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Some fixed point and coupled fixed point theorems in partially ordered probabilistic like (quasi) menger spaces

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Abstract

In this paper, we define the concept of probabilistic like Menger (probabilistic like quasi Menger) space (briefly, PLM-space (PL_qM-space)). We present some coupled fixed point and fixed point results for certain contraction type maps in partially order PLM-spaces (PL_qM-spaces).

Keywords: Coupled fixed point; Partially ordered PLM-space (PL_qM-space); Mixed monotone property.
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1. Introduction and Preliminaries

In 1931, Wilson [60] introduced the concept of a quasi metric space (where the condition of symmetry in dropped) and then Kelly [30] developed the concept of a quasi metric space. Many problems in theoretical computer science, topological algebra and approximation theory one can solved by the theory of quasi metric spaces, see [23, 33, 38]. Many fixed point theorems have extensions in generalized form of metric spaces, particularly in quasi metric spaces, partially quasi metric spaces and partially ordered metric spaces.

The common fixed point theorems for multi-valued mappings in quasi metric spaces was proved by Cho [11].

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Indeed, the study of fixed point theorems in partially ordered metric spaces is one of the most active research areas in fixed point theory. Many authors proved some new fixed point theorems for some contraction mappings in partially ordered metric spaces, for example, one can refer to [2, 5, 39, 40, 47].

Ran et al. in [47] and Nieto et al. in [39, 40] proved the existence and uniqueness of solutions for some matrix equations and differential equations respectively by some fixed point theorems in partially ordered metric spaces.

Opoitsev in [42, 43, 44] introduced and studied the notion of a coupled fixed point and then by Guo et al. [24]. Afterwards, the concept of coupled fixed point of a mapping in partially ordered set was introduced by Bhaskar et al. [5]. Later Lakshmikantham et al. [34] proved some coupled fixed point theorems in partially ordered sets. Recently, the results of Bhaskar et al. [5] was extended by Samet [51] for some mappings satisfying a generalized Meir-Keeler contractive condition.

For further existence results of a coupled and tripled fixed point in ordered metric and cone metric spaces one can refer to [1, 3, 4, 6, 7, 13, 15, 16, 17, 18, 27, 32, 34, 52, 59].

In 1942, probabilistic metric space (abbreviated, PM-space) was introduced by Karl Menger [35]. Schweizer and Sklar were two pioneers in the study of PM-space [54, 55].

PM-spaces are very useful in probabilistic functional analysis, quantum particle physics, ε^{∞} theory, nonlinear analysis and applications, see [8, 9, 19, 20, 21].

Indeed, the study of fixed point results in PM-spaces is one of the most active research areas in fixed point theory. Sehgal and Bharucha-Reid [57], were two pioneers in this study. For further existence results of a fixed point and common fixed point in PM-spaces, one can refer, for example, to [29, 41, 48].

The class of probabilistic quasi metric spaces was introduced by Kent et al. [31] and they proved some common fixed point theorems in this spaces.

For further existence results of a fixed point for single-valued mappings in probabilistic quasi metric spaces, one can refer to [10, 36, 37, 38, 46, 53, 56, 58].

Many authors investigated many fixed point theorems for contraction mappings in partially ordered probabilistic metric spaces. For further recent results on fixed point theory in partially ordered probabilistic metric spaces, for example, one can refer to [12, 50, 61]. Recently, the concept of monotone generalized contraction in partially ordered probabilistic metric spaces was introduced by Ciric et al. in [14] and they proved some fixed point and common fixed point theorems for this contraction mappings.

Next we shall recall some well-known definitions and results in the theory of probabilistic metric spaces which are used later in this paper.

Definition 1.1. [54] A function $F : \mathbb{R} \to [0, 1]$ is distribution function if F is nondecreasing and left continuous function on $\mathbb{R}$, $\inf_{t\in\mathbb{R}} F(t) = 0$ and $\sup_{t\in\mathbb{R}} F(t) = 1$.

Let Δ be all the distribution functions and Δ^+ be all distribution functions F such that $F(0) = 0$. It is easy to see that the space Δ^+ is partially ordered by the usual pointwise ordering of functions and the function $\epsilon_0 = \chi_{(0,\infty)}$ is maximal element of Δ^+.

Definition 1.2. [54] Let X be a nonempty set and $F : X \times X \to \Delta^+$ ($F(p, q)$ is denoted by $F_{p,q}$). The ordered pair (X, F) is a probabilistic metric space (abbreviated, PM-space) if the following three conditions are satisfied:

(PM1) $F_{p,q} = \epsilon_0$, iff $p = q$, (PM2) $F_{p,q} = F_{q,p}$,

(PM3) If $F_{p,q}(t) = 1$ and $F_{q,r}(s) = 1$, then $F_{p,r}(t+s) = 1$,

for every $p, q, r \in X$ and $t, s \geq 0$.

Definition 1.3. [54] A mapping $\tau : [0,1] \times [0,1] \to [0,1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:

(i) $\tau(a,b) = \tau(b,a)$,
(ii) $\tau(a, \tau(b,c)) = \tau(\tau(a,b), c)$,
(iii) $\tau(a,b) \geq \tau(c,d)$, whenever $a \geq c$ and $b \geq d$,
(iv) $\tau(a,1) = a$,

for every $a, b, c, d \in [0,1]$.

The mappings $\tau_p(a,b) = a \cdot b$ and $\tau_m(a,b) = \min\{a,b\}$ are two examples of continuous t-norms. It is easy to see that, as regards the pointwise ordering, $\tau \leq \tau_m$, for each t-norm τ.

Let us recall that, for a t-norm τ, the sequence $(\tau^n)_{n=0}^{\infty}$ is defined as follows:

$$\tau^0(a) = a \quad \text{and} \quad \tau^n(a) = \tau(a, \tau^{n-1}(a)), \quad (n \in \mathbb{N}, a \in [0,1]).$$

Definition 1.4. [25] Let τ be a t-norm. If the sequence of functions $(\tau^n(a))$ is equicontinuous at $a = 1$, that is

$$\forall\, \varepsilon \in (0,1), \quad \exists\, \delta \in (0,1) : \; a > 1 - \delta \Rightarrow \tau^n(a) > 1 - \varepsilon, \quad (n \in \mathbb{N}).$$

Then τ is called t-norm of Hadžić type (abbreviated, H-type).

Clearly the t-norm τ_m is an example of a t-norm of H-type, but there are t-norms τ of H-type with $\tau \neq \tau_m$, see [25]. It is easy to see that if τ is of H-type, then τ satisfies $\sup_{a \in (0,1)} \tau(a,a) = 1$.

