinear functional-integral equations. Finally Section 5, concludes the paper.

2. Preliminaries

Throughout the paper, let E be a Banach space, we write $\bar{B}_r = \{x \in E : \| x \| \leq r\}$ for the closed ball and $\partial \bar{B}_r = \{x \in E : \| x \| = r\}$ for the sphere in E around 0 with radius $r > 0$.
Measures of non compactness are very useful tools in functional analysis, for instance in metric fixed point theory and in the theory of operator equations in Banach spaces. Before stating our main results in the next section, we recall classical definitions and theorems.

Theorem 2.1. (Kuratowski [15]) If M is a bounded subset of a Banach space E, let $\alpha(M)$ denote the (Kuratowski) measure of non compactness of M, that is,

$$\alpha(M) = \inf\{\sigma > 0 : M \text{ may be covered by finitely many sets of diameter } \leq \sigma\}. \tag{2.1}$$

Other measures of non compactness were introduced by Goldenstein.

Theorem 2.2. (Goldenstein and Markus [13]) The Hausdorff (or ball) measure of non compactness

$$\mu(M) = \inf\{\sigma > 0 : \text{there exists a finite } \sigma\text{–net for } M \text{ in } E\}, \tag{2.2}$$

where by a finite σ-net for M in E we mean, as usual, a set $\{z_1, z_2, \ldots, z_m\} \subset E$ such that the balls $B_\sigma(E; z_1), B_\sigma(E; z_2), \ldots, B_\sigma(E; z_m)$ over M. These measures of non compactness are mutually equivalent in the sense that

$$\mu(M) \leq \alpha(M) \leq 2\mu(M)$$

for any bounded set $M \subset E$.

It is easy to see that the following properties hold for Kuratowski and Hausdorff measure of non compactness.

Theorem 2.3. (Petryshyn [22]) Let E be a Banach space, $\lambda \in \mathbb{R}$ and $M, N \subset E$ bounded. Then

(i) $\mu(M \cup N) = \max\{\mu(M), \mu(N)\}$;

(ii) $\mu(M + N) \leq \mu(M) + \mu(N)$;

(iii) $\mu(\lambda M) =\mid \lambda \mid \mu(M)$;

(iv) $\mu(M) \leq \mu(N)$ for $M \subseteq N$;

(v) $\mu(\bar{\text{co}} M) = \mu(M)$;

(vi) $\mu(M) = 0$ if and only if M is precompact.

In what follows, we will work in the space $C[0, a]$ consisting of all real–valued functions and continuous on the interval $[0, a]$. The space $C[0, a]$ is equipped with the standard norm

$$\|x\| = \sup\{|x(t)| : t \in [0, a]\}.$$

Recall that the modulus of continuity of a function $u \in C[0, a]$ is defined by

$$\omega(u, \sigma) = \sup\{|u(x) - u(y)| : |x - y| \leq \sigma\}.$$

We have then $w(u, \sigma) \to 0$, as $\sigma \to 0$, since u is uniformly continuous on $[0, a]$. More generally, if this limit relation holds uniformly for u running over some bounded set $M \subset C[0, a]$, then M is equicontinuous, and vice versa. Therefore the following result is not too surprising:

Theorem 2.4. (Goldenstein and Markus [13]) On the space $C[0, a]$, the measures of non compactness (2.2) is equivalent to

$$\mu(M) = \lim_{\sigma \to 0} \sup_{u \in M} \omega(u, \sigma) \tag{2.3}$$

for all bounded sets $M \subset C[0, a]$.

For our purpose we use equation (2.3) in the rest of the paper. Closely associated with the measures of non compactness is the concept of k–set contraction.

Theorem 2.5. Let $T : E \to E$ be a continuous mapping of a Banach space E. T is called a k-set contraction if for all $A \subset E$ with A bounded, $T(A)$ is bounded and $\alpha(TA) \leq k\alpha(A), 0 < k < 1$. If

$$\alpha(TA) < \alpha(A), \text{ for all } \alpha(A) > 0,$$

then T is called densifying or condensing map [20].

