



A numerical scheme for space-time fractional advection-dispersion equation

Shahnam Javadi*, Mostafa Jani, Esmail Babolian

Department of Mathematics, Faculty of Mathematical Sciences and Computer, Kharazmi University, Tehran, Iran

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Abstract

In this paper, we develop a numerical resolution of the space-time fractional advection-dispersion equation. The main idea of the present method is that we utilize spectral-collocation method combining with a product integration technique in order to discretize the terms involving spatial fractional order derivatives and it leads to a simple evaluation of the related terms. By using the Bernstein polynomial approximation, the problem is transformed into a linear system of algebraic equations. The error analysis and the order of convergence of the proposed algorithm are also discussed. Some numerical experiments are presented to demonstrate the effectiveness of the proposed method and to confirm the analytic results.

Keywords: advection-dispersion equation; space-time fractional PDE; Bernstein polynomials; product integration; Spectral-collocation.

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1. Introduction

In recent decades, many physical processes have been modeled in terms of fractional partial differential equations (FPDEs). This kind of mathematical modeling leads to better agreement with data obtained in lab experiments than the classical models involving integer order derivatives. For example, among recently developed models, Suzuki et al. [26] proposed a fractional advection-dispersion equation (FADE) for description of mass transport in a fractured reservoir. They also discussed a FADE model for the evaluation of the effects of cold-water injection into an advection-dominated

*Corresponding author

Email addresses: javadi@khu.ac.ir (Shahnam Javadi), mostafa.jani@gmail.com (Mostafa Jani), babolian@khu.ac.ir (Esmail Babolian)

geothermal reservoir in fault-related structures in geothermal areas [27]. Also, the particle's motion in crowded cellular environments which represent anomalous dispersion is formulated as a FPDE [24]. Uchaikin and Sibatov used a FPDE model for theoretical description of charge carrier transport in disordered semiconductors [28].

Recently, some numerical methods have been developed for the fractional advection-dispersion equations, most of them for time fractional advection-dispersion equations [13, 19, 22, 25, 17] and some for space fractional advection dispersion equations [18, 23, 30]. However, there are some physical problems that are modeled with both space and time fractional advection-dispersion equation like space-time fractional Fokker-Planck equation which is an effective tool for the processes with both traps and flights, in which the time fractional term characterizes the traps and the space fractional term characterizes the flights [6]. In spite of this and due to the fact that the numerical methods for these problems are very challenging, there are a few numerical schemes developed for these equations.

To the best of our knowledge, no numerical scheme based on product integration method and Bernstein collocation has been developed for the space-time fractional advection-dispersion equation. The main contribution of the current work is to implement a product integration technique for terms involving space fractional derivatives in the fractional advection-dispersion problem and to use Bernstein polynomials with the collocation method in order to transform the problem into an algebraic linear system.

In comparison with local approximation methods such as the finite difference and finite element method, the proposed method leads to a smaller size of coefficient matrix and less computational effort that is required for a specific accuracy.

This paper is concerned with providing a numerical scheme for the Caputo space-time fractional advection dispersion equation, given by

$$D_t^\alpha u(x, t) = \kappa_1 D_x^\beta u(x, t) - \kappa_2 D_x^\gamma u(x, t), \quad (x, t) \in \Omega = (0, L) \times (0, \infty), \quad (1.1)$$

where x is the spatial coordinate and t represents time, u is the concentration, $\alpha \in (0, 1)$ is the temporal order, $\beta \in (1, 2)$, $\gamma \in (0, 1)$ are the orders of spatial fractional derivatives. Moreover, the operators D_t^α , D_x^β and D_x^γ stand for time and space fractional derivatives in the sense of Caputo definition as described in Definition 2.1. The positive constants κ_1 and κ_2 denote the anomalous dispersion and advection coefficients, respectively. When $\alpha = \gamma = 1$ and $\beta = 2$, the classical advection dispersion equation is obtained. In [7] it is shown that the order of space fractional derivative regarding anomalous dispersion is mostly occurred between 1.4 and 2 to best fit the experimental data and in [15] it varies between 1.7 and 1.8.

Some authors discussed special cases of problem (1.1), for instance Gao and Sun [13] proposed a numerical method based on finite difference method for time fractional case, i.e., $\beta = 2, \gamma = 1$ and Deng [6] discussed finite element method in the case $\gamma = 1$ with Riemann-Liouville space fractional derivatives.

We consider equation (1.1) subject to the following initial and homogeneous boundary conditions

$$u(x, 0) = g(x), \quad x \in [0, L], \quad (1.2)$$

$$u(0, t) = 0, \quad u(L, t) = 0, \quad t > 0. \quad (1.3)$$

The paper is organized as follows. In section 2, we briefly introduce some background material on fractional derivatives. We also provide the basic idea of product integration rules. Section 3 is devoted to properties of Bernstein polynomials. In this section we derive two new representation for first and second order derivatives of Bernstein polynomials. Time discretization of the given problem is discussed in section 4. In section 5, we use time discretization results discussed in previous section

and we utilize Bernstein polynomials with product integration in order to transfer the given problem into solving a simple algebraic system. Matrix formulation, error analysis and convergence of the method are discussed in this section. Some numerical experiments are presented in section 6 to show the efficiency and numerical accuracy of the proposed method. Experimental rate of convergence is also provided in this section. We summarize the paper and present concluding remarks in the final section.

