Int. J. Nonlinear Anal. Appl. 12 (2021) No. 2, 293-296
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.10595.1513

A common fixed point theorem via measure of noncompactness

Alireza Valipour Baboli[a], Mohamad Bagher Ghaemi[b,*]

[a]Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran.
[b]Department of Mathematics, Iran University of Science and Technology, Tehran, Iran

(Communicated by Madjid Eshaghi Gordji)

Abstract

In this paper by applying the measure of noncompactness a common fixed point for the maps T and S is obtained, where T and S are self maps continuous or commuting continuous on a closed convex subset C of a Banach space E and also S is a linear map.

Keywords: Common fixed point theorem, The Kuratowski measure of noncompactness, Commuting map, Darbo's contraction conditions.
2010 MSC: Primary 26A25; Secondary 39B62.

1. Introduction and preliminaries

The compactness plays a major role in the Schauder's fixed point theorem so G.Darbo in 1955, extended the Schauder theorem to noncompact operators. The main aim of their study is defining a new class of operators which map any bounded set to a compact set. The first measure of noncompactness, was defined and studied by Kuratowski [10] in 1930. Suppose (X, d) be a metric space the Kuratowski measure of noncompactness of a subset $A \subset X$ defined as

$$\mu(A) = \inf\{\delta > 0; A = \bigcup_{i=1}^{n} A_i \ \ for\ some\ \ A_i \ \ with\ \ diam(A_i) \leq \delta for 1 \leq i \leq n < \infty\}, \tag{1.1}$$

where $diam(A)$ denotes the diameter of a set $A \subset X$ namely

$$diam(A) = \sup\{d(x, y); x, y \in A\}.$$

*Corresponding author
Email addresses: a.valipour@umz.ac.ir (Alireza Valipour Baboli), mghaemi@iust.ac.ir (Mohamad Bagher Ghaemi)

Received: December 2016 *Accepted:* February 2017

In this paper first some esential concept and result concerning measure of noncompactness is called. In the second section a common fixed point for the maps T and S where T and S are self map continuous or commuting continuous on a closed convex subset C of a Banach space E and also S is a linear map is showed. Now, we recall some basic facts concerning measures of noncompactness.
Suppose $(E, |.|)$ be a Banach space and $\overline{X}$, $ConvX$ be the closure and closed convex hull of a subset X of E, respectively. We denote $\mathfrak{M}_E$ is the family of all nonempty and bounded subsets of E and $\mathfrak{N}_E$ show the family of all nonempty and relatively compact subsets.
In 1955, G. Darbo [10] used measure of noncompactness to generalize Schauder's theorem to wide class of operators, called k-set contractive operators, which satisfy the following condition

$$\mu(T(A)) \leq k\mu(A)$$

for some $k \in [0, 1)$ and in 1967 Sadovskii generalized Darbo's theorem to set-condensing operators.

2. Common Fixed Point

Theorem 2.1. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T, S : C \longrightarrow C$ be continuous operators and S be a linear operator such that*

$$S(T(X)) \subseteq T(X)$$

and also

$$\mu(T(X)) \leq \varphi\big(\max\{\mu(X), \mu(S(X))\}\big),$$

for each $X \subseteq C$, where μ is an arbitrary measure of noncompactness and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-decreasing function such that $\varphi(t) < t$ for each $t \geq 0$ and $\varphi(0) = 0$. Then T, S have a common fixed point in C.

Proof . Set

$$C_0 = C$$

and

$$C_1 = ConvTC_0$$

in general, set

$$C_n = ConvTC_{n-1}$$

for $n = 1, 2, ...$
Then we have

$$C_n \subset C_{n-1} \qquad and \qquad S(C_n) \subset C_n \qquad (\star)$$

for ever $n = 1, 2, 3, ...$
Indeed it is clear that $C_1 \subset C_0$ and $S(C_1) \subset Conv(ST(C_0)) \subset Conv(T(C_0)) = C_1$.
So $(\star)$ holds for $n = 1$.
Assuming now that $(\star)$ is true for $n \geq 1$.