A k–set contraction with $k \in (0, 1)$, is densifying, but the converse is not true.

Now we state Petryshyn's fixed point theorem [22] which is used in the main results of this paper.

Theorem 2.6. (Petryshyn [22], see also Singh et al. [24]) Let B_r be an open ball about the origin in a Banach space E. If $T : \bar{B}_r \to E$ is a densifying mapping that satisfies the boundary condition,

$$\text{If } T(x) = kx, \text{ for some x in } \partial\bar{B}_r \text{ then } k \leq 1, \tag{P}$$

then $F(T)$, the set of fixed points of T in $\bar{B}_r$ is nonempty.

This property allows us to characterize solution of the integral Eq. (1.1) and will be used in the next section.

3. Main Results

In this section, we will study the existence of the nonlinear functional Eq. (1.1) for $x \in C[0,a]$ under the following assumptions:

(H1) $x \in C(I_a, \mathbb{R}), q \in C(I_a \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}, \mathbb{R}), h \in C(I_a \times [0,B] \times \mathbb{R}, \mathbb{R})$,
and,
$\alpha, \beta : I_a \to I_a, \varphi : I_a \to \mathbb{R}_+, \gamma : \mathbb{R}_+ \to I_a$ are continuous, $\varphi(t) \leq B$ for each $t \in I_a$;

(H2) There exist nonnegative constants c_1, c_2, c_3, $c_1 + c_2 < 1$ such that
$|q(t,u,v,w) - q(t,\bar{u},\bar{v},\bar{w})| \leq c_1|u-\bar{u}| + c_2|v-\bar{v}| + c_3|w-\bar{w}|$;

(H3) (Bounded condition) There exists $r_0 \geq 0$ such that q satisfies the following bounded condition
$\sup\{|q(t,u,v,w)| : t \in I_a, -r_0 \leq u \leq r_0, -r_0 \leq v \leq r_0, -BM_1 \leq w \leq BM_1\} \leq r_0$,
where,
$M_1 = \sup\{|h(t,s,x)| : \text{for all} \quad t \in I_a \quad, s \in [0,B] \quad \text{and} \quad x \in [-r_0, r_0]\}$.

The following result is obtained by using the above hypotheses.

Theorem 3.1. *Under the tacit assumption (H1)-(H3) above, Eq.* (1.1) *has at least one solution in the Banach space $E = C(I_a)$.*

Proof . To prove this result using Theorem 2.6 as our main tool, we need to define operator $T : B_{r_0} \to E$ in the following way

$$(Tx)(t) = q\left(t, x(\alpha(t)), x(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds\right).$$

Now, we show that the operator T is continuous on the ball B_{r_0}. To do this, consider $\sigma > 0$ and take arbitrary $x, y \in B_{r_0}$ such that $\| x - y \| \leq \sigma$. Then for $t \in I_a$, we get

$$\begin{aligned}
|(Tx)(t) - (Ty)(t)| = &\Big| q(t, x(\alpha(t)), x(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds) \\
&- q(t, y(\alpha(t)), y(\beta(t)), \int_0^{\varphi(t)} h(t,s,y(\gamma(s)))ds) \Big| \\
\leq &\Big| q(t, x(\alpha(t)), x(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds) \\
&- q(t, y(\alpha(t)), x(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds) \Big| \\
&+ |q(t, y(\alpha(t)), x(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds) \\
&- q(t, y(\alpha(t)), y(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds) \Big| \\
&+ \Big| q(t, y(\alpha(t)), y(\beta(t)), \int_0^{\varphi(t)} h(t,s,x(\gamma(s)))ds) \\
&- q(t, y(\alpha(t)), y(\beta(t)), \int_0^{\varphi(t)} h(t,s,y(\gamma(s)))ds) \Big| \\
\leq &\, c_1 \big| x(\alpha(t)) - y(\alpha(t)) \big| + c_2 \big| x(\beta(t)) - y(\beta(t)) \big|
\end{aligned}$$

