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Strong Differential Subordination and Superordination of Analytic Functions Associated with Komatu operator

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Abstract

Strong differential subordination and superordination properties are determined for some families analytic functions in the open unit disk which are associated with the Komatu operator by investigating appropriate classes of admissible functions. New strong differential sandwich-type results are also obtained.

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1. Introduction, Preliminaries and Definitions

Let $\mathcal{H}(\mathbb{U})$ denote the class of analytic functions in the open unit disk

$$\mathbb{U} := \{z \in \mathbb{C} : |z| < 1\}.$$

For $n \in \mathbb{N} = \{1, 2, 3, ...\}$ and $a \in \mathbb{C}$, let

$$\mathcal{H}[a, n] = \left\{f : f \in \mathcal{H}(\mathbb{U}) \quad \text{and} \quad f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots \right\},$$

with $\mathcal{H}_0 \equiv \mathcal{H}[0, 1]$ and $\mathcal{H} \equiv \mathcal{H}[1, 1]$. Let $\mathcal{A}$ denote the class of all normalized analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (z \in \mathbb{U}). \tag{1.1}$$

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Let f and F be members of $\mathcal{H}(\mathbb{U})$. The function f is said to be *subordinate* to F, or (equivalently) F is said to be *superordinate* to f, if there exists a Schwarz function w analytic in $\mathbb{U}$, with

$$w(0) = 0 \quad \text{and} \quad |w(z)| < 1 \quad (z \in \mathbb{U}),$$

such that

$$f(z) = F(w(z)) \quad (z \in \mathbb{U}).$$

In such a case, we write

$$f \prec F \quad \text{or} \quad f(z) \prec F(z) \quad (z \in \mathbb{U}).$$

If the function F is univalent in $\mathbb{U}$, then we have

$$f(z) \prec F(z) \quad (z \in \mathbb{U}) \quad \Longleftrightarrow \quad f(0) = F(0) \quad \text{and} \quad f(\mathbb{U}) \subset F(\mathbb{U}).$$

Let $H(z,\zeta)$ be analytic in $\mathbb{U} \times \overline{\mathbb{U}}$ and let $f(z)$ be analytic and univalent in $\mathbb{U}$. Then the function $H(z,\zeta)$ is said to be *strongly subordinate* to $f(z)$, or $f(z)$ is said to be *strongly superordinate* to $H(z,\zeta)$, written as

$$H(z,\zeta) \prec\prec f(z) \quad (z \in \mathbb{U}; \zeta \in \overline{\mathbb{U}}),$$

if, for $\zeta \in \overline{\mathbb{U}}$, $H(z,\zeta)$ as a function of z is subordinate to $f(z)$. We note that

$$H(z,\zeta) \prec\prec f(z)\ (z \in \mathbb{U}; \zeta \in \overline{\mathbb{U}}) \Longleftrightarrow H(0,\zeta) = f(0) \text{ and } H(\mathbb{U} \times \overline{\mathbb{U}}) \subset f(\mathbb{U}).$$

For a function f given by (1.1) and g given by

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n, \tag{1.2}$$

we denote by $(f * g)(z)$ the Hadamard product (or convolution) of f and g, defined by

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z). \tag{1.3}$$

For a function f in the class $\mathcal{A}$ given by (1.1), Komatu [4, 5] introduced the following operator:

$$\mathcal{K}_c^{\delta} f(z) = \frac{(c+1)^{\delta}}{\Gamma(\delta) z^c} \int_0^z t^{c-1} \left(\log \frac{z}{t}\right)^{\delta-1} f(t)dt \quad (\delta > 0; c > -1). \tag{1.4}$$

For $f \in \mathcal{A}$, it can be easily verified that

$$\mathcal{K}_c^{\delta} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{c+1}{c+k}\right)^{\delta} a_k z^k \quad (\delta \geq 0; c > -1). \tag{1.5}$$

Also, it is easily verified from (1.5) that

$$z\left(\mathcal{K}_c^{\delta} f\right)'(z) = (c+1)\mathcal{K}_c^{\delta-1} f(z) - c\mathcal{K}_c^{\delta} f(z). \tag{1.6}$$

Definition 1.1. *[10] Let*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

and let $h(z)$ be univalent in $\mathbb{U}$. If $p(z)$ is analytic in $\mathbb{U}$ and satisfies the following (second-order) strong differential subordination:

$$\phi(p(z), zp'(z), z^2p''(z); z, \zeta) \prec\prec h(z) \quad (z \in \mathbb{U}; \zeta \in \overline{\mathbb{U}}), \tag{1.7}$$

then $p(z)$ is called a solution of the strong differential subordination. The univalent function $q(z)$ is called a dominant of the solutions of the strong differential subordination or more simply a dominant if

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

for all $p(z)$ satisfying (1.7). A dominant $\tilde{q}(z)$ that satisfies

$$\tilde{q}(z) \prec q(z) \quad (z \in \mathbb{U})$$

for all dominants $q(z)$ of (1.7) is said to be the best dominant.

Recently, Oros [8] introduced the following notion of strong differential superordinations as the dual concept of strong differential subordinations.