2. Preliminaries

In this section, we provide some definitions and properties of fractional derivatives and product integration rules which are used further in the paper.

2.1. Fractional calculus

There are different definitions for fractional derivative. Since the Caputo definition allows imposing initial and boundary conditions, it is preferred in applications, so we focus on this definition in the paper.

Definition 2.1. [8] The Caputo fractional derivative of order $\alpha > 0$ of the function f is given by

$$D_x^\alpha f(x) = \frac{1}{\Gamma(m - \alpha)} \int_0^x \frac{1}{(x - s)^{\alpha - m + 1}} \frac{d^m f(s)}{ds^m} ds, \quad x > 0, \tag{2.1}$$

where $m = \lceil \alpha \rceil$.

According to Definition 2.1, the Caputo temporal and spatial fractional derivative of order α of the function $u(x, t)$ are respectively given by

$$D_t^\alpha u(x, t) = \frac{1}{\Gamma(m - \alpha)} \int_0^t \frac{1}{(t - s)^{\alpha - m + 1}} \frac{\partial^m u(x, s)}{\partial s^m} ds, \tag{2.2}$$

$$D_x^\alpha u(x, t) = \frac{1}{\Gamma(m - \alpha)} \int_0^x \frac{1}{(x - s)^{\alpha - m + 1}} \frac{\partial^m u(s, t)}{\partial s^m} ds. \tag{2.3}$$

2.2. Product integration

Product integration method, originally proposed by Young [32], is of convolution type quadrature. It is usually used for integral equations with singular kernels [1, 31]. Recently, this technique has been used for fractional ODEs [14]. Here, we introduce the basic idea behind this simple and powerful technique (see [2] for a detailed description).

Consider the integral operator

$$\mathcal{K}f(x) = \int_a^b H(x, s)L(x, s)f(s)ds, \quad a \leq x \leq b, \tag{2.4}$$

in which f is a smooth function, $K(x, t) = H(x, t)L(x, t)$ is the kernel of the integral operator such that \mathcal{K} is a compact operator, L is a well-behaved function and H is a weakly singular function on $[a, b] \times [a, b]$. Consider the grid space $\{t_i = a + ih : i = 0, \dots, n\}$ with grid length $h = \frac{b-a}{n}$. To provide an approximation for (2.4), the product integration's idea is to replace $L(x, s)f(s)$ with an interpolating polynomial based on nodes $t_i, i = 0, \dots, n$, lets denote by $[L(x, s)f(s)]_n$ and the remaining terms in (2.4) are then evaluated exactly. To be more precise, we use the common way

of approximating $L(x, s)f(s)$, namely Newton Cotes quadrature formulas. Consider the piecewise linear interpolation

$$[L(x, s)f(s)]_n = \frac{1}{h} ((s - t_j) L(x, t_{j+1})f(t_{j+1}) - (s - t_{j+1}) L(x, t_j)f(t_j)), \quad (2.5)$$

for $t_j \leq s \leq t_{j+1}$, $j = 0, \dots, n - 1$, and $a \leq x \leq b$. Putting (2.5) into (2.4) gives the following approximation

$$\mathcal{K}_n f(x) = \sum_{j=0}^n w_j(t) L(x, t_j) f(t_j)$$

where the quadrature weights are obtained as

$$\begin{aligned} w_0(t) &= \frac{1}{h} \int_{t_0}^{t_1} (t_1 - s) H(x, s) ds, \\ w_j(t) &= \frac{1}{h} \left(\int_{t_{j-1}}^{t_j} (s - t_{j-1}) H(x, s) ds + \int_{t_j}^{t_{j+1}} (t_{j+1} - s) H(x, s) ds \right), \quad 1 \leq j \leq n - 1, \\ w_n(t) &= \frac{1}{h} \int_{t_{n-1}}^{t_n} (s - t_{n-1}) H(x, s) ds. \end{aligned}$$

The function H is selected such that the weights can be evaluated exactly. In this paper, since we use fractional operators, it is assumed that $H(x, s) = \frac{1}{(x-s)^\eta}$ for some $0 < \eta < 1$.