$$
\begin{aligned}
&+ c_3 \int_0^{\varphi(t)} \big|h(t,s,x(\gamma(s))) - h(t,s,y(\gamma(s)))\big| ds \\
&\le (c_1 + c_2) \parallel x - y \parallel + c_3 B\omega(h,\sigma),
\end{aligned}
$$

where for $\sigma > 0$ we define

$$\omega(h,\sigma) = \sup\{|h(t,s,x) - h(t,s,y)| : t \in I_a, s \in [0,B], x,y \in [-r_0, r_0], \|x - y\| \le \sigma\}.$$

Since we know that $h = h(t,s,x)$ is uniformly continuous on the subset $[0,a] \times [0,B] \times \mathbb{R}$, we infer that $\omega(h,\sigma) \to 0$ as $\sigma \to 0$. Thus, the above estimate shows that the operator T is continuous on B_{r_0}.

Now, we will prove that the operator T satisfies densifying condition with respect to the measure μ as defined in (2.3). To do this, we choose a fixed arbitrary $\sigma > 0$. Let us take $x \in M$ and M is bounded subset of E, $t_1, t_2 \in I_a$ such that without loss of generality we may assume that $t_1 \le t_2$ with $t_2 - t_1 \le \sigma$, we obtain

$$
\begin{aligned}
|(Tx)(t_2) - (Tx)(t_1)| &= \Big|q(t_2, x(\alpha(t_2)), x(\beta(t_2)), \int_0^{\varphi(t_2)} h(t_2,s,x(\gamma(s)))ds)) \\
&\quad - q(t_1, x(\alpha(t_1)), x(\beta(t_1)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds))\Big| \\
&\le \Big|q(t_2, x(\alpha(t_2)), x(\beta(t_2)), \int_0^{\varphi(t_2)} h(t_2,s,x(\gamma(s)))ds)) \\
&\quad - q(t_2, x(\alpha(t_2)), x(\beta(t_2)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds))\Big| \\
&\quad + \Big|q(t_2, x(\alpha(t_2)), x(\beta(t_2)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds)) \\
&\quad - q(t_2, x(\alpha(t_2)), x(\beta(t_1)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds))\Big| \\
&\quad + \Big|q(t_2, x(\alpha(t_2)), x(\beta(t_1)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds)) \\
&\quad - q(t_2, x(\alpha(t_1)), x(\beta(t_1)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds))\Big| \\
&\quad + \Big|q(t_2, x(\alpha(t_1)), x(\beta(t_1)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds))) \\
&\quad - q(t_1, x(\alpha(t_1)), x(\beta(t_1)), \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds)))\Big| \\
&\le c_3 \Big| \int_0^{\varphi(t_2)} h(t_2,s,x(\gamma(s)))ds - \int_0^{\varphi(t_1)} h(t_1,s,x(\gamma(s)))ds \Big| \\
&\quad + c_2\big|x(\beta(t_2)) - x(\beta(t_1))\big| + c_1\big|x(\alpha(t_2)) - x(\alpha(t_1))\big| + \omega_{q_1}(I_a,\sigma) \\
&\le c_3 \Big| \int_0^{\varphi(t_1)} (h(t_2,s,x(\gamma(s)) - h(t_1,s,x(\gamma(s))))ds \\
&\quad + \int_{\varphi(t_1)}^{\varphi(t_2)} h(t_2,s,x(\gamma(s)))ds \Big| \\
&\quad + c_2\omega(x,\omega(\beta,\sigma)) + c_1\omega(x,\omega(\alpha,\sigma)) + \omega_{q_1}(I_a,\sigma).
\end{aligned}
$$