Definition 1.2. *[7, 8] Let*

$$\varphi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

and let $h(z)$ be analytic in $\mathbb{U}$. If

$$p(z) \quad \text{and} \quad \varphi(p(z), zp'(z), z^2p''(z); z, \zeta)$$

are univalent in $\mathbb{U}$ for $\zeta \in \overline{\mathbb{U}}$ and satisfy the following (second-order) strong differential superordination:

$$h(z) \prec\prec \varphi(p(z), zp'(z), z^2p''(z); z, \zeta) \quad (z \in \mathbb{U}; \zeta \in \overline{\mathbb{U}}), \tag{1.8}$$

then $p(z)$ is called a solution of the strong differential superordination. An analytic function $q(z)$ is called a subordinant of the solution of the strong differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for all $p(z)$ satisfying (1.8). A univalent subordinant $\tilde{q}(z)$ that satisfies

$$q(z) \prec \tilde{q}(z) \quad (z \in \mathbb{U})$$

for all subordinants $q(z)$ of (1.8) is said to be the best subordinant.

We denote by $\mathcal{Q}$ the class of functions q that are analytic and injective on $\overline{\mathbb{U}} \setminus E(q)$, where

$$E(q) = \left\{ \xi \in \partial\mathbb{U} : \lim_{z \to \xi} q(z) = \infty \right\},$$

and are such that $q'(\xi) \neq 0$ for $\xi \in \partial\mathbb{U} \setminus E(q)$. Further, let the subclass of $\mathcal{Q}$ for which $q(0) = a$ be denoted by $\mathcal{Q}(a)$, $\mathcal{Q}(0) \equiv \mathcal{Q}_0$ and $\mathcal{Q}(1) \equiv \mathcal{Q}_1$.

Definition 1.3. *[10] Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}$ and $n \in \mathbb{N}$. The class of admissible functions $\Psi_n[\Omega, q]$ consists of those functions*

$$\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the following admissibility condition:

$$\psi(r, s, t; z, \zeta) \notin \Omega$$

whenever

$$r = q(\xi), \quad s = k\xi q'(\xi) \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \geq k\,\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geq n).$$

We simply write $\Psi_1[\Omega, q]$ as $\Psi[\Omega, q]$.

Definition 1.4. *[8] Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{H}[a, n]$ with $q'(z) \neq 0$. The class of admissible functions $\Psi'_n[\Omega, q]$ consists of those functions*

$$\psi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the following admissibility condition:

$$\psi(r, s, t; \xi, \zeta) \in \Omega$$

whenever

$$r = q(z), \quad s = \frac{zq'(z)}{m}, \quad \text{and} \quad \Re\left(\frac{t}{s} + 1\right) \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geq n \geq 1).$$

In particular, we write $\Psi'_1[\Omega, q]$ as $\Psi'[\Omega, q]$.

For the above two classes of admissible functions, G.I. Oros and G. Oros [10] proved the following result.

Lemma 1.5. *[10] Let $\psi \in \Psi_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{H}[a, n]$ satisfies*

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \in \Omega,$$

then

$$p(z) \prec q(z) \quad (z \in \mathbb{U}).$$

G.I. Oros [8], on the other hand proved Lemma 1.6

Lemma 1.6. *[8] Let $\psi \in \Psi'_n[\Omega, q]$ with $q(0) = a$. If $p \in \mathcal{Q}(a)$ and*

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta)$$

is univalent in $\mathbb{U}$ for $\zeta \in \overline{\mathbb{U}}$, then

$$\Omega \subset \{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\}$$

implies the following subordination relationship:

$$q(z) \prec p(z) \quad (z \in \mathbb{U}).$$

In this present investigation, by making use of the strong differential subordination results and the strong superordination results of G. I. Oros and G. Oros [8, 10], we consider certain suitable classes of admissible functions and investigate some strong differential subordination and strong differential superordination properties of analytic functions associated with the Komatu operator $\mathcal{K}_c^\delta$ defined by (1.5). New strong differential sandwich-type results associated with the Komatu operator are also obtained. In recent years, several authors obtained many interesting results in strong differential subordination and superordination [1, 2, 3, 8, 9, 10, 11].

2. The main subordination results

We first define the following class of admissible functions that are required in our first result.

Definition 2.1. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}$. The class of admissible functions $\Phi_{\mathcal{K}}[\Omega, q]$ consists of those functions*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; z, \zeta) \notin \Omega$$

whenever

$$u = q(\xi), \quad v = \frac{k\xi q'(\xi) + cq(\xi)}{(c+1)} \quad (c > -1),$$

and

$$\Re\left\{\frac{(c+1)^2w - c^2u}{(c+1)v - cu} - 2c\right\} \geq k\,\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geq 1).$$

Theorem 2.2. *Let $\phi \in \Phi_{\mathcal{K}}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\left\{\phi\left(\mathcal{K}_c^\delta f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\} \subset \Omega, \tag{2.1}$$

then

$$\mathcal{K}_c^\delta f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Proof . Define the function p in $\mathbb{U}$ by

$$p(z) := \mathcal{K}_c^\delta f(z). \tag{2.2}$$

A simple calculation yields

$$\mathcal{K}_c^{\delta-1} f(z) = \frac{1}{(c+1)} \left[cp(z) + zp'(z) \right]. \tag{2.3}$$

Further computations show that

$$\mathcal{K}_c^{\delta-2} f(z) = \frac{c^2 p(z) + (2c+1) zp'(z) + z^2 p''(z)}{(c+1)^2}. \tag{2.4}$$

We now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{cr+s}{c+1}, \quad w = \frac{c^2 r + (2c+1)s + t}{(c+1)^2}. \tag{2.5}$$

Let

$$\psi(r,s,t;z,\zeta) = \phi(u,v,w;z,\zeta) = \phi\left(r, \frac{s+cr}{c+1}, \frac{t+(2c+1)s+c^2 r}{(c+1)^2}; z, \zeta\right). \tag{2.6}$$