3. Bernstein polynomial basis

Bernstein polynomials provide a flexible tool in solving differential and integral equations [3, 9, 16, 20] as well as in computer graphics [10]. Bernstein polynomials of degree N are defined on the interval $[a, b]$ as follows:

$$B_{i,N}(x) = \frac{\binom{N}{i} (x-a)^i (b-x)^{N-i}}{(b-a)^N}, \quad 0 \leq i \leq N. \quad (3.1)$$

In this paper, we suppose that these polynomials are zero for the cases $i < 0$ and $i > N$. The set $\{B_{i,N}(x) : i = 0, \dots, N\}$ forms a basis for \mathcal{P}_n , the set of polynomials of degree up to N . This basis has many advantages in comparison with other bases in some applications. Farouki [11] showed that the Bernstein polynomial basis on a given interval is optimally stable in the sense that no other nonnegative basis yields systematically smaller condition numbers for the values or roots of arbitrary polynomials on that interval. Moreover, we have the following properties [12]

$$B_{i,N}(a) = \delta_{i,0}, \quad B_{i,N}(b) = \delta_{i,N}, \quad (3.2)$$

$$0 \leq B_{i,N}(x) \leq 1, \quad \sum_{i=0}^N B_{i,N}(x) = 1, \quad (3.3)$$

$$B'_{i,N}(x) = \frac{N}{b-a} (B_{i-1,N-1}(x) - B_{i,N-1}(x)), \quad (3.4)$$

$$B_{i,N-1}(x) = \frac{1}{N} [(N-i) B_{i,N}(x) + (i+1) B_{i+1,N}(x)], \quad (3.5)$$

for $0 \leq i \leq N$. The symbol δ is the Kronecker symbol.

For a better formulation of our method, by combining (3.5) and (3.4) and some simplifications, we derive the following result to express derivatives of Bernstein basis of degree N in terms of the same basis.

Theorem 3.1. For the first and second order derivatives of Bernstein polynomials, we have the following three-term and five-terms properties

$$B'_{i,N}(x) = \frac{1}{b-a} \sum_{k=-1}^1 d_{k,i}^{(1)} B_{i+k,N}(x), \tag{3.6}$$

$$B''_{i,N}(x) = \frac{1}{(b-a)^2} \sum_{k=-2}^2 d_{k,i}^{(2)} B_{i+k,N}(x), \tag{3.7}$$

for $0 \leq i \leq N$, where the coefficients are given by

$$\begin{cases} d_{-1,i}^{(1)} = N - i + 1, & d_{0,i}^{(1)} = -(N - 2i), & d_{1,i}^{(1)} = -(i + 1), \\ d_{-2,i}^{(2)} = (N - i + 2)(N - i + 1), & d_{-1,i}^{(2)} = -2(N - i + 1)(N - 2i + 1), \\ d_{0,i}^{(2)} = N^2 - 6Ni + 6i^2 - N, & d_{1,i}^{(2)} = 2(i + 1)(N - 2i - 1), \\ d_{2,i}^{(2)} = (i + 2)(i + 1). \end{cases}$$

Remark 3.2. Corresponding to the non-orthogonal Bernstein polynomial basis, the associated orthonormalized basis, denoted by $\phi_{i,N}(x)$, is obtained by using Gram-Schmidt algorithm. Bellucci [4] derived an explicit formula for $\phi_{i,N}(x)$ on the unit interval $[0, 1]$ as

$$\phi_{i,N}(x) = \sqrt{2(N-i)+1} (1-x)^{N-i} \sum_{k=0}^j (-1)^k \binom{2N+1-k}{i-k} \binom{i}{k} x^{i-k}.$$

Although it has many advantages due to orthogonality, this basis does not have the three- and five-term formula similar to (3.5) and (3.7). For example, for $i = 4$ and $N = 4$, we obtain

$$\phi'_{i,N}(x) = \sum_{i=0}^N c_i \phi_{i,N}(x)$$

with all nonzero coefficient c_0, c_1, c_2, c_3, c_4 . It is worth noting that even Chebyshev and Legendre polynomials also do not have three- and five-terms relation for derivatives (see Sections 3.3 and 3.4 of [21] for these polynomials and derivative relations). We choose the Bernstein polynomials to formulate our method in Section 5.

It is a well-known result that for a continuous function f on the unit interval, the Bernstein approximation polynomial defined by

$$p_n(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) B_{k,n}(x),$$

has the following asymptotic property when $f''(x) \neq 0$ [29]

$$\lim_{n \rightarrow \infty} n(f(x) - p_n(x)) = \frac{x(1-x)}{2} f''(x).$$

The next two sections are devoted to providing a numerical method for the problem (1.1)-(1.3) using Bernstein polynomials with collocation method utilizing a product integration technique.

4. Time discretization

In this section, we describe discretization of the time fractional derivative. The fractional derivative uses function information on a continuous interval, this discretization only evaluates the function at some nude points so it leads to less computational complexity. This scheme is a common way in the numerical methods for time dependent FPDEs [6, 13, 19].