For simplicity we use the following notations:

$$\omega_{h_1}(I_a,\sigma)=\sup\left\{|h(t,s,x)-h(\bar{t},s,x)|:|t-\bar{t}|\le\sigma, t\in I_a,\ s\in[0,B],\ x\in[-r_0,r_0]\right\},$$

$$\begin{aligned}\omega_{q_1}(I_a,\sigma)=\sup\big\{&(t,u,v,w)-q(\bar{t},u,v,w)|:\\ &|t-\bar{t}|\le\sigma, t\in I_a,\ u\in[-r_0,r_0],\ v\in[-r_0,r_0],\ w\in[-BM_1,BM_1]\big\}\end{aligned}$$

and

$$k=\sup\{|h(t,s,x)|:\quad t\in I_a,\quad s\in[0,B],\quad x\in[-r_0,r_0]\}.$$

Then using above relation we obtain the estimate

$$|(Tx)(t)-(Ty)(t)|\le c_3B\omega_{h_1}(I_a,\sigma)+c_3k\omega(\varphi,\sigma)+c_2\omega(x,\omega(\beta,\sigma))+c_1\omega(x,\omega(\alpha,\sigma))+\omega_{q_1}(I_a,\sigma).$$

Taking limit as $\sigma\to 0$ we obtain

$$\omega(Tx,\sigma)\le(c_1+c_2)\omega(x,\sigma),\qquad x\in M.$$

This yields the following estimate:

$$\mu(TM)\le(c_1+c_2)\mu(M).$$

This means T is a densifying map. Finally, investigation of condition (P) is remained. Suppose $x\in\partial\bar{B}_{r_0}$. If $Tx=kx$ then we have $kr_0=k\|x\|=\|Tx\|$ and by condition (H3) we concluded that

$$|Tx(t)|=|q(t,x(\alpha(t)),x(\beta(t)),\int_0^{\varphi(t)}h(t,s,x(\gamma(s)))ds)|\le r_0$$

for all $t\in I_a$, hence $\|Tx\|\le r_0$, so this shows $k\le 1$. The proof is complete. □

The following theorem is a generalization of Theorem 3.1.

Theorem 3.2. *Assume that*

(H1)' $x\in C(I_a,\mathbb{R}), q\in C(I_a\times\mathbb{R}^n\times\mathbb{R}^m,\mathbb{R}), h_j\in C(I_a\times[0,b_j]\times\mathbb{R},\mathbb{R})$, *for* $j=0,\dots,m$, *and*
$\alpha_i:I_a\to I_a,\varphi_j:I_a\to\mathbb{R}_+,\gamma_j:\mathbb{R}_+\to I_a$ *are continuous,* $\varphi_j(t)\le B$, *for each* $t\in I_a$, *where*
$B=\max_{j=1,\dots,m}\{b_j\}$,

(H2)' There exist nonnegative constants $c_i, i=1,\dots,n+m$ *such that* $c_1+c_2+\cdots+c_n<1$ *and*

$$|q(t,u_1,\dots,u_n,w_1,\dots,w_m)-q(t,\bar{u}_1,\dots,\bar{u}_n,\bar{w}_1,\dots,\bar{w}_m)|\le\sum_{i=1}^n c_i|u_i-\bar{u}_i|+\sum_{j=1}^m c_{n+j}|w_j-\bar{w}_j|$$

(H3)' There exists $r_0\ge 0$ *such that* q *satisfies the following bounded condition*

$$\begin{aligned}\sup\{|q(t,u_1,\dots,u_n,w_1,\dots,w_m)|:t\in I_a,\ -r_0\le u_i\le r_0,\ i=0,\dots,n,\\ -BM_j\le w_j\le BM_j,\ j=0,\dots,m\}\le r_0,\end{aligned}$$

where

$$M_j=\sup\{|h_j(t,s,x)|;\textit{for all}\quad t\in I_a,\ s\in[0,B],x\in[-r_0,r_0]\},\quad j=1,\dots,m.$$

Then

$$x(t) = q\left(t, x(\alpha_1(t)), x(\alpha_2(t)), \dots, x(\alpha_n(t)), \int_0^{\varphi_1(t)} h_1(t,s,x(\gamma_1(s)))ds\ ,\right.$$
$$\left. \dots, \int_0^{\varphi_m(t)} h_m(t,s,x(\gamma_m(s)))ds\right), \tag{3.1}$$

has at least one solution in the Banach space $E = C(I_a)$.