Lemma 1.5. *If τ is a t-norm, then $\tau(a,a) \geq a$, for all $a \in [0,1]$, if and only if $\tau = \tau_m$.*

Proof . For an arbitrary t-norm τ we get $\tau \leq \tau_m$. Let $a, b \in [0,1]$ such that $a \leq b \leq 1$, so we have

$$a \leq \tau(a,a) \leq \tau(a,b) \leq \tau_m(a,b) = a,$$

then $\tau(a,b) = \tau_m(a,b)$. $\square$

Definition 1.6. [25] Let (X, F) be a PM-space and τ be a t-norm. Then the triplet (X, F, τ) is called a Menger space if

$$F_{p,r}(t+s) \geq \tau\left(F_{p,q}(t), F_{q,r}(s)\right),$$

for all $p, q, r \in X$ and for all $t, s \geq 0$.

Definition 1.7. Let X be a nonempty set, τ be a t-norm and $F : X \times X \to \Delta^+$ ($F(p,q)$ is denoted by $F_{p,q}$) be a mapping. The triplet (X, F, τ) is a probabilistic like quasi Menger space (abbreviated, PL_qM-space) if the following two conditions are satisfied:

(i) $F_{p,q} = \epsilon_0 = F_{q,p} \Rightarrow p = q$,

(ii) $F_{p,r}(t+s) \geq \tau\left(F_{p,q}(t), F_{q,r}(s)\right)$,

for every $p, q, r \in X$ and $t, s \geq 0$.

Example 1.8. Let $X = \mathbb{R}$, define

$$F_{x,y}(t) = \frac{1}{e^{\frac{|x-y|+2|x|+|y|}{t}}},$$

for all $x, y \in X, t > 0$. It is clear that if $F_{x,y}(t) = \epsilon_0(t) = F_{y,x}(t)$, for every $x, y \in \mathbb{R}$ and $t > 0$, then $x = y$. We know that

$$|x-z|+2|x|+|z| \leq \left(\frac{t+s}{t}\right)(|x-y|+2|x|+|y|) + \left(\frac{t+s}{s}\right)(|y-z|+2|y|+|z|)\,,$$

i.e.

$$\frac{|x-z|+2|x|+|z|}{t+s} \leq \frac{|x-y|+2|x|+|y|}{t} + \frac{|y-z|+2|y|+|z|}{s},$$

therefore

$$e^{\frac{|x-z|+2|x|+|z|}{t+s}} \leq e^{\frac{|x-y|+2|x|+|y|}{t}}\, e^{\frac{|y-z|+2|y|+|z|}{s}}.$$

Thus $F_{x,y}(t)F_{y,z}(s) \leq F_{x,z}(t+s)$. Hence (X, F, τ_p) is a PL_qM-space.

A quasi metric space is a nonempty set X with a function $d : X \times X \to [0, \infty)$ which satisfies the following two conditions:

(i) $d(x, y) \geq 0$ for all $x, y \in X$ and if $d(x, y) = 0 = d(y, x)$, then $x = y$,

(ii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Example 1.9. Let (X, d) be a quasi metric space. Then it is easy to see that, (X, F, τ) is a PL_qM-space, where τ is any continuous t-norm and F is defined by

$$F_{x,y}(t) = \begin{cases} 1, & d(x,y) < t \text{ and } \ 0 < t, \\ 0, & d(x,y) \geq t \text{ or } 0 \geq t, \end{cases}$$

for all $x, y \in X$.

Definition 1.10. A probabilistic like Menger space (abbreviated, PLM-space) is a PL_qM-space (X, F, τ), such that for all $p, q \in X$, $F_{p,q} = F_{q,p}$.

Let (X, F, τ) be a PL_qM-space and $F^{\ddagger}_{p,q}(t) = \min\{F_{p,q}(t), F_{q,p}(t)\}$ ($p, q \in X$ and $t \in [0, \infty)$), then it is easy to see that, $(X, F^{\ddagger}, \tau)$ is a PLM-space.

Definition 1.11. Let (X, F, τ) be a PL_qM-space. A left (right) open ball (abbreviated, L-open (R-open) ball) with center x and radius r ($0 < r < 1$) in X is the set $B_L(x, r, t) = \{y \in X : F_{x,y}(t) > 1 - r\}$ ($B_R(x, r, t) = \{y \in X : F_{y,x}(t) > 1 - r\}$), for all $t > 0$. Moreover, an open ball with center x and radius r ($0 < r < 1$) in X is the set $B(x, r, t) = \{y \in X : F^{\ddagger}_{x,y}(t) > 1 - r\}$, for all $t > 0$.

Definition 1.12. A sequence (x_n) in a PL_qM-space (X, F, τ) is said to be left (right) convergent to a point $x \in X$ if and only if $\lim_{n\to\infty} F_{x_n,x}(t) = 1$ ($\lim_{n\to\infty} F_{x,x_n}(t) = 1$) for all $t > 0$. Also, a sequence (x_n) is said to be bi-convergent to a point $x \in X$ if and only if $\lim_{n\to\infty} F^{\ddagger}_{x_n,x}(t) = 1$ for all $t > 0$, in this case we say that limit of the sequence (x_n) is x. A sequence (x_n) is said to be left (right) Cauchy sequence if and only if $\lim_{n\to\infty} F_{x_n,x_{n+p}}(t) = 1$ ($\lim_{n\to\infty} F_{x_{n+p},x_n}(t) = 1$) for all $t > 0,\ p \in \mathbb{N}$. Also, a sequence (x_n) is said to be bi-Cauchy if and only if $\lim_{n\to\infty} F^{\ddagger}_{x_n,x_{n+p}}(t) = 1$ for all $t > 0,\ p \in \mathbb{N}$.

The concept of left (right) Cauchy sequence is inspired from that of G-Cauchy sequence (it belongs to Grabiec [22]).

Definition 1.13. A PL_qM-space (X, F, τ) is said to be left (right) complete if and only if every left (right) Cauchy sequence in X, is left (right) convergent. Also, a PL_qM-space (X, F, τ) is said to be bi-complete if and only if every bi-Cauchy sequence in X, is *bi*-convergent.

Clearly a sequence (x_n) in a PL_qM-space (X, F, τ) is bi-Cauchy sequence if and only if sequence (x_n) is a Cauchy sequence in the PLM-space $(X, F^{\ddagger}, \tau)$. Also, a PL_qM-space (X, F, τ) is bi-Complete if and only if the PLM-space $(X, F^{\ddagger}, \tau)$ is complete.

Proposition 1.14. *Let (X, F, τ) be a PL_qM-space. If the t-norm τ is continuous at $(1, 1)$ (or τ of H-type). Then limit of a bi-convergent sequence is unique.*

Proof . It is obvious. □

Definition 1.15. Let (X, F, τ) be a PL_qM-space and $T : X \to X$ be a mapping. The mapping T is said to be continuous at a point $x \in X$ if for every sequence (x_n) in X, which bi-converges to x, the sequence (Tx_n) in X bi-converges to Tx.

Let Φ denote all the functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy $\varphi(0) = 0$, $\varphi(t) < t$ and $\lim\limits_{n\to\infty} \varphi^n(t) = 0$, for all $t > 0$.