Without loss of generality, we consider the problem (1.1)-(1.3) on the bounded domain $\Omega = [0, 1] \times [0, T]$. Let $u_k(x) := u(x, t_k)$ for $k = 0, 1, \dots, M$, where $t_k = k\tau$ and $\tau = \frac{T}{M}$ is the time step length. By using (2.2), the time fractional derivative at time t_{k+1} in the left side of the equation (1.1) is formulated as

$$\begin{aligned} D_t^\alpha u(x, t_{k+1}) &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \int_{t_j}^{t_{j+1}} \frac{\frac{\partial u}{\partial s}(x, s)}{(t_{k+1}-s)^\alpha} ds \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{j=0}^k \frac{u(x, t_{j+1}) - u(x, t_j)}{\tau} \int_{t_j}^{t_{j+1}} \frac{ds}{(t_{k+1}-s)^\alpha} + r_\tau^{k+1}. \end{aligned}$$

from which it follows that

$$D_t^\alpha u(x, t_{k+1}) = \mu_\tau^\alpha \sum_{j=0}^k a_{k,j}^\alpha (u(x, t_{j+1}) - u(x, t_j)) + r_\tau^{k+1}, \tag{4.1}$$

where $\mu_\tau^\alpha = \frac{1}{\tau^\alpha \Gamma(2-\alpha)}$, $a_{k,j}^\alpha = (k+1-j)^{1-\alpha} - (k-j)^{1-\alpha}$. The error bound for truncation error is given by

$$r_\tau^{k+1} \leq \tilde{c}_u \tau^{2-\alpha}, \tag{4.2}$$

where the coefficient \tilde{c}_u is a constant depending only on u [6].

Similar to [6, 13], we consider the time discrete fractional differential operator L_t^α as

$$L_t^\alpha u(x, t_{k+1}) = \mu_\tau^\alpha \sum_{j=0}^k a_{k,j}^\alpha (u_{j+1}(x) - u_j(x)), \tag{4.3}$$

in which $u_j(x) := u(x, t_{j+1})$. So we have

$$D_t^\alpha u(x, t_{k+1}) = L_t^\alpha u(x, t_{k+1}) + r_\tau^{k+1}.$$

From (4.2), it is seen that the order of convergence is $O(\tau^{2-\alpha})$. The discretized operator L_t^α is called the L1 approximation operator [13].

5. Formulation and algorithm for space-time fractional advection-dispersion equation (1.1)

We rewrite the advection-dispersion equation (1.1) at horizontal time line $t = t_{k+1}$ as

$$D_t^\alpha u(x, t_{k+1}) = \kappa_1 D_x^\beta u(x, t_{k+1}) - \kappa_2 D_x^\gamma u(x, t_{k+1}).$$

Using $L_t^\alpha u(x, t_{k+1})$ given by (4.3) as an approximation for $D_t^\alpha u(x, t_{k+1})$, it yields the following time-discrete scheme for (1.1)

$$\mu_\tau^\alpha \sum_{j=0}^k a_{k,j}^\alpha (u_{j+1}(x) - u_j(x)) = \kappa_1 D_x^\beta u_{k+1}(x) - \kappa_2 D_x^\gamma u_{k+1}(x), \tag{5.1}$$

for $k = 0, \dots, M - 1$. In the following, we discuss about approximating the terms involving spatial fractional derivatives.

Let N be a positive integer and $h = \frac{1}{N}$ be the grid size in the x -direction and $x_r = \frac{r}{h}$, $r = 0, \dots, N$. We have the following theorem for a simple evaluation of spatial fractional derivatives at collocation points.

Theorem 5.1. *Suppose that $\partial_x^4 u(x, t)$ is continuous on Ω , $0 \leq k \leq M - 1$, $1 < \beta < 2$. Then for $r = 1, \dots, N - 1$, we have*

$$D_x^\beta u_{k+1}(x_r) = \nu_h^\beta \sum_{j=0}^r w_{j,r}^\beta u''_{k+1}(x_j) + R_h^\beta(r), \tag{5.2}$$

where $\nu_h^\beta = \frac{h^{2-\beta}}{\Gamma(2-\beta)}$. The weighting coefficients are given by

$$\begin{aligned} w_{j,r}^\beta &= \tilde{w}_{j-1,0}^\beta - \tilde{w}_{j,1}^\beta, & j = 0, \dots, r, \\ \tilde{w}_{r-j,\rho}^\beta &= c_j^{3-\beta} - (j - \rho) c_j^{2-\beta}, & \rho = 0, 1, j = 0, \dots, r, (j, \rho) \neq (0, 1), \end{aligned}$$

with $\tilde{w}_{-1,0}^\beta = 0$ and $\tilde{w}_{r,1}^\beta = 0$. The error is bounded as

$$\|R_h^\beta\|_\infty \leq \frac{h^2 \mathcal{M}_u}{8\Gamma(3 - \beta)}, \tag{5.3}$$

where \mathcal{M}_u depends only on u .