Proof . The proof is similar as previous theorem, and we can omit the details. □

The following corollaries which are the main results of Maleknejad et al. [17] and Özdemir et al. [21], would be obtained from Theorem 3.1.

Corollary 1. (Maleknejad et al. [17, Theorem 3]) *Assume that*

(M1) $g \in C(I_a \times \mathbb{R}, \mathbb{R}), f \in C(I_a \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, *and there exist nonnegative constants* μ, γ, λ *such that*
$|g(t,0)| \le \mu$,
$|f(t,0,x(\alpha(t))| \le \gamma + \lambda |x(t)|$,

(M2) There exist the continuous functions $a_1, a_2, a_3 : I_a \to I_a$ *such that*
$|g(t,x_1) - g(t,x_2)| \le a_1(t)|x_1 - x_2|$,
$|f(t,y_1,x) - f(t,y_2,x)| \le a_2(t)|y_1 - y_2|$,
$|f(t,y,x_1) - f(t,y,x_2)| \le a_3(t)|x_1 - x_2|$,
for all $x_i, y_i \in \mathbb{R}, i = 1,2\ ,\ t \in I_a$ *and let* $k = \max\{|a_j(t)| : t \in I_a, j = 1,2,3\}$,

(M3) (Sublinear condition) $u(t,s,x) \in C([0,a] \times [0,a] \times \mathbb{R}, \mathbb{R})$ *and satisfies in sublinear condition, so that there exist the constants* α *and* β *such that:*
$|u(t,s,x)| \le \alpha + \beta|x|$ *for all* $t,s \in [0,a]$ *and* $x \in \mathbb{R}$,

(M4) $k < \frac{1-\lambda}{2(1+a\beta)}$.

Then the equation

$$x(t) = g(t,x(t)) + f\left(t, \int_0^t u(t,s,x(s))ds, x(\alpha(t))\right), \qquad t \in I_a$$

has at least one solution in the Banach space $E = C(I_a)$.

Proof . Let $r_0 = \frac{L_2}{1-L_1}$ where

$$L_1 = k + \lambda + ka\beta, L_2 = \mu + \gamma + ka\alpha$$

and

$$q(t,u,v,w) = g(t,v) + f(t,w,u),$$

where $v = x(\beta(t))$, $w = \int_0^{\varphi(t)} u(t,s,x(\gamma(s)))ds$ and $u = x(\alpha(t))$ and $\alpha(t) = t$, $\gamma(s) = s$, $\varphi(t) = t$. By (M4) we have $L_1 = k + \lambda + ka\beta < 1 - (k + ka\beta) < 1$, so r_0 is a positive real number. In addition, it is easy to check that (H2) is concluded from (M2) and (M4). Now we show that (H3) is also holds. Setting $M_1 = \alpha + \beta r_0$, then we have

$$\begin{aligned} |x(t)| &= \left| g(t, x(t)) + f(t, \int_0^t u(t, s, x(s)))ds, x(\alpha(t)) \right| \\ &\leq \Big| g(t, x(t)) - g(t, 0)| + |g(t, 0) \Big| \\ &\quad + \left| f(t, \int_0^t u(t, s, x(s))ds, x(\alpha(t))) - f(t, 0, x(\alpha(t))) \right| + |f(t, 0, x(\alpha(t))| \\ &\leq k|x| + \mu + ka(\alpha + \beta|x|) + \gamma + \lambda|x| \\ &\leq (k + \lambda + ka\beta)|x| + \mu + \gamma + ka\alpha \end{aligned}$$

for all $t \in I_a$, consequently

$$\sup_{t \in I_a} |q(t, u, v, w)| \leq L_1 r_0 + L_2 = L_1 \frac{L_2}{1 - L_1} + L_2 = r_0.$$