Definition 1.16. [5] Let X be a nonempty set and $T : X \times X \to X$ be a mapping. An element $(x, y) \in X \times X$ is called a coupled fixed point of the mapping T if

$$T(x, y) = x, \quad \textit{and} \quad T(y, x) = y.$$

Definition 1.17. A partially ordered probabilistic like quasi Menger space (abbreviated, partially ordered PL_qM-space) is 4-tuple $(X, F, \tau, \leq)$ such that (X, F, τ) is a PL_qM-space and $\leq$ is a partially ordered on X.

Let $(X, \leq)$ be a partially ordered set. A self-map $T : X \to X$ is said to be nondecreasing if the condition $x \leq y$ implies $Tx \leq Ty$ for all $x, y \in X$.

Definition 1.18. [5] Let $(X, \leq)$ be a partially ordered set and $T : X \times X \to X$ be a mapping. The mapping T is said to have the mixed monotone property if $T(x, y)$ is nondecreasing in x and is nonincreasing in y, that is, for any $x, y \in X$

$$x_1, x_2 \in X, \qquad x_1 \leq x_2 \Longrightarrow T(x_1, y) \leq T(x_2, y),$$

and

$$y_1, y_2 \in X, \qquad y_1 \leq y_2 \Longrightarrow T(x, y_1) \geq T(x, y_2).$$

Let $(X, \leq)$ be a partially ordered set. We endow the product space $X \times X$ with the following partial order:

$$(x, y) \leq (u, v) \;\Leftrightarrow\; x \leq u \;\; and \;\; y \geq v,$$

for all $(x, y), (u, v) \in X \times X$.

The following lemma has been proved by Jachymski in [28] for mappings $g_n : (0, \infty) \to (0, \infty)$, but it is also valid for mappings $g_n : [0, \infty) \to [0, \infty)$.

Lemma 1.19. *Let $n \in \mathbb{N}$, $g_n : [0, \infty) \to [0, \infty)$ and $F_n, F : \mathbb{R} \to [0, 1]$. Assume that $\sup\{F(t) : t > 0\} = 1$ and for any $t > 0$, $\lim_{n\to\infty} g_n(t) = 0$ and $F_n(g_n(t)) \geq F(t)$. If each F_n is nondecreasing, then $\lim_{n\to\infty} F_n(t) = 1$ for any $t > 0$.*

Proof . Fix $t > 0$ and $\varepsilon > 0$. By hypothesis, there is $t_0 > 0$ such that $F(t_0) > 1 - \varepsilon$. Since $g_n(t_0) \to 0$, there is $N \in \mathbb{N}$ such that $g_n(t_0) < t$ for all $n \geq N$. By monotonicity

$$F_n(t) \geq F_n(g_n(t_0)) \geq F(t_0) > 1 - \varepsilon \quad (\forall\, n \geq N).$$

Hence we get $\lim_{n\to\infty} F_n(t) = 1$. □

In this paper, we define the concept of PLM-space (PL_qM-space). We show that if $(X, F, \tau, \leq)$ is a partially ordered bi-complete PL_qM-space with a t-norm τ of H-type and $T : X \to X$ is a nondecreasing mapping with respect to the order $\leq$ on X such that for all elements $x, y \in X$ are comparable and for all $t > 0$

$$F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t),$$

where $\varphi \in \Phi$. Then the mapping T has a fixed point in X, under certain conditions. We also prove that if $(X, F, \tau, \leq)$ is a partially ordered complete PLM-space with a t-norm τ of H-type and $T : X \times X \to X$ be a mapping having the mixed monotone property on X such that for all elements $(x, y), (u, v) \in X \times X$ are comparable and for all $t > 0$

$$\begin{aligned}\tau\left(F_{T(x,y),T(u,v)}(\varphi(t)), F_{T(y,x),T(v,u)}(\varphi(t))\right) \geq \min\{&\tau\left(F_{x,u}(t), F_{y,v}(t)\right), \tau\left(F_{x,T(x,y)}(t), F_{y,T(y,x)}(t)\right),\\ &\tau\left(F_{u,T(u,v)}(t), F_{v,T(v,u)}(t)\right)\}\end{aligned}$$

or

$$\tau\left(F_{T(x,y),T(u,v)}(\varphi(t)), F_{T(y,x),T(v,u)}(\varphi(t))\right) \geq \tau\left(F_{x,u}(t), F_{y,v}(t)\right)$$

where $\varphi \in \Phi$. Then under certain conditions the mapping T has a coupled fixed point in X. Finally, we give some examples to illustrate the theorems.

2. Main results

We first bring the following lemma, then we will bring the main results of this paper.

Lemma 2.1. *Let (X, F, τ) be a PL_qM-space with a t-norm τ of H-type and (x_n) be a sequence in X. If there exists a function $\varphi \in \Phi$ such that*

$$F_{x_n,x_m}(\varphi(t)) \geq F_{x_{n-1},x_{m-1}}(t), \qquad (m > n), \tag{2.1}$$

then (x_n) is a left Cauchy sequence in X.

Proof . Clearly, $F_{x_n,x_{n+1}}(\varphi(t)) \geq F_{x_{n-1},x_n}(t)$, for any $t > 0$, so the sequence $(F_{x_n,x_{n+1}}(\varphi^n(t)))$ is nondecreasing. Indeed, given $n \in \mathbb{N}$, so by (2.1), we get

$$F_{x_n,x_{n+1}}(\varphi^n(t)) = F_{x_n,x_{n+1}}(\varphi(\varphi^{n-1}(t))) \geq F_{x_{n-1},x_n}(\varphi^{n-1}(t)), \qquad (t > 0).$$

Hence, we infer that $F_{x_n,x_{n+1}}(\varphi^n(t)) \geq F_{x_0,x_1}(t)$, so by Lemma 1.19

$$\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1, \qquad \text{for any } t > 0. \tag{2.2}$$

Now let $n \in \mathbb{N}$ and $t > 0$. We show by induction that for any $k \in \mathbb{N}$,

$$F_{x_n,x_{n+k}}(t) \geq \tau^k \left(F_{x_n,x_{n+1}}(t - \varphi(t))\right). \tag{2.3}$$

This is obvious for $k = 1$, since $F_{x_n,x_{n+1}}(t) \geq F_{x_n,x_{n+1}}(t - \varphi(t)) = \tau^1 \left(F_{x_n,x_{n+1}}(t - \varphi(t))\right)$. Assume that (2.3) hold for some $k \geq 1$. Hence by (2.1) and the monotonicity of τ, we have

$$\begin{aligned} F_{x_n,x_{n+k+1}}(t) =& F_{x_n,x_{n+k+1}}((t - \varphi(t)) + \varphi(t)) \\ \geq& \tau \left(F_{x_n,x_{n+1}}(t - \varphi(t)), F_{x_{n+1},x_{n+k+1}}(\varphi(t))\right) \\ \geq& \tau \left(F_{x_n,x_{n+1}}(t - \varphi(t)), F_{x_n,x_{n+k}}(t)\right) \\ \geq& \tau \left(F_{x_n,x_{n+1}}(t - \varphi(t)), \tau^k \left(F_{x_n,x_{n+1}}(t - \varphi(t))\right)\right) \\ =& \tau^{k+1} \left(F_{x_n,x_{n+1}}(t - \varphi(t))\right), \end{aligned}$$

which complete the induction. Let $t > 0$ and $\lambda > 0$. Since τ is a t-norm of H-type and $\tau^n(1) = 1$, so there is $\delta > 0$ such that

$$a > 1 - \delta \implies \tau^n(a) > 1 - \lambda, \qquad (n \in \mathbb{N}). \tag{2.4}$$