The proof will make use of Lemma 1.5. Using (2.2), (2.3), and (2.4), from (2.6) we obtain

$$\psi(p(z), zp'(z), z^2 p''(z); z, \zeta) = \phi\left(\mathcal{K}_c^\delta f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right). \tag{2.7}$$

Hence (2.1) becomes

$$\psi(p(z), zp'(z), z^2 p''(z); z, \zeta) \in \Omega.$$

A computation using (2.5) yields

$$\frac{t}{s} + 1 = \frac{(c+1)^2 w - c^2 u}{(c+1)v - cu} - 2c.$$

Thus the admissibility condition for $\phi \in \Phi_{\mathcal{K}}[\Omega, q]$ in Definition 2.1 is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.5

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

or, equivalently,

$$\mathcal{K}_c^\delta f(z) \prec q(z) \quad (z \in \mathbb{U}),$$

which evidently completes the proof of Theorem 2.2. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathcal{K}}[h(\mathbb{U}), q]$ is written as $\Phi_{\mathcal{K}}[h, q]$. The following result is an immediate consequence of Theorem 2.2.

Theorem 2.3. *Let $\phi \in \Phi_{\mathcal{K}}[h, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\mathcal{K}_c^\delta f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right) \prec\prec h(z), \quad (z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}) \tag{2.8}$$

then

$$\mathcal{K}_c^\delta f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Our next result in an extension of Theorem 2.2 to the case in which the behavior of q on $\partial\mathbb{U}$ is not known.

Theorem 2.4. *Let h and q be univalent in $\mathbb{U}$ with $q(0) = 0$, and set $q_\rho(z) = q(\rho z)$ and $h_\rho(z) = h(\rho z)$. Let $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ satisfies one of the following conditions:*

(i) *$\phi \in \Phi_{\mathcal{K}}[h, q_\rho]$ for some $\rho \in (0, 1)$, or*
(ii) *there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_{\mathcal{K}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (2.8), then

$$\mathcal{K}_c^\delta f(z) \prec q(z) \quad (z \in \mathbb{U}).$$

Proof . The proof of Theorem 2.4 is similar to that of a known result [6, Theorem 2.3d, page 30] and so it is omitted here. □

Our next theorem yields the best dominant of the strong differential subordination (2.8).

Theorem 2.5. *Let h be univalent in $\mathbb{U}$, and $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the following differential equation*

$$\phi\left(q(z), \frac{zq'(z) + cq(z)}{c+1}, \frac{z^2q''(z) + (2c+1)zq'(z) + c^2q(z)}{(c+1)^2}; z, \zeta\right) = h(z) \tag{2.9}$$

has a solution q with $q(0) = 0$ and satisfies one of the following conditions:

(i) *$q \in \mathcal{Q}_0$ and $\phi \in \Phi_{\mathcal{K}}[h, q]$,*
(ii) *q is univalent in $\mathbb{U}$ and $\phi \in \Phi_{\mathcal{K}}[h, q_\rho]$ for some $\rho \in (0, 1)$, or*
(iii) *q is univalent in $\mathbb{U}$ and there exists $\rho_0 \in (0, 1)$ such that $\phi \in \Phi_{\mathcal{K}}[h_\rho, q_\rho]$ for all $\rho \in (\rho_0, 1)$.*

If $f \in \mathcal{A}$ satisfies (2.8), then

$$\mathcal{K}_c^\delta f(z) \prec q(z) \quad (z \in \mathbb{U}),$$

and q is the best dominant.

Proof . Following the same arguments as in [6, Theorem 2.3e, page 31], we deduce that q is a dominant from Theorem 2.3 and Theorem 2.4. Since q satisfies (2.9), it is also a solution of (2.8) and therefore q will be dominated by all dominants. Hence q is the best dominant. □

We will apply Theorem 2.2 to a specific case for $q(z) = Mz$, $M > 0$.

In the particular case $q(z) = Mz, M > 0$, and in view of Definition 2.1, the class of admissible functions $\Phi_{\mathcal{K}}[\Omega, q]$, denoted by $\Phi_{\mathcal{K}}[\Omega, M]$, is described below.

Definition 2.6. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{\mathcal{K}}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{c+k}{c+1}Me^{i\theta}, \frac{L + [c(2k+1) + k]Me^{i\theta}}{(c+1)^2} : z, \zeta\right) \notin \Omega, \tag{2.10}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and $\Re\{Le^{-i\theta}\} \geq (k-1)kM$ for all θ, $\zeta \in \overline{\mathbb{U}}$ and $k \geq 1$.

Corollary 2.7. *Let $\phi \in \Phi_{\mathcal{K}}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right) \in \Omega$$

$$(z \in \mathbb{U};\ \zeta \in \overline{\mathbb{U}}),$$

then

$$\left|\mathcal{K}_c^{\delta} f(z)\right| < M.$$

For the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_{\mathcal{K}}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{K}}[M]$.