Now, the desired result obtained from Theorem 3.1. □

Corollary 2. (Özdemir et al. [21, Theorem 5]) *Assume that*

(K1) $\alpha, \beta \in C(I_a, I_a), \varphi \in C(I_a, \mathbb{R}_+), \gamma \in C(\mathbb{R}_+, I_a), \varphi(t) \leq C,$

(K2) $f, g \in C(I_a \times \mathbb{R}, \mathbb{R})$ *and There exist positive constants* k *and* l *such that*
$|f(t, x_1) - f(t, x_2)| \leq l|x_1 - x_2|,$
$|g(t, x_1) - g(t, x_2)| \leq k|x_1 - x_2|,$
for all $x_1, x_2 \in \mathbb{R}, t \in I_a$ *, and* $|f(t, 0)| \leq N, |g(t, 0)| \leq M,$

(K3) $u(t, s, x) \in C(I_a \times [0, C] \times \mathbb{R}, \mathbb{R})$ *and there exist positive constants* m, n *and* p *such that* $|u(t, s, x)| \leq m + n|x|^p$ *for all* $t \in I_a, s \in [0, C]$ *and* $x \in \mathbb{R},$

(K4) $M + C(m + n)(l + N) + k < 1.$

Then the equation

$$x(t) = g(t, x(\beta(t))) + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))ds, \quad t \in [0, a], \tag{3.2}$$

has at least one solution in the Banach space $E = C(I_a)$.

Proof . Let

$$q(t, u, v, w) = g(t, v) + f(t, u)w,$$

where $v = x(\beta(t))$, $w = \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))ds$ and $u = x(\alpha(t))$. It is easy to cheek that (H2) is concluded by (K2). Now we show that (H3) is also holds. Suppose that $\|x\| \leq \rho, \rho > 0$ and setting $M_1 = m + n\rho^p$, then we have

$$\begin{aligned} |x(t)| &= \sup\{|q(t, u, v, w)| : t \in I_a, -\rho \leq u \leq \rho, -\rho \leq v \leq \rho, -BM_1 \leq w \leq BM_1\} \\ &= \left| g(t, x(\beta(t))) + f(t, x(\alpha(t))) \int_0^{\varphi(t)} u(t, s, x(\gamma(s)))ds \right| \\ &\leq k\|x\| + M + C(N + l\|x\|)(m + n\|x\|^p). \end{aligned}$$

Hence, r_0 in (H3) is real number that satisfies

$$\rho \leq k\rho + M + C(N + l\rho)(m + n\rho^p).$$

Similar argument as in the first paragraph of the proof of [21, Theorem 5] shows that this inequality has a solution in $(0, 1)$. The proof is complete. □

Remark 3.3. Like the similar argument as the above two corollaries, also one can easily prove that Theorem 2 of [18] and Theorem 3 of [10] can be obtained from Theorem 3.1.

4. Examples

In this section, we present some examples of functional-integral equations to illustrate the usefulness of our results.