By (2.2), $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t - \varphi(t)) = 1$, so there is $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$, $F_{x_n,x_{n+1}}(t - \varphi(t)) > 1 - \delta$. Hence, by (2.3) and (2.4), we get $F_{x_n,x_{n+k}}(t) > 1 - \lambda$ for any $k \in \mathbb{N}$. Thus $\lim_{n\to\infty} F_{x_n,x_{n+k}}(t) = 1$, this means that (x_n) is a left Cauchy sequence in X. □

Lemma 2.2. *Let (X, F, τ) be a PL_qM-space with a t-norm τ of H-type and (x_n) be a sequence in X. If there exists a mapping $\varphi \in \Phi$ such that*

$$F_{x_m,x_n}(\varphi(t)) \geq F_{x_{m-1},x_{n-1}}(t), \qquad (m > n),$$

then (x_n) is a right Cauchy sequence in X.

Proof . By using a similar argument as in the proof of the above lemma, the result follows. □

Theorem 2.3. *Let $(X, F, \tau, \leq)$ be a partially ordered bi-complete PL_qM-space with a t-norm τ of H-type. Suppose that $T : X \to X$ is a nondecreasing mapping with respect to the order $\leq$ on X. If the following conditions hold:*

(i) there is a $\varphi \in \Phi$ such that

$$F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t), \tag{2.5}$$

for all elements $x, y \in X$ are comparable and for all $t > 0$,

(ii) there exists an $x_0 \in X$ such that $x_0 \leq Tx_0$,

(iii) either

(a) T is a continuous mapping or
(b) if a nondecreasing sequence (x_n) in X is bi-convergent to x, then x_n and x are comparable for all n.

Then the mapping T has a fixed point in X. Furthermore, if for each $x, y \in X$, there exists $z \in X$ which is comparable to x and y, then the fixed point of T is unique.

Proof . Define a sequence $(x_n) \subseteq X$ by $x_{n+1} = Tx_n$, $n = 0, 1, \cdots$. Since $x_0 \leq Tx_0$ and T is nondecreasing mapping, we have

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots .$$

If there exists n_0 such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0} = Tx_{n_0}$ and x_{n_0} is a fixed point of T. Therefore the result trivially holds. Suppose now that $x_{n+1} \neq x_n$ for all n. Following the assumption (i), we see that

$$\begin{aligned} F_{x_n,x_m}(\varphi(t)) &= F_{Tx_{n-1},Tx_{m-1}}(\varphi(t)) \geq F_{x_{n-1},x_{m-1}}(t), \\ F_{x_m,x_n}(\varphi(t)) &= F_{Tx_{m-1},Tx_{n-1}}(\varphi(t)) \geq F_{x_{m-1},x_{n-1}}(t), \end{aligned}$$

for every $m > n$. Hence by Lemma 2.1 and Lemma 2.2, (x_n) is a bi-Cauchy sequence in X. Then by the bi-completeness of X, there is $x^* \in X$ such that $\lim_{n\to\infty} F^{\ddagger}_{x_n,x^*}(t) = 1$. Suppose (a) holds. It follows from $x_{n+1} = Tx_n$, that

$$x^* = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tx^*.$$

Then the mapping T has a fixed point in X. Suppose (b) in the assumption (iii) holds, then x_n and x^* are comparable so we have

$$\begin{aligned} F_{Tx^*,x^*}(t) &\geq \tau\left(F_{Tx^*,Tx_n}(\varphi(t)), F_{Tx_n,x^*}(t-\varphi(t))\right) \\ &\geq \tau\left(F_{x^*,x_{n+1}}(t), F_{x_{n+1},x^*}(t-\varphi(t))\right). \end{aligned}$$

Letting $n \to \infty$, it follows that $F_{Tx^*,x^*}(t) = 1$. Similarly we can show that $F_{x^*,Tx^*}(t) = 1$, thus $F^{\ddagger}_{x^*,Tx^*}(t) = 1$, therefore $x^* = Tx^*$. Let $y^* \in X$ such that $y^* = Ty^*$, then following the assumption there exists $z \in X$ which is comparable to x^* and y^*. By the monotony of T, we infer that $T^n(z)$ is comparable to $T^n(x^*) = x^*$ and $T^n(y^*) = y^*$. Hence we have

$$F_{x^*,T^n(z)}(\varphi^n(t)) = F_{T^n(x^*),T^n(z)}(\varphi^n(t)) \geq F_{T^{n-1}(x^*),T^{n-1}(z)}(\varphi^{n-1}(t)) = F_{x^*,T^{n-1}(z)}(\varphi^{n-1}(t)), \qquad (t > 0).$$

Hence, we infer that $F_{x^*,T^n(z)}(\varphi^n(t)) \geq F_{x^*,z}(t)$, so by Lemma 1.19

$$\lim_{n\to\infty} F_{x^*,T^n(z)}(t) = 1, \quad \forall\, t > 0.$$

Similarly we can show that $\lim_{n\to\infty} F_{T^n(z),y^*}(t) = 1$, for any $t > 0$, and we have

$$F_{x^*,y^*}(t) \geq \tau\left(F_{x^*,T^n(z)}(\frac{t}{2}), F_{T^n(z),y^*}(\frac{t}{2})\right).$$

Letting n go to infinity, it follows that $F_{x^*,y^*}(t) = 1$. Similarly we can show that $F_{y^*,x^*}(t) = 1$, thus $F^{\ddagger}_{x^*,y^*}(t) = 1$, therefore $x^* = y^*$. $\square$

Lemma 2.4. *If (X, F, τ) is a bi-complete PL_qM-space such that τ is a continuous t-norm at $(1,1)$, then (X^2, M, τ) is also a bi-complete PL_qM-space, where for every $x, y, u, v \in X$,*

$$M_{(x,y),(u,v)}(t) = \tau\left(F_{x,u}(t), F_{y,v}(t)\right). \tag{2.6}$$

Proof . If $M_{(x,y),(u,v)}(t) = 1$, then $F_{x,u}(t) = F_{y,v}(t) = 1$, for any $t > 0$, otherwise if there is a $t_0 > 0$ such that $F_{x,u}(t_0) < 1$ or $F_{y,v}(t_0) < 1$, then we have

$$\begin{aligned} 1 = \tau\left(F_{x,u}(t_0), F_{y,v}(t_0)\right) &\leq \min\{\tau\Big(F_{x,u}(t_0), 1\Big), \tau\Big(1, F_{y,v}(t_0)\Big)\} \\ &= \min\{F_{x,u}(t_0), F_{y,v}(t_0)\} < 1, \end{aligned}$$

a contradiction. Thus for all $t > 0$, $F_{x,u}(t) = F_{y,v}(t) = 1$. Similarly if $M_{(u,v),(x,y)}(t) = 1$, then $F_{u,x}(t) = F_{v,y}(t) = 1$, for any $t > 0$, so $(x, y) = (u, v)$. We now prove that for $(x, y), (u, v), (z, w) \in X^2$ and $t, s \geq 0$,

$$M_{(x,y),(z,w)}(t+s) \geq \tau\left(M_{(x,y),(u,v)}(t), M_{(u,v),(z,w)}(s)\right).$$