Corollary 2.8. *Let $\phi \in \Phi_{\mathcal{K}}[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right)\right| < M,$$

then

$$\left|\mathcal{K}_c^{\delta} f(z)\right| < M.$$

Definition 2.9. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_0 \cap \mathcal{H}$. The class of admissible functions $\Phi_{\mathcal{K},1}[\Omega, q]$ consists of those functions*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; z, \zeta) \notin \Omega$$

whenever

$$u = q(\xi), \quad v = \frac{1}{c+1}\left(k\xi q'(\xi) + (c+1)q(\xi)\right) \quad (c > -1),$$

and

$$\Re\left\{\frac{[w - 2v + u](c+1)}{v - u}\right\} \geq k\,\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geq 1).$$

Theorem 2.10. *Let $\phi \in \Phi_{\mathcal{K},1}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\left\{\phi\left(\frac{\mathcal{K}_c^{\delta} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\} \subset \Omega, \tag{2.11}$$

then

$$\frac{\mathcal{K}_c^{\delta} f(z)}{z} \prec q(z) \quad (z \in \mathbb{U}).$$

Proof . Define the analytic function p in $\mathbb{U}$ by

$$p(z) := \frac{\mathcal{K}_c^{\delta} f(z)}{z}. \tag{2.12}$$

By making use of (1.5) in (2.12), we get

$$\frac{\mathcal{K}_c^{\delta-1} f(z)}{z} = \frac{1}{c+1}\left(zp'(z) + (c+1)p(z)\right). \tag{2.13}$$

Further computations show that

$$\frac{\mathcal{K}_c^{\delta-2} f(z)}{z} = \frac{1}{(c+1)^2}\left(z^2p''(z) + (2c+3)zp'(z) + (c+1)^2p(z)\right). \tag{2.14}$$

We now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{s+(c+1)r}{c+1}, \quad w = \frac{t+(2c+3)s+(c+1)^2r}{(c+1)^2}. \tag{2.15}$$

Let

$$\psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi\left(r, \frac{s+(c+1)r}{c+1}, \frac{t+(2c+3)s+(c+1)^2r}{(c+1)^2}; z, \zeta\right). \tag{2.16}$$

The proof shall make use of Lemma 1.5. Using (2.12), (2.13), (2.14), from (2.16), we obtain

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta) = \phi\left(\frac{\mathcal{K}_c^{\delta} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right). \tag{2.17}$$

Hence (2.11) becomes

$$\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \in \Omega.$$

A computation using (2.15) yields

$$\frac{t}{s} + 1 = \frac{[w-2v+u](c+1)}{v-u}.$$

Thus the admissibility condition for $\phi \in \Phi_{\mathcal{K},1}[\Omega, q]$ in Definition 2.9 is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi \in \Psi[\Omega, q]$ and by Lemma 1.5

$$p(z) \prec q(z) \quad (z \in \mathbb{U})$$

or, equivalently,

$$\frac{\mathcal{K}_c^{\delta} f(z)}{z} \prec q(z) \quad (z \in \mathbb{U}),$$

which evidently completes the proof of Theorem 2.10. □

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathcal{K},1}[h(\mathbb{U}), q]$ is written as $\Phi_{\mathcal{K},1}[h, q]$. The following result is an immediate consequence of Theorem 2.10.

Theorem 2.11. *Let $\phi \in \Phi_{\mathcal{K},1}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathcal{K}_c^{\delta} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right) \prec\prec h(z), \tag{2.18}$$

then

$$\frac{\mathcal{K}_c^{\delta} f(z)}{z} \prec q(z) \quad (z \in \mathbb{U}).$$

We will apply Theorem 2.10 to a specific case for $q(z) = Mz$, $M > 0$.

In the particular case $q(z) = Mz, M > 0$, and in view of Definition 2.9, the class of admissible functions $\Phi_{\mathcal{K},1}[\Omega, q]$, denoted by $\Phi_{\mathcal{K},1}[\Omega, M]$, is described below.

Definition 2.12. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{\mathcal{K},1}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ such that*

$$\phi\left(Me^{i\theta}, \frac{k+c+1}{c+1} Me^{i\theta}, \frac{L + [(2c+3)k + (c+1)^2]Me^{i\theta}}{(c+1)^2} : z, \zeta\right) \notin \Omega, \tag{2.19}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and $\Re\{Le^{-i\theta}\} \geq (k-1)kM$ for all θ, $\zeta \in \overline{\mathbb{U}}$ and $k \geq 1$.

Corollary 2.13. *Let $\phi \in \Phi_{\mathcal{K},1}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathcal{K}_c^{\delta} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right) \in \Omega \quad (z \in \mathbb{U};\ \zeta \in \overline{\mathbb{U}}),$$

then

$$\left|\frac{\mathcal{K}_c^{\delta} f(z)}{z}\right| < M.$$

For the special case $\Omega = q(\mathbb{U}) = \{w : |w| < M\}$, the class $\Phi_{\mathcal{K},1}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{K},1}[M]$.

Corollary 2.14. *Let $\phi \in \Phi_{\mathcal{K},1}[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi\left(\frac{\mathcal{K}_c^{\delta} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right)\right| < M,$$

then

$$\left|\frac{\mathcal{K}_c^{\delta} f(z)}{z}\right| < M.$$

Definition 2.15. *Let Ω be a set in $\mathbb{C}$ and $q \in \mathcal{Q}_1 \cap \mathcal{H}$. The class of admissible functions $\Phi_{\mathcal{K},2}[\Omega, q]$ consists of those functions*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; z, \zeta) \notin \Omega$$

whenever

$$u = q(\xi), \quad v = \frac{1}{c+1}\left((c+1)q(\xi) + \frac{k\xi q'(\xi)}{q(\xi)}\right) \quad (q(\xi) \neq 0; c > -1),$$

and

$$\Re\left\{\frac{[vw - 3uv + 2u^2](c+1)}{v-u}\right\} \geq k\,\Re\left\{\frac{\xi q''(\xi)}{q'(\xi)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U} \setminus E(q);\ \zeta \in \overline{\mathbb{U}};\ k \geq 1).$$