Proof . Based on Definition 2.1, and utilizing product integration method as explained in Section 2.2, we can write

$$\begin{aligned} D_x^\beta u_{k+1}(x_r) &= \frac{1}{\Gamma(2 - \beta)} \int_0^{x_r} \frac{1}{(x_r - s)^{\beta-1}} \frac{d^2 u_{k+1}(s)}{ds^2} ds \\ &\approx \frac{1}{\Gamma(2 - \beta)} \int_0^{x_r} \frac{1}{(x_r - s)^{\beta-1}} \left[\frac{d^2 u_{k+1}(s)}{ds^2} \right]_r ds \\ &= \frac{1}{\Gamma(2 - \beta)} \sum_{j=0}^{r-1} \int_{x_j}^{x_{j+1}} \frac{1}{(x_r - s)^{\beta-1}} \frac{(s - x_j) u''_{k+1}(x_{j+1}) - (s - x_{j+1}) u''_{k+1}(x_j)}{h} ds \\ &= \frac{h^{2-\beta}}{\Gamma(2 - \beta)} \left(- (c_r^{3-\beta} - (r - 1) c_r^{2-\beta}) u''_{k+1}(x_0) \right. \\ &\quad + \sum_{j=1}^{r-1} \left((c_{r-j+1}^{3-\beta} - (r - j + 1) c_{r-j+1}^{2-\beta}) - (c_{r-j}^{3-\beta} - (r - j - 1) c_{r-j}^{2-\beta}) \right) u''_{k+1}(x_j) \\ &\quad \left. + \left((c_1^{3-\beta} - c_1^{2-\beta}) \right) u''_{k+1}(x_r) \right), \end{aligned}$$

where $c_j^\eta := \frac{(j-1)^{\eta-j^\eta}}{\eta}$ for $\eta = 2 - \beta$ and $\eta = 3 - \beta$. This is simply written as (5.2) by applying the

notations in the theorem. Using Lagrange interpolation error formula, we have

$$\begin{aligned}
 |R_h^\beta(r)| &= \left| \frac{1}{\Gamma(2-\beta)} \sum_{j=0}^{r-1} \int_{x_j}^{x_{j+1}} \frac{(s-x_j)(s-x_{j+1})u_{k+1}^{(4)}(\xi_j(s))}{2(x_r-s)^{\beta-1}} ds \right| \\
 &= \frac{h^{4-\beta}}{\Gamma(2-\beta)} \sum_{j=0}^{r-1} \left| u_{k+1}^{(4)}(\bar{\xi}_j) \int_{r-j-1}^{r-j} \frac{(r-j-t)(r-j-1-t)}{2t^{\beta-1}} dt \right| \\
 &\leq \frac{h^{4-\beta}}{8\Gamma(3-\beta)} \|u_{k+1}^{(4)}\|_\infty \sum_{j=0}^{r-1} (j+1)^{2-\beta} - j^{2-\beta} \\
 &\leq \frac{h^{4-\beta}r^{2-\beta}\mathcal{M}_u}{8\Gamma(3-\beta)} \\
 &\leq \frac{h^2\mathcal{M}_u}{8\Gamma(3-\beta)} \left(\frac{N-1}{N}\right)^{2-\beta} \\
 &\leq \frac{h^2\mathcal{M}_u}{8\Gamma(3-\beta)},
 \end{aligned}$$

for $r = 1, 2, \dots, N - 1$ and the continuity of $\frac{\partial^4 u}{\partial x^4}(x, t)$ on the closed square Ω implies that it is bounded, i.e., $\mathcal{M}_u = \sup_{(x,t) \in \Omega} |\frac{\partial^4 u}{\partial x^4}(x, t)| < \infty$. Due to the continuity of $\frac{\partial^4 u}{\partial x^4}(x, t)$ and noting that $(s-x_j)(s-x_{j+1})/(x_r-s)^{\beta-1}$ does not change sign in the interval of integration, we used the first mean value theorem for integrals in the second equality. This concludes the proof. \square

Next we present the following theorem which can be proved analogous to Theorem 5.1. So we omit the proof.

Theorem 5.2. Suppose that $\partial_x^3 u(x, t)$ is continuous on Ω , $0 \leq k \leq M - 1$, $1 < \beta < 2$. Then for $r = 1, \dots, N - 1$, we have

$$D_x^\gamma u_{k+1}(x_r) = \nu_h^\gamma \sum_{j=0}^r w_{j,r}^\gamma u'_{k+1}(x_j) + R_h^\gamma(r), \tag{5.4}$$

where $\nu_h^\gamma = \frac{h^{1-\gamma}}{\Gamma(1-\gamma)}$ and the weighting coefficients are given by

$$\begin{aligned}
 w_{j,r}^\gamma &= \tilde{w}_{j-1,0}^\gamma - \tilde{w}_{j,1}^\gamma, & j &= 0, \dots, r, \\
 \tilde{w}_{r-j,\rho}^\gamma &= c_j^{2-\gamma} - (j-\rho)c_j^{1-\gamma}, & \rho &= 0, 1, j = 0, \dots, r, (j, \rho) \neq (0, 1),
 \end{aligned}$$

and as before we set $\tilde{w}_{-1,0}^\beta = 0$ and $\tilde{w}_{r,1}^\beta = 0$ and the error is bounded as

$$\|R_h^\gamma\|_\infty \leq \frac{h^2\bar{\mathcal{M}}_u}{8\Gamma(2-\gamma)}. \tag{5.5}$$

where $\bar{\mathcal{M}}_u = \sup_{(x,t) \in \Omega} |\frac{\partial^3 u}{\partial x^3}(x, t)| < \infty$.