By definition of M we have

$$\begin{aligned} M_{(x,y),(z,w)}(t+s) =& \tau\left(F_{x,z}(t+s), F_{y,w}(t+s)\right) \\ \geq& \tau\left(\tau\Big(F_{x,u}(t), F_{u,z}(s)\Big), \tau\left(F_{y,v}(t), F_{v,w}(s)\right)\right) \\ =& \tau\left(F_{x,u}(t), \tau\left(F_{u,z}(s), \tau\left(F_{y,v}(t), F_{v,w}(s)\right)\right)\right) \\ =& \tau\left(F_{x,u}(t), \tau\left(F_{u,z}(s), \tau\left(F_{v,w}(s), F_{y,v}(t)\right)\right)\right) \\ =& \tau\left(F_{x,u}(t), \tau\left(\tau\Big(F_{u,z}(s), F_{v,w}(s)\Big), F_{y,v}(t)\right)\right) \\ =& \tau\left(F_{x,u}(t), \tau\left(F_{y,v}(t), \tau\Big(F_{u,z}(s), F_{v,w}(s)\Big)\right)\right) \\ =& \tau\left(\tau\left(F_{x,u}(t), F_{y,v}(t)\right), \tau\Big(F_{u,z}(s), F_{v,w}(s)\Big)\right) \\ =& \tau\left(M_{(x,y),(u,v)}(t), M_{(u,v),(z,w)}(s)\right). \end{aligned}$$

If a sequence (x_n, y_n) is a bi-Cauchy sequence in (X^2, M, τ), then for all $t > 0$ and $\lambda \in (0, 1)$ there is a positive integer $N(\lambda, t)$ such that

$$\tau\left(F_{x_n,x_{n+p}}(t), F_{y_n,y_{n+p}}(t)\right) = M_{(x_n,y_n),(x_{n+p},y_{n+p})}(t) \geq M^{\ddagger}_{(x_n,y_n),(x_{n+p},y_{n+p})}(t) > 1 - \lambda$$

$$\left(\tau\left(F_{x_{n+p},x_n}(t), F_{y_{n+p},y_n}(t)\right) = M_{(x_{n+p},y_{n+p}),(x_n,y_n)}(t) \geq M^{\ddagger}_{(x_n,y_n),(x_{n+p},y_{n+p})}(t) > 1 - \lambda\right),$$

for all $n > N(\lambda, t)$ and $p \in \mathbb{N}$. Then it is easy to see that $F_{x_n,x_{n+p}}(t) > 1 - \lambda$ $\left(F_{x_{n+p},x_n}(t) > 1 - \lambda\right)$ and $F_{y_n,y_{n+p}}(t) > 1 - \lambda$ $\left(F_{y_{n+p},y_n}(t) > 1 - \lambda\right)$. Thus both (x_n) and (y_n) are bi-Cauchy sequences in (X, F, τ). Then by the bi-completeness of X, there are $x, y \in X$ such that $\lim_{n\to\infty} F^{\ddagger}_{x_n,x}(t) = 1$ and $\lim_{n\to\infty} F^{\ddagger}_{y_n,y}(t) = 1$. By

$$M_{(x_n,y_n),(x,y)}(t) = \tau\left(F_{x_n,x}(t), F_{y_n,y}(t)\right),$$

and continuouty of τ at $(1,1)$ we get $\lim_{n\to\infty} M_{(x_n,y_n),(x,y)}(t) = 1$. Similarly we can show that $\lim_{n\to\infty} M_{(x,y),(x_n,y_n)}(t) = 1$, thus $\lim_{n\to\infty} M^{\ddagger}_{(x_n,y_n),(x,y)}(t) = 1$, therefore (X^2, M, τ) is a bi-complete PL_qM-space. The proof is complete. □

Theorem 2.5. *Let $(X, F, \tau_m, \leq)$ be a partially ordered bi-complete PL_qM-space. Let $T : X\times X \to X$ be a mapping having the mixed monotone property on X and $\varphi \in \Phi$ such that*

$$F_{T(x,y),T(u,v)}(\varphi(t)) \geq \tau_m\left(F_{x,u}(t), F_{y,v}(t)\right), \tag{2.7}$$

for all elements $(x, y), (u, v) \in X \times X$ are comparable and for all $t > 0$. Assume that either

(i) T is continuous or

(ii) if a nondecreasing sequence $((x_n, y_n))$ in $X \times X$ is bi-convergent to (x, y), then (x_n, y_n) and (x, y) are comparable for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq T(x_0, y_0) \quad \textit{and} \quad y_0 \geq T(y_0, x_0),$$

then there exist $x^*, y^* \in X$ such that

$$x^* = T(x^*, y^*) \qquad \textit{and} \qquad y^* = T(y^*, x^*).$$

Furthermore, if for each $(x, y), (z, t) \in X \times X$, there exists $(u, v) \in X \times X$ that is comparable to (x, y) and (z, t), then the coupled fixed point (x^*, y^*) of T is unique and $x^* = y^*$.

Proof . Suppose that $M_{(x,y),(u,v)}(t) = \tau_m\left(F_{x,u}(t), F_{y,v}(t)\right)$, for each $(x, y), (u, v) \in X \times X$ and $t \geq 0$. By Lemma 2.4, (X^2, M, τ_m) is a bi-complete PL_qM-space. Let $S : X^2 \to X^2$ be defined by $S(x, y) = (T(x, y), T(y, x))$. The mapping S is nondecreasing because T has the mixed monotone property. For each $u = (u_1, u_2), v = (v_1, v_2) \in X^2$ are comparable and (2.7), we have

$$\begin{aligned} M_{Su,Sv}(\varphi(t)) =& M_{(T(u_1,u_2),T(u_2,u_1)),(T(v_1,v_2),T(v_2,v_1))}(\varphi(t)) \\ =& \tau_m\left(F_{T(u_1,u_2),T(v_1,v_2)}(\varphi(t)), F_{T(u_2,u_1),T(v_2,v_1)}(\varphi(t))\right) \\ \geq& \tau_m\left(\tau_m\left(F_{u_1,v_1}(t), F_{u_2,v_2}(t)\right), \tau_m\left(F_{u_2,v_2}(t), F_{u_1,v_1}(t)\right)\right) \\ =& \tau_m\left(F_{u_1,v_1}(t), F_{u_2,v_2}(t)\right) \\ =& M_{u,v}(t), \end{aligned}$$

hence $M_{Su,Sv}(\varphi(t)) \geq M_{u,v}(t)$. By our assumptions either S is continuous or if a nondecreasing sequence $u_n \to u$, $u_n \in X^2$, then u_n and u are comparable for all n. Since $x_0 \leq T(x_0, y_0)$ and $y_0 \geq T(y_0, x_0)$, then $(x_0, y_0) \leq S(x_0, y_0)$. Also, for each $u, v \in X^2$, there exists $w \in X^2$ which is comparable to u and v. Then from Theorem 2.3 we deduce that S has a unique fixed point $u^* = (x^*, y^*)$. Then, (x^*, y^*) is the unique coupled fixed point of T . Since (x^*, y^*) is a coupled fixed point of T then by the definition we have that (y^*, x^*) is a coupled fixed point too. Then by the uniqueness, we get $(x^*, y^*) = (y^*, x^*)$, and so $x^* = y^*$. □