Theorem 2.16. *Let $\phi \in \Phi_{\mathcal{K},2}[\Omega, q]$. If $f \in \mathcal{A}$ satisfies*

$$\left\{\phi\left(\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)}, \frac{\mathcal{K}_c^{\delta-2}f(z)}{\mathcal{K}_c^{\delta-1}f(z)}, \frac{\mathcal{K}_c^{\delta-3}f(z)}{\mathcal{K}_c^{\delta-2}f(z)}; z, \zeta\right) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\} \subset \Omega, \tag{2.20}$$

then

$$\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)} \prec q(z) \quad (z \in \mathbb{U}).$$

Proof . Define the analytic function p in $\mathbb{U}$ by

$$p(z) := \frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)}. \tag{2.21}$$

Using (2.21), we get

$$\frac{zp'(z)}{p(z)} := \frac{z\left(\mathcal{K}_c^{\delta-1}f(z)\right)'}{\mathcal{K}_c^{\delta-1}f(z)} - \frac{z\left(\mathcal{K}_c^{\delta}f(z)\right)'}{\mathcal{K}_c^{\delta}f(z)}. \tag{2.22}$$

By making use of (1.5) in (2.22), we get

$$\frac{\mathcal{K}_c^{\delta-2}f(z)}{\mathcal{K}_c^{\delta-1}f(z)} = \frac{1}{c+1}\left(\frac{zp'(z)}{p(z)} + (c+1)p(z)\right). \tag{2.23}$$

Further computations show that

$$\frac{\mathcal{K}_c^{\delta-3}f(z)}{\mathcal{K}_c^{\delta-2}f(z)} = \frac{1}{c+1}\left\{(c+1)p(z) + \frac{zp'(z)}{p(z)} + \frac{(c+1)zp'(z) + \frac{zp'(z)}{p(z)} + \frac{z^2p''(z)}{p(z)} - \left(\frac{zp'(z)}{p(z)}\right)^2}{(c+1)p(z) + \frac{zp'(z)}{p(z)}}\right\}. \tag{2.24}$$

We now define the transformations from $\mathbb{C}^3$ to $\mathbb{C}$ by

$$u = r, \quad v = \frac{1}{c+1}\left((c+1)r + \frac{s}{r}\right),$$

$$w = \frac{1}{c+1}\left\{(c+1)r + \frac{s}{r} + \frac{(c+1)s + \frac{t}{r} + \frac{s}{r} - \left(\frac{s}{r}\right)^2}{(c+1)r + \frac{s}{r}}\right\}. \tag{2.25}$$

Let $\psi(r,s,t;z,\zeta)=\phi(u,v,w;z,\zeta)$
$$=\phi\left(r, \frac{1}{c+1}\left((c+1)r+\frac{s}{r}\right),\right.$$
$$\left.\frac{1}{c+1}\left\{(c+1)r+\frac{s}{r}+\frac{(c+1)s+\frac{t}{r}+\frac{s}{r}-\left(\frac{s}{r}\right)^2}{(c+1)r+\frac{s}{r}}\right\};z,\zeta\right). \tag{2.26}$$

The proof shall make use of Lemma 1.5. Using (2.21), (2.23), (2.24), from (2.26), we obtain

$$\psi(p(z),zp'(z),z^2p''(z);z,\zeta)=\phi\left(\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)},\frac{\mathcal{K}_c^{\delta-2}f(z)}{\mathcal{K}_c^{\delta-1}f(z)},\frac{\mathcal{K}_c^{\delta-3}f(z)}{\mathcal{K}_c^{\delta-2}f(z)};z,\zeta\right). \tag{2.27}$$

Hence (2.20) becomes

$$\psi(p(z),zp'(z),z^2p''(z);z,\zeta)\in\Omega.$$

A computation using (2.25) yields

$$\frac{t}{s}+1=\frac{[vw-3uv+2u^2](c+1)}{v-u}.$$

Thus the admissibility condition for $\phi\in\Phi_{\mathcal{K},2}[\Omega,q]$ in Definition 2.15 is equivalent to the admissibility condition for ψ as given in Definition 1.3. Hence $\psi\in\Psi[\Omega,q]$ and by Lemma 1.5

$$p(z)\prec q(z)\quad(z\in\mathbb{U})$$

or, equivalently,

$$\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)}\prec q(z)\quad(z\in\mathbb{U}),$$

which evidently completes the proof of Theorem 2.16. □

If $\Omega\neq\mathbb{C}$ is a simply connected domain, then $\Omega=h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω. In this case, the class $\Phi_{\mathcal{K},2}[h(\mathbb{U}),q]$ is written as $\Phi_{\mathcal{K},2}[h,q]$. The following result is an immediate consequence of Theorem 2.16.

Theorem 2.17. *Let $\phi\in\Phi_{\mathcal{K},2}[\Omega,q]$. If $f\in\mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)},\frac{\mathcal{K}_c^{\delta-2}f(z)}{\mathcal{K}_c^{\delta-1}f(z)},\frac{\mathcal{K}_c^{\delta-3}f(z)}{\mathcal{K}_c^{\delta-2}f(z)};z,\zeta\right)\prec\prec h(z), \tag{2.28}$$

then

$$\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)}\prec q(z)\quad(z\in\mathbb{U}).$$

We will apply Theorem 2.16 to a specific case for $q(z)=1+Mz$, $M>0$.

In the particular case $q(z)=1+Mz, M>0$, and in view of Definition 2.15, the class of admissible functions $\Phi_{\mathcal{K},2}[\Omega,q]$, denoted by $\Phi_{\mathcal{K},2}[\Omega,M]$, is described below.