Using Theorems 5.1 and 5.2, we define the discrete spatial fractional operators $L_x^\beta u$ and $L_x^\gamma u$ by

$$L_x^\beta u(x_r, t_{k+1}) = \nu_h^\beta \sum_{j=0}^r w_{j,r}^\beta u''_{k+1}(x_j), \quad L_x^\gamma u(x_r, t_{k+1}) = \nu_h^\gamma \sum_{j=0}^r w_{j,r}^\gamma u'_{k+1}(x_j),$$

as approximations for $D_x^\beta u$ and $D_x^\gamma u$. Plugging these approximations into equation (5.1) at nodes $x_r, r = 1, \dots, N - 1$, we obtain the following recursive relation

$$\mu_\tau^\alpha \sum_{j=0}^k a_{k,j}^\alpha (u_{j+1}(x_r) - u_j(x_r)) = \kappa_1 \nu_h^\beta \sum_{j=0}^r w_{j,r}^\beta u_{k+1}''(x_j) - \kappa_2 \nu_h^\gamma \sum_{j=0}^r w_{j,r}^\gamma u_{k+1}'(x_j),$$

for $k \geq 0$, from which, with simplification, we obtain

$$\mu_\tau^\alpha u_{k+1}(x_r) - \kappa_1 \nu_h^\beta \sum_{j=0}^r w_{j,r}^\beta u_{k+1}''(x_j) + \kappa_2 \nu_h^\gamma \sum_{j=0}^r w_{j,r}^\gamma u_{k+1}'(x_j) = f_{k+1}(x_r), \tag{5.6}$$

where $f_{k+1}(x) = \mu_\tau^\alpha \left(u_k(x) - \sum_{j=0}^{k-1} a_{k,j}^\alpha (u_{j+1}(x) - u_j(x)) \right)$.

From the error formulas (4.2), (5.3) and (5.5), the perturbation error for the solution of problem (1.1) on the domain Ω , when applying spatial fractional approximations (5.2), (5.4) in conjunction with time fractional approximation (4.3), is obtained as

$$\max_{k \geq 0} \|u(x, t_{k+1}) - u_{k+1}(x)\|_\infty \leq \tilde{c}_u \tau^{2-\alpha} + \frac{h^2}{8} \left(\frac{\kappa_1}{\Gamma(3-\beta)} + \frac{\kappa_2}{\Gamma(2-\gamma)} \right) \mathcal{M}, \tag{5.7}$$

where $\mathcal{M} = \max(\mathcal{M}_u, \bar{\mathcal{M}}_u)$ depends only on u . This shows the convergence of the proposed method and also it is seen that the method has order of convergence two for space and $2 - \alpha$ for time.

In order to obtain the approximate solution of the problem (1.1)-(1.3), we solve (5.6) using Bernstein polynomial basis. Consider

$$u_{k+1}(x) \approx \sum_{i=0}^N c_{i,k+1} B_{i,N}(x), \quad k \geq 0$$

as an approximate solution at time step t_{k+1} . Using the boundary conditions (1.3), we have $u_{k+1}(0) = 0$ and $u_{k+1}(1) = 0$. Now from (3.2), we obtain $c_{0,k+1} = 0, c_{N,k+1} = 0$ and so

$$u_{k+1}(x) \approx \sum_{i=1}^{N-1} c_{i,k+1} B_{i,N}(x). \tag{5.8}$$

Substituting (5.8) accompanied with (3.6) and (3.7) into (5.6), we obtain the following linear system of equations

$$\sum_{i=1}^{N-1} \mathcal{A}_{r,i} c_{i,k+1} = f_{k+1}(x_r), \quad 1 \leq r < N, \tag{5.9}$$

at each time level $k = 0, 1, \dots, M - 1$, where the coefficients are given by

$$\mathcal{A}_{r,i} = \mu_\tau^\alpha B_{i,N}(x_r) - \sum_{j=0}^r \left\{ \kappa_1 \nu_h^\beta w_{j,r}^\beta \sum_{s=-2}^2 d_{s,i}^{(2)} B_{i+s,N}(x_j) + \kappa_2 \nu_h^\gamma w_{j,r}^\gamma \sum_{s=-1}^1 d_{s,i}^{(1)} B_{i+s,N}(x_j) \right\}$$