Lemma 2.6. *Let $F, G \in \Delta^+$ and $\varphi \in \Phi$. If*

$$F(\varphi(t)) \geq \min\{F(t), G(t)\}, \qquad \forall t \geq 0,$$

then $F(\varphi(t)) \geq G(t)$.

Proof . By way of contradiction, we assume that the conclusion is false. Hence, there exists $t_0 > 0$ such that $G(t_0) > F(\varphi(t_0))$. So by the hypothesis we have $F(\varphi(t_0)) \geq F(t_0)$. As F is nondecreasing and $\varphi(t_0) < t_0$, one then has that $F(t) = F(t_0)$ for all $\varphi(t_0) \leq t \leq t_0$. So in fact $G(t_0) > F(t_0)$. Let $m = \sup\{t > 0 : F(t) = F(t_0)\}$, by the hypothesis we have $m < \infty$. Choose $t_1 \in (\varphi(m), m]$ and $t_2 > m$ such that $\varphi(t_2) \leq t_1$, so we have, as F is nondecreasing and $t_1 \leq m$,

$$F(\varphi(t_2)) \leq F(t_1) = F(t_0) < F(t_2).$$

This implies $F(\varphi(t_2)) \geq G(t_2)$, (as $F(\varphi(t_2)) \geq \min\{F(t_2), G(t_2)\}$). Since $G(t_0) > F(t_0)$, we have

$$G(t_0) > F(t_0) \geq F(\varphi(t_2)) \geq G(t_2) \geq G(t_0),$$

a contradiction, the result follows. □

Theorem 2.7. *Let $(X, F, \tau, \leq)$ be a partially ordered complete PLM-space with a t-norm τ of H-type. Assume that $T : X \to X$ is a nondecreasing mapping and $\varphi \in \Phi$ such that $\varphi : (0, \infty) \to (0, \infty)$ is a strictly increasing mapping and*

$$F_{Tx,Ty}(\varphi(t)) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\}, \tag{2.8}$$

for all elements $x, y \in X$ are comparable and for all $t > 0$. Assume that either

(i) T is continuous or

(ii) if a nondecreasing sequence $x_n \to x$, then (x_n) and x are comparable for all n.

If there exists $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point.

Proof . Define a sequence $(x_n) \subseteq X$ by $x_{n+1} = Tx_n$, $n = 0, 1, \cdots$. Since $x_0 \leq Tx_0$ and T is nondecreasing mapping, we have

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots .$$

If there exists n_0 such that $x_{n_0+1} = x_{n_0}$, then $x_{n_0} = Tx_{n_0}$ and x_{n_0} is a fixed point of T. Then the result trivially holds. Suppose now that $x_{n+1} \neq x_n$ for all n. Following the assumption (2.8), we see that

$$\begin{aligned} F_{x_{n+1},x_n}(\varphi(t)) = F_{x_n,x_{n+1}}(\varphi(t)) =& F_{Tx_{n-1},Tx_n}(\varphi(t)) \\ \geq& \min\{F_{x_{n-1},x_n}(t), F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\} \\ =& \min\{F_{x_{n-1},x_n}(t), F_{x_n,x_{n+1}}(t)\}. \end{aligned}$$

By Lemma 2.6, we have

$$F_{x_{n+1},x_n}(\varphi(t)) = F_{x_n,x_{n+1}}(\varphi(t)) \geq F_{x_{n-1},x_n}(t) = F_{x_n,x_{n-1}}(t), \tag{2.9}$$

thus by Lemma 2.1 or Lemma 2.2, (x_n) is a Cauchy sequence in X. Since X is a complete, (x_n) has a limit u in X. By (2.9), for all $t > 0$, we get $F_{x_n,x_{n+1}}(t) \geq F_{x_0,x_1}(\varphi^{-n}(t))$, since $\lim_{t\to\infty} F_{x_0,x_1}(t) = 1$, we have $\lim_{n\to\infty} F_{x_n,x_{n+1}}(t) = 1$, for all $t > 0$.
If (i) holds then clearly $Tu = u$. Now suppose (ii) holds. We show that $F_{u,Tu}(t)$ is a constant function on $[t_0, \varphi^{-1}(t_0))$ for every $t_0 > 0$. Let $t \in (t_0, \varphi^{-1}(t_0))$, then

$$\begin{aligned} F_{u,Tu}(t_0) &\geq \tau\left(F_{u,Tx_n}((t_0 - \varphi(t)), F_{Tx_n,Tu}(\varphi(t))\right) \\ &\geq \tau\left(F_{u,Tx_n}((t_0 - \varphi(t)), \min\{F_{x_n,u}(t), F_{x_n,x_{n+1}}(t), F_{u,Tu}(t)\}\right). \end{aligned}$$

Now letting $n \to \infty$, we obtain

$$F_{u,Tu}(t_0) \geq \tau\left(1, \min\{1, 1, F_{u,Tu}(t)\}\right) = \tau\left(1, F_{u,Tu}(t)\right) = F_{u,Tu}(t) \geq F_{u,Tu}(t_0),$$

since $\varphi(t) < t_0$ (φ is strictly increasing mapping), so $F_{u,Tu}(t_0) = F_{u,Tu}(t)$ for all $t \in [t_0, \varphi^{-1}(t_0))$, hence $F_{u,Tu}(t)$ is a constant function on $[t_0, \varphi^{-1}(t_0))$ and so on $\mathbb{R}$. Since $F \in \Delta^+$, we get $F_{u,Tu}(t) = 1$, for all $t > 0$, then $u = Tu$. $\square$

Corollary 2.8. Let $(X, F, \tau, \leq)$ be a partially ordered complete PLM-space with a t-norm τ of H-type. Assume that $T : X \to X$ is a nondecreasing mapping and there is a $k \in (0, 1)$ with

$$F_{Tx,Ty}(kt) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\}, \tag{2.10}$$

for all elements $x, y \in X$ are comparable and for all $t > 0$. Assume that either

(i) T is continuous or

(ii) if a nondecreasing sequence $x_n \to x$, then (x_n) and x are comparable for all n.