Definition 2.18. *Let Ω be a set in $\mathbb{C}$ and $M > 0$. The class of admissible functions $\Phi_{\mathcal{K},2}[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$ such that*

$$\phi\Bigg(1 + Me^{i\theta}, 1 + \frac{k+1+Me^{i\theta}}{(c+1)(1+Me^{i\theta})}Me^{i\theta}, 1 + \frac{k+1+Me^{i\theta}}{(c+1)(1+Me^{i\theta})}Me^{i\theta} + \frac{(M+e^{-i\theta})[Le^{-i\theta} + kM(c+1)(1+Me^{i\theta}) + kM] - k^2M^2}{(c+1)(M+e^{-i\theta})\left\{(c+1)[M^2e^{i\theta} + 2M + e^{-i\theta}] + kM\right\}} : z, \zeta\Bigg) \notin \Omega, \tag{2.29}$$

whenever $z \in \mathbb{U}$, $\theta \in \mathbb{R}$ and $\Re\{Le^{-i\theta}\} \geq (k-1)kM$ for all θ, $\mu > 0$, $\zeta \in \overline{\mathbb{U}}$ and $k \geq 1$.

Corollary 2.19. *Let $\phi \in \Phi_{\mathcal{K},2}[\Omega, M]$. If $f \in \mathcal{A}$ satisfies*

$$\phi\left(\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)}, \frac{\mathcal{K}_c^{\delta-2}f(z)}{\mathcal{K}_c^{\delta-1}f(z)}, \frac{\mathcal{K}_c^{\delta-3}f(z)}{\mathcal{K}_c^{\delta-2}f(z)}; z, \zeta\right) \in \Omega \qquad (z \in \mathbb{U};\ \zeta \in \overline{\mathbb{U}}),$$

then

$$\left|\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)} - 1\right| < M.$$

For the special case $\Omega = q(\mathbb{U}) = \{w : |w-1| < M\}$, the class $\Phi_{\mathcal{K},2}[\Omega, M]$ is simply denoted by $\Phi_{\mathcal{K},2}[M]$.

Corollary 2.20. *Let $\phi \in \Phi_{\mathcal{K},2}[M]$. If $f \in \mathcal{A}$ satisfies*

$$\left|\phi\left(\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)}, \frac{\mathcal{K}_c^{\delta-2}f(z)}{\mathcal{K}_c^{\delta-1}f(z)}, \frac{\mathcal{K}_c^{\delta-3}f(z)}{\mathcal{K}_c^{\delta-2}f(z)}; z, \zeta\right) - 1\right| < M$$

$$(z \in \mathbb{U};\ \zeta \in \overline{\mathbb{U}}),$$

then

$$\left|\frac{\mathcal{K}_c^{\delta-1}f(z)}{\mathcal{K}_c^{\delta}f(z)} - 1\right| < M \quad (z \in \mathbb{U}).$$

3. Superordination and Sandwich-type Results

In this section, we investigate the dual problem of strong differential subordination (that is, strong differential superordination). For this purpose, the class of admissible functions is given in the following definition.

Definition 3.1. *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{H}$. The class of admissible functions $\Phi'_{\mathcal{K}}[\Omega, q]$ consists of those functions*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; \xi, \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{zq'(z) + cmq(z)}{m(c+1)} \quad (c > -1),$$

and

$$\Re\left\{\frac{(c+1)^2w - c^2u}{(c+1)v - cu} - 2c\right\} \leq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geq 1).$$

Theorem 3.2. *Let* $\phi \in \Phi'_{\mathcal{K}}[\Omega, q]$*. If* $f \in \mathcal{A}$*,* $\mathcal{K}_c^\delta f(z) \in \mathcal{Q}_0$ *and*

$$\phi\left(\mathcal{K}_c^\delta f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$*, then*

$$\Omega \subset \left\{\phi\left(\mathcal{K}_c^\delta f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right) : z, \zeta\right\} \quad (z \in \mathbb{U};\ \zeta \in \overline{\mathbb{U}}) \tag{3.1}$$

implies

$$q(z) \prec \mathcal{K}_c^\delta f(z) \quad (z \in \mathbb{U}). \tag{3.2}$$

Proof . With $p(z) = \mathcal{K}_c^\delta f(z)$ and

$$\psi(r, s, t; z, \zeta) = \phi\left(r, \frac{s + cr}{c+1}, \frac{t + (2c+1)s + c^2r}{(c+1)^2}; \xi, \zeta\right) = \phi(u, v, w; \xi, \zeta),$$

equation (2.7) and (3.1) yields

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\}.$$

Since

$$\frac{t}{s} + 1 = \frac{(c+1)^2w - c^2u}{(c+1)v - cu} - 2c,$$

the admissibility condition for $\phi \in \Phi'_{\mathcal{K}}[\Omega, q]$ in Definition 3.1 is equivalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.6

$$q(z) \prec p(z) \quad (z \in \mathbb{U})$$

or

$$q(z) \prec \mathcal{K}_c^\delta f(z) \quad (z \in \mathbb{U}).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, then $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω with $\Phi'_{\mathcal{K}}[h(\mathbb{U}), q]$ as $\Phi'_{\mathcal{K}}[h, q]$, Theorem 3.2 can be written in the following form.