Now we write (5.9) in the matrix form as

$$\left(\mu_\tau^\alpha \mathcal{B} - \kappa_1 \nu_h^\beta \mathcal{W}_\beta \mathcal{D}_2 + \kappa_2 \nu_h^\gamma \mathcal{W}_\gamma \mathcal{D}_1 \right) \mathbf{c}_{k+1} = \mathbf{f}_{k+1}, \quad k \geq 0$$

where $\mathbf{c}_{k+1}^T = [c_{1,k+1}, \dots, c_{N-1,k+1}]$, $\mathbf{f}_{k+1}^T = [f_{1,k+1}, \dots, f_{N-1,k+1}]$ and $f_{r,k+1} := f_{k+1}(x_r)$, $1 \leq r < N$. Let $\phi^T(x) = [B_{1,N}(x), \dots, B_{N-1,N}(x)]$ be the vector of basis. The solution (5.8) is written as $u_{k+1}(x) = \mathbf{c}_{k+1}^T \phi(x)$ and the square matrices \mathcal{B} , \mathcal{W}_β and \mathcal{W}_γ are defined as

$$\mathcal{B} = \begin{bmatrix} B_{1,N}(x_1) & B_{2,N}(x_1) & \cdots & B_{N-1,N}(x_1) \\ B_{1,N}(x_2) & B_{2,N}(x_2) & \cdots & B_{N-1,N}(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ B_{1,N}(x_{N-1}) & B_{2,N}(x_{N-1}) & \cdots & B_{N-1,N}(x_{N-1}) \end{bmatrix}, \tag{5.10}$$

$$\mathcal{W}_\eta = \begin{bmatrix} w_{0,1}^\eta & w_{1,1}^\eta & 0 & \cdots & 0 \\ w_{0,2}^\eta & w_{1,2}^\eta & w_{2,2}^\eta & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{0,N-2}^\eta & w_{1,N-2}^\eta & w_{2,N-2}^\eta & \cdots & w_{N-2,N-2}^\eta \\ w_{0,N-1}^\eta & w_{1,N-1}^\eta & w_{2,N-1}^\eta & \cdots & w_{N-2,N-1}^\eta \end{bmatrix}, \tag{5.11}$$

for $\eta = \beta, \gamma$. The matrix \mathcal{B} is well known as Bernstein collocation matrix that is a totally positive matrix, i.e., all its minors are nonnegative [5], and the matrices \mathcal{W}_β and \mathcal{W}_η are lower Hessenberg matrices. Also \mathcal{D}_1 and \mathcal{D}_2 are as follows:

$$\mathcal{D}_p = \sum_{s=-p}^p d_{s,i}^{(p)} \begin{bmatrix} B_{1+s,N}(x_0) & B_{2+s,N}(x_0) & \cdots & B_{N-1+s,N}(x_0) \\ B_{1+s,N}(x_1) & B_{2+s,N}(x_1) & \cdots & B_{N-1+s,N}(x_1) \\ \vdots & \vdots & \ddots & \vdots \\ B_{1+s,N}(x_{N-1}) & B_{2+s,N}(x_{N-1}) & \cdots & B_{N-1+s,N}(x_{N-1}) \end{bmatrix},$$

where $p = 1, 2$. Note that each matrix in this summation can be written as a permutation of the Bernstein collocation matrix.

Note that for the initial time step, the solution is given by the initial condition (1.2) as $u_0(x) = g(x)$ and for the remaining steps, the linear system (5.9) is solved to obtain the vector \mathbf{c}_{k+1} and then the solution is obtained as (5.8) at time step $k + 1$. The procedure is repeated for $k = 0, 1, \dots$ until the desired time step is reached.

6. Numerical experiments and discussion

Due to nonlocal feature of fractional derivatives, even local-based numerical techniques like finite element method applied to space-time fractional problems do not lead to sparse linear systems [6]. But for the cases that only one of time or space derivatives is fractional, it is possible to develop some methods that lead to a system of equations with sparse coefficient matrix. For instance, authors of [13] obtained a triple-tridiagonal structure for time-fractional advection-dispersion equation. The matrix formulation of the method proposed in this paper shows that the resultant matrix is not sparse but the following numerical experiments show that acceptable results are obtained with relatively small resultant matrices.

We provide some numerical tests to show the efficiency of the proposed method and to examine the theoretical results of the paper. Without loss of generality, we add a forcing term $h(x, t)$ to the space-time fractional advection-dispersion equation (1.1).

As the first example, consider the problem (1.1) with initial condition $g(x) = x^2(1 - x)^2$, homogeneous boundary conditions, the time and space fractional orders $\alpha = 0.4$, $\gamma = 0.5$ and anomalous dispersion and advection coefficients $\kappa_1 = 0.001$, $\kappa_2 = 2$, respectively with the exact solution $u = x^2(1 - x)^2 \exp(-t)$. Table 1 shows $E_2^T = \|u(x, T) - u_h^T(x)\|_2$ and $E_\infty^T = \|u(x, T) - u_h^T(x)\|_\infty$

Table 1: L_2 and L_∞ error norms for the scheme (5.9) for different values of β

β	h	E_2^T	$Rate_2$	E_∞^T	$Rate_\infty$
1.2	1/4	3.685109E-03		4.927313E-03	
	1/8	9.198267E-04	2.0022	1.257808E-03	1.9698
	1/16	2.380429E-04	1.9501	3.278200E-04	1.9399
1.4	1/4	3.684945E-03		4.927401E-03	
	1/8	9.198071E-04	2.0022	1.257786E-03	1.9699
	1/16	2.314130E-04	1.9908	3.211054E-04	1.9697
1.6	1/4	3.683697E-03		4.924700E-03	
	1/8	9.195621E-04	2.0021	1.257504E-03	1.9694
	1/16	2.168473E-04	2.0842	3.028411E-04	2.0539
1.8	1/4	3.681100E-03		4.924700E-03	
	1/8	9.187786E-04	2.0023	1.256610E-03	1.9704
	1/16	2.121292E-04	2.1147	2.994302E-04	2.0692

Table 2: Error norms and experimental order of convergence for different values of α at $T = 1$.