Example 1. Consider the following nonlinear Volterra integral equation

$$x(t) = \frac{1}{4}(te^{-t} + t^2x(t)) + \frac{1}{2(e^t + \sin(|x(t^3)|))}\int_0^{t^3} e^{-2s}(e^s + t\cos(s) + \frac{1}{2}(x(s^2)))ds, \quad t \in [0, 1] \quad (4.1)$$

Observe that equation (4.1) is a special case of Eq. (1.1). Let us take $q : [0, 1] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \alpha, \beta, \gamma, \varphi : [0, 1] \to [0, 1]$ and $h : [0, 1] \times [0, 1] \times \mathbb{R} \to \mathbb{R}$ and comparing (4.1) with Eq. (1.1), we get

$$\alpha(t) = t, \varphi(t) = \beta(t) = t^3, \gamma(t) = t^2, \qquad \text{for all} \qquad t \in [0, 1],$$

$$q(t, x(\alpha(t)), x(\beta(t)), w) = q_1(t, x(\alpha(t))) + q_2(t, x(\beta(t)), w),$$

where

$$q_1(t, x(\alpha(t))) = \frac{1}{4}(te^{-t} + t^2x(t)),\ q_2(t, x(\beta(t)), w) = \frac{w}{2(e^t + \sin(|x(t^3)|))},\ w = \int_0^{\varphi(t)} h(t, s, x(\gamma(s)))ds,$$

and

$$h(t, s, x(\gamma(s))) = e^{-2s}(e^s + t\cos(s) + \frac{1}{2}(x(s^2))).$$

We investigate the solution in $C[0, 1]$. It is easy to prove that these functions satisfy the assumptions (H1) and (H2). We show that (H3) also holds. Choose $r_0 = \frac{3}{2} + e$ then we have $M_1 \leq \frac{7}{4} + \frac{3}{2}e$ and

$$\begin{aligned}
&\sup\{|q(t, u, v, w)| : t \in [0, 1], -r_0 \leq u \leq r_0, -r_0 \leq v \leq r_0\} \\
&\leq \sup\{|\frac{1}{4}(te^{-t} + t^2x(t)) + \frac{1}{2}\omega|; t \in [0, 1], -(\frac{7}{4} + \frac{3}{2}e) \leq \omega \leq (\frac{7}{4} + \frac{3}{2}e)\} \\
&\leq \frac{3}{2} + e.
\end{aligned}$$

Hence, from Theorem 3.1 equation (4.1) has at least one solution in Banach space $C[0, 1]$.

Example 2. Consider the following nonlinear integral equation

$$x(t) = \left(\frac{t^2}{2 + 2t^2}\ln(1 + |x(t)|) + \int_0^t \frac{se^{-t}\sin(x(s))}{1 + |\cos(x(s))|}ds\right)$$

$$\times\left(\frac{t^4}{4+4t^4}\ln(1+|x(t)|)+\int_0^t \frac{se^{-t}\sin(x(s))}{2+|\cos(x(s))|}ds\right). \qquad t\in[0,1]$$

Observe that Equation (**??**) is a special case of Eq. (3.1). Let us take $q:[0,1]\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}, \alpha_1,\alpha_2,\gamma_1,\gamma_2,\varphi_1,\varphi_2:[0,1]\to[0,1]$ and $h_1,h_2:[0,1]\times[0,1]\times\mathbb{R}\to\mathbb{R}$ and comparing (**??**) with Eq. (3.1), we get

$$\alpha_1(t)=\alpha_2(t)=\varphi_1(t)=\varphi_2(t)=\gamma_1(t)=\gamma_2(t)=t, \quad \text{for all} \quad t\in[0,1],$$

$$q(t,x(\alpha_1(t)),x(\alpha_2(t)),w_1,w_2)=q_1(t,x(\alpha_1(t)),w_1)\times q_2(t,x(\alpha_2(t)),w_2),$$

where

$$q_1(t,x(\alpha_1(t)),w_1)=\frac{t^2}{2+2t^2}\ln(1+|x(t)|)+w_1, \quad w_1=\int_0^{\varphi_1(t)} h_1(t,s,x(\gamma_1(s)))ds,$$

$$q_2(t,x(\alpha_2(t)),w_2)=\frac{t^4}{4+4t^4}\ln(1+|x(t)|)+w_2, \quad w_2=\int_0^{\varphi_2(t)} h_2(t,s,x(\gamma_2(s)))ds,$$

and

$$h_j(t,s,x(\gamma(s)))=\frac{se^{-t}\sin(x(s))}{j+|\cos(x(s))|}ds. \qquad j=1,2.$$