Theorem 3.3. *Let $q \in \mathcal{H}$, h be analytic in $\mathbb{U}$ and $\phi \in \Phi'_{\mathcal{K}}[h,q]$. If $f \in \mathcal{A}$, $\mathcal{K}_c^{\delta} f(z) \in \mathcal{Q}_0$ and*

$$\phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right) \tag{3.3}$$

implies

$$q(z) \prec \mathcal{K}_c^{\delta} f(z) \quad (z \in \mathbb{U}).$$

Theorem 3.2 and Theorem 3.3 can only be used to obtain subordinants of differential superordination of the form (3.1) or (3.3). The following theorem proves the existence of the best subordinant of (3.3) for an appropriate ϕ.

Theorem 3.4. *Let h be analytic in $\mathbb{U}$ and $\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$. Suppose that the differential equation*

$$\phi\left(q(z), \frac{zq'(z) + cq(z)}{c+1}, \frac{z^2 q''(z) + (2c+1)zq'(z) + c^2 q(z)}{(c+1)^2}; z, \zeta\right) = h(z) \quad (c > -1) \tag{3.4}$$

has a solution $q \in \mathcal{Q}_0$. If $\phi \in \Phi'_{\mathcal{K}}[h,q]$, $f \in \mathcal{A}$, $\mathcal{K}_c^{\delta} f(z) \in \mathcal{Q}_0$ and

$$\phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right)$$

implies

$$q(z) \prec \mathcal{K}_c^{\delta} f(z) \quad (z \in \mathbb{U})$$

and q is the best subordinant.

Proof . The proof is similar to that of Theorem 2.5, and so it is being omitted here. □

By combining Theorem 2.3 and Theorem 3.3 , we obtain the following sandwich-type theorem.

Corollary 3.5. *Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{\mathcal{K}}[h_2, q_2] \cap \Phi'_{\mathcal{K}}[h_1, q_1]$. If $f \in \mathcal{A}$, $\mathcal{K}_c^{\delta} f(z) \in \mathcal{H} \cap \mathcal{Q}_0$ and*

$$\phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec\prec \phi\left(\mathcal{K}_c^{\delta} f(z), \mathcal{K}_c^{\delta-1} f(z), \mathcal{K}_c^{\delta-2} f(z); z, \zeta\right) \prec\prec h_2(z)$$

implies

$$q_1(z) \prec \mathcal{K}_c^{\delta} f(z) \prec q_2(z) \quad (z \in \mathbb{U}).$$

Definition 3.6. *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}_0 \cap \mathcal{H}$ with $q(z) \neq 0$. The class of admissible functions $\Phi'_{\mathcal{K},1}[\Omega, q]$ consists of those functions*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; \xi, \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{zq'(z) + m(c+1)q(z)}{m(c+1)} \quad (c > -1),$$

and

$$\Re\left\{\frac{[w - 2v + u](c+1)}{v - u}\right\} \geq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geq 1).$$

Theorem 3.7. *Let $\phi \in \Phi'_{\mathcal{K},1}[\Omega, q]$. If $f \in \mathcal{A}$, $\dfrac{\mathcal{K}_c^\delta f(z)}{z} \in \mathcal{Q}_0$ and*

$$\phi\left(\frac{\mathcal{K}_c^\delta f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{\mathcal{K}_c^\delta f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right) : z, \zeta\right\} \quad (z \in \mathbb{U};\ \zeta \in \overline{\mathbb{U}}) \tag{3.5}$$

implies

$$q(z) \prec \frac{\mathcal{K}_c^\delta f(z)}{z} \quad (z \in \mathbb{U}). \tag{3.6}$$

Proof . From (2.17) and (3.5), we have

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\}.$$

In view of (2.16), the admissibility condition for $\phi \in \Phi'_{\mathcal{K},1}[\Omega, q]$ in Definition 3.6 is equivalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.6

$$q(z) \prec p(z) \quad (z \in \mathbb{U})$$

or

$$q(z) \prec \frac{\mathcal{K}_c^\delta f(z)}{z} \quad (z \in \mathbb{U}).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, the $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω with $\Phi'_{\mathcal{K},1}[h(\mathbb{U}), q]$ as $\Phi'_{\mathcal{K},1}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.7.

Theorem 3.8. *Let $q \in \mathcal{H}$, h be analytic in $\mathbb{U}$ and $\phi \in \Phi'_{\mathcal{K},1}[h,q]$. If $f \in \mathcal{A}$, $\frac{\mathcal{K}_c^\delta f(z)}{z} \in \mathcal{Q}_0$ and*

$$\phi\left(\frac{\mathcal{K}_c^\delta f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi\left(\frac{\mathcal{K}_c^\delta f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right) \tag{3.7}$$

implies

$$q(z) \prec \frac{\mathcal{K}_c^\delta f(z)}{z} \quad (z \in \mathbb{U}).$$

By combining Theorem 2.11 and Theorem 3.8 we obtain the following sandwich-type theorem.