Then

$$C_{n+1} = Conv(T(C_n)) \subset Conv(T(C_{n-1})) = C_n$$

and

$$S(C_{n+1}) = S(Conv(T(C_n))) \subset Conv(S(T(C_n))) \subset ConvT(C_n) = C_{n+1}.$$

We obtain

$$C_0 \supseteq C_1 \supseteq C_2 \supseteq \cdots$$

If there exists an integer $N \geq 0$ so $\mu(C_N) = 0$, then C_N is relatively compact and since $TC_N \subseteq ConvTC_N = C_{N+1} \subseteq C_N$, Schauder's fixed point theorem implies that T has a fixed point. So we assume that $\mu(C_n) \neq 0$ for $n \geq 0$. By assumptions we have

$$\begin{aligned}
\mu(C_{n+1}) &= \mu(ConvTC_n) \\
&= \mu(TC_n) \\
&\leq \varphi(\max\{\mu(TC_n), \mu(STC_n)\}) \\
&\leq \varphi(\mu(TC_n)) \\
&\leq \mu(TC_n) \\
&\leq \mu(C_n)
\end{aligned}$$

which implies that $\mu(C_n)$ is a positive decreasing sequence of real numbers, thus, there is an $r \geq 0$ so that $\mu(C_n) \longrightarrow r$ as $n \longrightarrow \infty$. We show that $r = 0$. Suppose, in the contrary, that $r \neq 0$. Then we have

$$\begin{aligned}
\mu(C_{n+1}) &= \mu(ConvTC_n) \\
&= \mu(TC_n) \\
&\leq \varphi(\mu(TC_n)) \\
&\leq \varphi(\mu(C_n)) \\
&= \varphi(\mu(ConvTC_{n-1})) \\
&\leq \varphi(\mu(TC_{n-1})) \\
&\leq \varphi^2(\mu(C_{n-1})) \\
&\cdot \\
&\cdot \\
&\cdot \\
&\leq \varphi^n(\mu(C_0)).
\end{aligned}$$

By Lemma 2.1 [3] and assumption with choose $\mu(C_0) = t$, we have

$$r = \lim_{n \longrightarrow \infty} \mu(C_{n+1}) \leq \lim_{n \longrightarrow \infty} \varphi^n(\mu(C_0)) = \lim_{n \longrightarrow \infty} \varphi^n(t) = 0$$

for any $t > 0$. Then $r = 0$ and so $\mu(C_n) \longrightarrow 0$, when $n \longrightarrow \infty$. Since $C_{n+1} \subseteq C_n$ and $TC_n \subseteq C_n$ for all $n \geq 1$, by use definition of the measure of noncompactness given in [8], we have $C_\infty = \bigcap_{n=1}^{\infty} C_n$ is a non empty convex closed set, and $C_\infty \subset C$. Moreover, the set C_∞ is invariant under the operator T and belongs to $Ker\mu$. Thus, applying Schauder's fixed point theorem, T has a fixed point. Now, suppose that $F_T = \{x \in C : Tx = x\}$. The set F_T is closed by the continuity of T, by assumption we have $SF_T \subset F_T$ then Sx is a fixed point of T for any $x \in F_T$ and

$$\begin{aligned} \mu(F_T) = \mu(TF_T) &\leq \varphi(\max\{\mu(F_T), \mu(SF_T)\}) \\ &= \varphi(\mu(F_T)) \\ &< \mu(F_T) \end{aligned}$$

then $\mu(F_T) = 0$ and have F_T is compact. Then by Schauder's fixed point theorem we deduce that S has a fixed point and set $F_S = \{x \in C, Sx = x\}$ is closed by the continuity of S. Also, since $SF_T \subset F_T$ by Schauder's fixed point theorem we have Tx is a fixed point of S for each $x \in F_S$. Since $F_T \cap F_S \subseteq F_T \subset C$ is a compact subset, $T, S : F_T \cap F_S \longrightarrow F_T \cap F_S$ are continuous self maps, now by Schauder's fixed point theorem we have a common fixed point in C. □

Corollary 2.2. *Let C be a nonempty, bounded, closed and convex subset of a Banach space E and let $T, S : C \longrightarrow C$ be continuous operators and S be a linear operator such that T and S be two commuting map and*

$$\mu(T(X)) \leq \varphi\big(\max\{\mu(X), \mu(S(X))\}\big),$$

for each $X \subseteq C$, where μ is a measure of noncompactness and $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a nondecreasing function such that $\varphi(t) < t$ for each $t \geq 0$ and $\varphi(0) = 0$. Then T, S have a common fixed point in C.

Proof . The proof is similar to proof of Theorem 2.1. □

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