We investigate the solution in $C[0,1]$. It is easy to prove that these functions satisfy the assumptions (H1)' and (H2)'. We show that (H3)' also holds. Suppose that $\|x\|\le\rho,\rho>0$, then we have

$$|x(t)|=\left|\left(\frac{t^2}{2+2t^2}\ln(1+|x(t)|)+\int_0^t \frac{se^{-t}\sin(x(s))}{1+|\cos(x(s))|}ds\right)\right.$$
$$\left.\times\left(\frac{t^4}{4+4t^4}\ln(1+|x(t)|)+\int_0^t \frac{se^{-t}\sin(x(s))}{2+|\cos(x(s))|}ds\right)\right|\le(\frac{1}{2}\rho+1)(\frac{1}{4}\rho+\frac{1}{2}),$$

for all $t\in I_a$. Hence (H3)' holds if, $(\frac{1}{2}\rho+1)(\frac{1}{4}\rho+\frac{1}{2})\le\rho$. This shows that $r_0=2$. The result is followed from Theorem 3.2.

Example 3. Consider the following nonlinear Volterra integral equation

$$x(t)=\frac{1}{2}\sin(\frac{t}{2})x(t)+\int_0^{t^2} ts\sin(x(s))ds+\frac{1}{4}\int_0^t x(s)ds$$
$$+\left(\int_0^{t^2} ts\sin(\frac{x(s)}{1+x(s)})ds\right)\left(\int_0^t \frac{x^2(s)}{1+x^2(s)}ds\right), \quad t\in[0,1]. \tag{4.2}$$

Observe that Equation (4.2) is a special case of Eq. (3.1). It is easy to investigate the assumptions (H1)' and (H2)'. We show that (H3)' also holds. Suppose that $\|x\|\le\rho,\rho>0$, then we have

$$|x(t)|=\left|\frac{1}{2}\sin(\frac{t}{2})x(t)+\int_0^{t^2} ts\sin(x(s))ds+\frac{1}{4}\int_0^t x(s)ds\right.$$
$$\left.+\left(\int_0^{t^2} ts\sin(\frac{x(s)}{1+x(s)})ds\right)\left(\int_0^t \frac{x^2(s)}{1+x^2(s)}ds\right)\right|\le\frac{3}{4}\rho+2$$

for all $t\in I_a$. Hence (H3)' holds if, $\frac{3}{4}\rho+2\le\rho$. This shows we can choose $r_0\ge 8$. So, from Theorem 3.2 equation (4.2) has at least one solution in Banach space $C[0,1]$.

Example 4. Consider the following nonlinear Volterra integral equation studied in [10],

$$x(t) = \left(u(t, x(t)) + f(t, x(\alpha(t)), \int_0^t p(t, s, x(s))ds)\right) g\left(t, x(\beta(t)), \int_0^a q(t, s, x(s))ds\right), \quad t \in I_a. \tag{4.3}$$

Observe that Equation (4.3) is a special case of Eq. (3.1). Also for $u(t, x) = 0$, we obtain nonlinear functional equation studied [16, 6].

5. Conclusion and Perspective

By unifying and extending the previous results of [17, 18, 21, 5, 16] and applying Petryshyn's fixed point theorem (Theorem 2.6), in the third section, we obtained a new method to prove the existence of solutions for some nonlinear functional-integral equations. The advantage of Theorem 2.6 among the others (Darbo and Schauder fixed point theorems) lies in that in applying the theorem, one does not need to verify the involved operator maps a closed convex subset onto itself. Also condition (P) deals with the eigenvalue of nonlinear operator T (see [3, 12, 11, 8] for definitions and results) which the authors hope that this can be constitute to further study in this area of research.

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