Corollary 3.9. *Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_0$ with $q_1(0) = q_2(0) = 0$ and $\phi \in \Phi_{\mathcal{K},1}[h_2, q_2] \cap \Phi'_{\mathcal{K},1}[h_1, q_1]$. If $f \in \mathcal{A}$, $\frac{\mathcal{K}_c^\delta f(z)}{z} \in \mathcal{H} \cap \mathcal{Q}_0$ and*

$$\phi\left(\frac{\mathcal{K}_c^\delta f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec\prec \phi\left(\frac{\mathcal{K}_c^\delta f(z)}{z}, \frac{\mathcal{K}_c^{\delta-1} f(z)}{z}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{z}; z, \zeta\right) \prec\prec h_2(z)$$

implies

$$q_1(z) \prec \frac{\mathcal{K}_c^\delta f(z)}{z} \prec q_2(z) \quad (z \in \mathbb{U}).$$

Definition 3.10. *Let Ω be a set in $\mathbb{C}$, $q \in \mathcal{Q}_1 \cap \mathcal{H}$ with $q(z) \neq 0$. The class of admissible functions $\Phi'_{\mathcal{K},2}[\Omega, q]$ consists of those functions*

$$\phi : \mathbb{C}^3 \times \mathbb{U} \times \overline{\mathbb{U}} \to \mathbb{C}$$

that satisfy the admissibility condition:

$$\phi(u, v, w; \xi, \zeta) \in \Omega$$

whenever

$$u = q(z), \quad v = \frac{1}{c+1}\left((c+1)q(z) + \frac{zq'(z)}{mq(z)}\right) \quad (c > -1),$$

and

$$\Re\left\{\frac{[vw - 3uv + 2u^2](c+1)}{v-u}\right\} \geq \frac{1}{m}\Re\left\{\frac{zq''(z)}{q'(z)} + 1\right\},$$

$$(z \in \mathbb{U};\ \xi \in \partial\mathbb{U};\ \zeta \in \overline{\mathbb{U}};\ m \geq 1).$$

Theorem 3.11. *Let $\phi \in \Phi'_{\mathcal{K},2}[\Omega, q]$. If $f \in \mathcal{A}$, $\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{\mathcal{K}_c^{\delta-1} f(z)}, \frac{\mathcal{K}_c^{\delta-3} f(z)}{\mathcal{K}_c^{\delta-2} f(z)}; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$\Omega \subset \left\{\phi\left(\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{\mathcal{K}_c^{\delta-1} f(z)}, \frac{\mathcal{K}_c^{\delta-3} f(z)}{\mathcal{K}_c^{\delta-2} f(z)}; z, \zeta\right) : z, \zeta\right\} \quad (z \in \mathbb{U};\ \zeta \in \overline{\mathbb{U}}) \tag{3.8}$$

implies

$$q(z) \prec \frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)} \quad (z \in \mathbb{U}). \tag{3.9}$$

Proof . From (2.27) and (3.8), we have

$$\Omega \subset \left\{\psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in \mathbb{U}, \zeta \in \overline{\mathbb{U}}\right\}.$$

In view of (2.26), the admissibility condition for $\phi \in \Phi'_{\mathcal{K},2}[\Omega, q]$ in Definition 3.10 is equivalent to the admissibility condition for ψ as given in Definition 1.4. Hence $\psi \in \Psi'[\Omega, q]$, and by Lemma 1.6

$$q(z) \prec p(z) \quad (z \in \mathbb{U})$$

or

$$q(z) \prec \frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)} \quad (z \in \mathbb{U}).$$

□

If $\Omega \neq \mathbb{C}$ is a simply connected domain, the $\Omega = h(\mathbb{U})$ for some conformal mapping h of $\mathbb{U}$ onto Ω with $\Phi'_{\mathcal{K},2}[h(\mathbb{U}), q]$ as $\Phi'_{\mathcal{K},2}[h, q]$. Proceeding similarly as in the previous section, the following result is an immediate consequence of Theorem 3.11.

Theorem 3.12. *Let $q \in \mathcal{H}$, h be analytic in $\mathbb{U}$ and $\phi \in \Phi'_{\mathcal{K},2}[h, q]$. If $f \in \mathcal{A}$, $\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)} \in \mathcal{Q}_1$ and*

$$\phi\left(\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{\mathcal{K}_c^{\delta-1} f(z)}, \frac{\mathcal{K}_c^{\delta-3} f(z)}{\mathcal{K}_c^{\delta-2} f(z)}; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h(z) \prec\prec \phi\left(\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{\mathcal{K}_c^{\delta-1} f(z)}, \frac{\mathcal{K}_c^{\delta-3} f(z)}{\mathcal{K}_c^{\delta-2} f(z)}; z, \zeta\right) \tag{3.10}$$

implies

$$q(z) \prec \frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)} \quad (z \in \mathbb{U}).$$

By combining Theorem 2.17 and Theorem 3.12 we obtain the following sandwich-type theorem.

Corollary 3.13. *Let h_1 and q_1 be analytic functions in $\mathbb{U}$, h_2 be univalent function in $\mathbb{U}$, $q_2 \in \mathcal{Q}_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_{\mathcal{K},2}[h_2, q_2] \cap \Phi'_{\mathcal{K},2}[h_1, q_1]$. If $f \in \mathcal{A}$, $\dfrac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)} \in \mathcal{H} \cap \mathcal{Q}_1$ and*

$$\phi\left(\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{\mathcal{K}_c^{\delta-1} f(z)}, \frac{\mathcal{K}_c^{\delta-3} f(z)}{\mathcal{K}_c^{\delta-2} f(z)}; z, \zeta\right)$$

is univalent in $\mathbb{U}$, then

$$h_1(z) \prec\prec \phi\left(\frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)}, \frac{\mathcal{K}_c^{\delta-2} f(z)}{\mathcal{K}_c^{\delta-1} f(z)}, \frac{\mathcal{K}_c^{\delta-3} f(z)}{\mathcal{K}_c^{\delta-2} f(z)}; z, \zeta\right) \prec\prec h_2(z)$$

implies

$$q_1(z) \prec \frac{\mathcal{K}_c^{\delta-1} f(z)}{\mathcal{K}_c^{\delta} f(z)} \prec q_2(z) \quad (z \in \mathbb{U}).$$

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