α	h	E_2^T	$Rate_2$	E_∞^T	$Rate_\infty$
0.2	1/4	3.315112E-03		4.475943E-03	
	1/8	8.253774E-04	2.0059	1.134993E-03	1.9795
	1/16	2.198645E-04	1.9084	2.997409E-04	1.9209
0.4	1/4	3.684620E-03		4.927243E-03	
	1/8	9.198521E-04	2.0020	1.257693E-03	1.9700
	1/16	2.661032E-04	1.7894	3.548821E-04	1.8254
0.6	1/4	4.141701E-03		5.482632E-03	
	1/8	1.042701E-03	1.9899	1.415220E-03	1.9538
	1/16	3.255528E-04	1.6794	4.256264E-04	1.7334
0.8	1/4	4.769215E-03		6.246503E-03	
	1/8	1.225821E-03	1.9600	1.646025E-03	1.9241
	1/16	3.157964E-04	1.9567	4.427624E-04	1.8944

errors at time $T = 1$ with time step size $\tau = 0.05$ for different values of $\beta \in (1, 2)$ and with different space mesh sizes. Also experimental rate of convergence is provided in this table. It is seen that the rate of convergence is about two as it is expected from theoretical results. Tables 2 and 3 show the computational errors and experimental rate of convergence for the same parameters as 1 but for $\beta = 1.5$ and varying $\alpha \in (0, 1)$ and also $\alpha = 0.5, \beta = 1.5$ and different values of $\gamma \in (0, 1)$, respectively.

Note that both in formulation of the method and in numerical tests, for convenience of formulation, the collocation points are assumed to be equidistant on the unit interval. However, it can be done using any arbitrary set of collocation points.

As the second example, consider the advection dispersion equation (1.1) with the coefficients $\kappa_1 = 0.1, \kappa_2 = 5$ and fractional orders as $\alpha = 0.5, \beta = 1.5$ and $\gamma = 0.5$ with initial condition $u(x, 0) = 0$ and the forcing term, $h(x, s)$ such that the exact solution is $u = \sin(\pi x)t^2$. Table 4 provides the L_2 error norms with overall CPU time (seconds) consumed in the algorithm for obtaining the numerical solution at $T = 1$.

Table 3: Error norms and experimental order of convergence for different values of γ at $T = 1$.

γ	h	E_2^T	$Rate_2$	E_∞^T	$Rate_\infty$
0.2	1/4	3.9274E-03		5.3811E-03	
	1/8	9.8523E-04	1.9950	1.3545E-03	1.9901
	1/16	2.5973E-04	1.9234	3.6105E-04	1.9075
0.4	1/4	3.9810E-03		5.3830E-03	
	1/8	9.9690E-04	1.9976	1.3653E-03	1.9792
	1/16	2.6503E-04	1.9113	3.6183E-04	1.9158
0.6	1/4	3.7012E-03		4.8086E-03	
	1/8	9.2097E-04	2.0068	1.2447E-03	1.9498
	1/16	2.4507E-04	1.9099	3.2997E-04	1.9154
0.8	1/4	2.8985E-03		3.6448E-03	
	1/8	6.6410E-04	2.1259	8.8712E-04	2.0387
	1/16	1.7361E-04	1.9355	2.3733E-04	1.9022

Table 4: L_2 norm error and CPU time after $M = \frac{1}{\tau}$ steps

h	τ	E_2^T	CPU time (s)
1/4	1/10	3.4898E-02	0.16
1/6	1/20	1.3529E-02	0.843
1/8	1/30	7.5908E-03	2.359
1/10	1/40	4.8497E-03	6.266

7. Concluding remarks

In this paper, we proposed a numerical approach for solving space and time fractional advection dispersion equations. After time discretization, we used product integration technique to derive some explicit formulas for terms involving space fractional derivatives. By utilizing Bernstein polynomial basis, we transformed the problem into solving a linear system of algebraic equations. It is worth to mention that the proposed method is different from the well-known method of line in the sense that in our method, the problem at first is discretized in time leading to a boundary value problem instead of an initial value problem in order to utilize collocation method with Bernstein basis. We also discussed the error estimation of the approximating relations for fractional derivatives and using some numerical experiments, we showed that the method is efficient and simple to implement for solving fractional advection dispersion equations on a bounded domain. From these numerical tests, it is seen that the associated experimental order of convergence is consistent with the analysis. The proposed method can be applied to a wide range of space and time fractional partial differential equations on bounded domains.

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