If there exists $x_0 \in X$ with $x_0 \leq T(x_0)$, then T has a fixed point.

By the similar argument as we did in the proof of Theorem 2.5, we deduce the following corollary from Theorem 2.7.

Corollary 2.9. Let $(X, F, \tau, \leq)$ be a partially ordered complete PLM-space with a t-norm τ of H-type. Assume that $T : X \times X \to X$ be a mapping having the mixed monotone property on X and $\varphi \in \Phi$ such that $\varphi : (0, \infty) \to (0, \infty)$ is a strictly increasing mapping and

$$\begin{aligned}\tau\left(F_{T(x,y),T(u,v)}(\varphi(t)), F_{T(y,x),T(v,u)}(\varphi(t))\right) \geq \min\{&\tau\left(F_{x,u}(t), F_{y,v}(t)\right), \tau\left(F_{x,T(x,y)}(t), F_{y,T(y,x)}(t)\right), \\ &\tau\left(F_{u,T(u,v)}(t), F_{v,T(v,u)}(t)\right)\},\end{aligned}$$

for all elements $(x, y), (u, v) \in X \times X$ are comparable and for all $t > 0$. Assume that either

(i) T is continuous or

(ii) if a nondecreasing sequence $(x_n, y_n) \to (x, y)$, then (x_n, y_n) and (x, y) are comparable for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq T(x_0, y_0) \quad \textit{and} \quad y_0 \geq T(y_0, x_0),$$

then there exist $x^*, y^* \in X$ such that

$$x^* = T(x^*, y^*) \qquad \textit{and} \qquad y^* = T(y^*, x^*).$$

Corollary 2.10. Let $(X, F, \tau, \leq)$ be a partially ordered complete PLM-space with a t-norm τ of H-type. Assume that $T : X \times X \to X$ be a mapping having the mixed monotone property on X and there is a $k \in (0, 1)$ with

$$\begin{aligned}\tau\left(F_{T(x,y),T(u,v)}(kt), F_{T(y,x),T(v,u)}(kt)\right) \geq \min\{&\tau\left(F_{x,u}(t), F_{y,v}(t)\right), \tau\left(F_{x,T(x,y)}(t), F_{y,T(y,x)}(t)\right), \\ &\tau\left(F_{u,T(u,v)}(t), F_{v,T(v,u)}(t)\right)\},\end{aligned} \tag{2.11}$$

for all elements $(x, y), (u, v) \in X \times X$ are comparable and for all $t > 0$. Assume that either

(i) T is continuous or

(ii) if a nondecreasing sequence $(x_n, y_n) \to (x, y)$, then (x_n, y_n) and (x, y) are comparable for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq T(x_0, y_0) \quad \textit{and} \quad y_0 \geq T(y_0, x_0),$$

then there exist $x^*, y^* \in X$ such that

$$x^* = T(x^*, y^*) \qquad \textit{and} \qquad y^* = T(y^*, x^*).$$

From Corollary 2.10, we immediately get the following result.

Corollary 2.11. *Let $(X, F, \tau, \leq)$ be a partially ordered complete PLM-space with a t-norm τ of H-type. Assume that $T : X \times X \to X$ be a mapping having the mixed monotone property on X and $\varphi \in \Phi$ such that $\varphi : (0, \infty) \to (0, \infty)$ is a strictly increasing mapping and*

$$\tau\left(F_{T(x,y),T(u,v)}(\varphi(t)), F_{T(y,x),T(v,u)}(\varphi(t))\right) \geq \tau\left(F_{x,u}(t), F_{y,v}(t)\right),$$

for all elements $(x, y), (u, v) \in X \times X$ are comparable and for all $t > 0$. Assume that either

(i) T is continuous or

(ii) if a nondecreasing sequence $(x_n, y_n) \to (x, y)$, then (x_n, y_n) and (x, y) are comparable for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq T(x_0, y_0) \quad \textit{and} \quad y_0 \geq T(y_0, x_0),$$

then there exist $x^*, y^* \in X$ such that

$$x^* = T(x^*, y^*) \qquad \textit{and} \qquad y^* = T(y^*, x^*).$$

Proof . Since

$$\begin{aligned}\tau\left(F_{T(x,y),T(u,v)}(\varphi(t)), F_{T(y,x),T(v,u)}(\varphi(t))\right) &\geq \tau\left(F_{x,u}(t), F_{y,v}(t)\right)\\ &\geq \min\{\tau\left(F_{x,u}(t), F_{y,v}(t)\right), \tau\left(F_{x,T(x,y)}(t), F_{y,T(y,x)}(t)\right),\\ &\qquad \tau\left(F_{u,T(u,v)}(t), F_{v,T(v,u)}(t)\right)\},\end{aligned}$$

the result follows from Corollary 2.10. □

Corollary 2.12. Let $(X, F, \tau, \leq)$ be a partially ordered complete PLM-space with a t-norm τ of H-type. Assume that $T : X \times X \to X$ be a mapping having the mixed monotone property on X and there is a $k \in (0, 1)$ with

$$\tau\left(F_{T(x,y),T(u,v)}(kt), F_{T(y,x),T(v,u)}(kt)\right) \geq \tau\left(F_{x,u}(t), F_{y,v}(t)\right), \tag{2.12}$$

for all elements $(x, y), (u, v) \in X \times X$ are comparable and for all $t > 0$. Assume that either

(i) T is continuous or

(ii) if a nondecreasing sequence $(x_n, y_n) \to (x, y)$, then (x_n, y_n) and (x, y) are comparable for all n.

If there exist $x_0, y_0 \in X$ such that

$$x_0 \leq T(x_0, y_0) \quad and \quad y_0 \geq T(y_0, x_0),$$

then there exist $x^*, y^* \in X$ such that

$$x^* = T(x^*, y^*) \qquad and \qquad y^* = T(y^*, x^*).$$

Example 2.13. Let $X = [0,1]$ with the usual order and $F_{x,y}(t) = \epsilon_0(t - d(x,y))$, for all $x, y \in X$ and $t > 0$ where $d(x,y) = |x-y| + \min\{|\frac{1}{2} - x|, |\frac{3}{4} - x|\} + \min\{|\frac{1}{2} - y|, |\frac{3}{4} - y|\}$. It is easy to see that $F_{x,y}(t) = F_{y,x}(t)$ and if $F_{x,y}(t) = \epsilon_0(t)$, then $x = y$. Now we show that for all $x, y, z \in X$ and $t, s > 0$

$$F_{x,y}(t+s) \geq \tau_m(F_{x,z}(t), F_{z,y}(s)). \tag{2.13}$$

If $F_{x,y}(t+s) = 1$, then (2.13) holds. If $F_{x,y}(t+s) = 0$, then at least one of the $F_{x,z}(t), F_{z,y}(s)$ should be equal 0. Since if $F_{x,z}(t) = 1 = F_{z,y}(s)$, then by defnition of F we get $t > d(x,z)$ and $s > d(z,y)$. Also we have

$$t + s \leq d(x,y) \leq d(x,z) + d(z,y) < t + s,$$

which is a contradiction. Therefore (2.13) holds. Thus (X, F, τ_m) is a complete PLM-space. Let $T : X \times X \to X$ be defined as

$$T(x,y) = \begin{cases} \frac{1}{4}, & y = 1, \\ \frac{1}{2}, & otherwise. \end{cases}$$

Then T satisfies condition (2.11) but does not satisfy condition (2.12). Indeed, assume that there exists $0 < k < 1$ such that the condition (2.12) holds. If $x = v = 1$ and $y = u = \frac{3}{4}$, then

$$\begin{aligned} \epsilon_0(kt - \frac{1}{2}) = \min\{\epsilon_0(kt - \frac{1}{2}), \epsilon_0(kt - \frac{1}{2})\} &= \min\{F_{T(x,y),T(u,v)}(kt), F_{T(y,x),T(v,u)}(kt)\} \\ &\geq \min\{F_{x,u}(t), F_{y,v}(t)\} \\ &= \min\{\epsilon_0(t - \frac{1}{2}), \epsilon_0(t - \frac{1}{2})\} \\ &= \epsilon_0(t - \frac{1}{2}), \end{aligned}$$

i.e. $\epsilon_0(kt - \frac{1}{2}) \geq \epsilon_0(t - \frac{1}{2})$, thus $k \geq 1$, a contradiction. To verify that (2.11) holds with $k = \frac{2}{3}$, we need to consider several possible cases.
Case 1. Let $x = u = v = y = 1$. Then we have

$$d(T(x,y), T(u,v)) = \frac{1}{2} < \frac{10}{12} = \frac{2}{3} \times \frac{5}{4} = \frac{2}{3} d(x, T(x,y)),$$

hence (2.11) is true.
Case 2. Let $x = y = u = 1$and $v \neq 1$. Then we have

$$d(T(x,y), T(u,v)) = \frac{1}{2} < \frac{10}{12} = \frac{2}{3} \times \frac{5}{4} = \frac{2}{3} d(x, T(x,y)).$$

In the same way we can show that when one of the numbers x, y, u, v againsts 1, and the rest are equal to 1, then for $k = \frac{2}{3}$, (2.11) is true.
Case 3. Let $x = v = 1$ and $y, u \neq 1$. Then we have

$$d(T(x,y), T(u,v)) = \frac{1}{2} = \frac{2}{3} \times \frac{3}{4} = \frac{2}{3} d(x, T(x,y)).$$

In the same way we can show that when two of the numbers x, y, u, v against 1, and the rest are equal to 1, then for $k = \frac{2}{3}$, (2.11) is true.
Case 4. Let $x = 1$ and $v, u, y \neq 1$. Then we have

$$d(T(y,x), T(v,u)) = \frac{1}{2} < \frac{2}{3} \times \frac{3}{4} = \frac{2}{3} d(x, T(x,y)).$$

In the same way we can show that when one of the numbers x, y, u, v is equal to 1 and the rest are against 1, then for $k = \frac{2}{3}$, (2.11) is true. Therefore

$$\max\{d(T(x,y), T(u,v)), d(T(y,x), T(v,u))\} \leq \frac{2}{3} \max\{d(x,u), d(y,v), d(x, T(x,y)), d(y, T(y,x)), d(u, T(u,v)), d(v, T(v,u))\},$$

or in other words

$$\begin{aligned}\tau_m \left(F_{T(x,y),T(u,v)}(\frac{2}{3}t), F_{T(y,x),T(v,u)}(\frac{2}{3}t) \right) &\geq \min\{F_{x,u}(t), F_{y,v}(t), F_{x,T(x,y)}(t), F_{y,T(y,x)}(t), \\ &\qquad F_{u,T(u,v)}(t), F_{v,T(v,u)}(t)\} \\ &= \min\{\tau_m\left(F_{x,u}(t), F_{y,v}(t)\right), \tau_m\left(F_{x,T(x,y)}(t), F_{y,T(y,x)}(t)\right), \\ &\qquad \tau_m\left(F_{u,T(u,v)}(t), F_{v,T(v,u)}(t)\right)\}.\end{aligned}$$

Now we show that T is a continuous mapping, to do this, suppose that $((x_n, y_n))$ be a sequence in $X \times X$ such that $(x_n, y_n) \to (x, y)$, then $\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(y_n, y)$. It is easy to see that $y = \frac{1}{2}$ or $y = \frac{3}{4}$ and $\lim_{n\to\infty} |y_n - y| = 0$, so there exists $N_0 \in \mathbb{N}$ such that $0 \leq y_n \leq \frac{3}{4}$ for all $n \geq N_0$. For all $n \geq N_0$, we have $T(x_n, y_n) = T(x, y) = \frac{1}{2}$ and hence we get $\lim_{n\to\infty} d(T(x_n, y_n), T(x, y)) = d(\frac{1}{2}, \frac{1}{2}) = 0$, therefore T is continuous. Also we show that T has mixed monotone property. To see this, let $x, y, x_1, x_2, y_1, y_2 \in X$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$. If $y = 1$, then $T(x_1, y) = \frac{1}{4} = T(x_2, y)$. If $y \neq 1$, then $T(x_1, y) = \frac{1}{2} = T(x_2, y)$. Now if $y_1 = 1$, then $T(x, y_1) = \frac{1}{4} = T(x, y_2)$. If $y_1 \neq 1$, then $T(x, y_1) = \frac{1}{2} \geq T(x, y_2)$. Therefore T has mixed monotone property. Also, note that $0 \leq T(0, 1)$, $1 \geq T(1, 0)$. Hence, all the conditions of Corollary 2.10 hold and $(\frac{1}{2}, \frac{1}{2})$ is the required coupled fixed point of T.

Example 2.14. [26] Let $\tau(1, x) = 1 = \tau(x, 1)$ for all $x \in [0, 1]$, $\tau(x, y) = \tau_p(x, y) = x \cdot y$ for all $x, y \in [0, 1]$ with $\max\{x, y\} \in [0, \frac{1}{2}]$ and $\tau(x, y) = \tau_m(x, y) = \min\{x, y\}$ for all $x, y \in [0, 1]$ with $\max\{x, y\} \in (\frac{1}{2}, 1]$, then τ is a t-norm of H-type.

Example 2.15. Let $X = [-\frac{\pi}{2}, \frac{\pi}{2}]$ with the usual order and $F_{x,y}(t) = \frac{t}{t+|x|}$ for all $x, y \in X$ and $t > 0$. It is clear to see that if $F_{x,y}(t) = \epsilon_0(t) = F_{y,x}(t)$, then $x = y = 0$. Now we show that for all $x, y, z \in X$ and $t, s > 0$

$$F_{x,z}(t + s) \geq \tau(F_{x,y}(t), F_{y,z}(s)), \tag{2.14}$$

where τ is defined in Example 2.14. If $\max\{F_{x,y}(t), F_{y,z}(s)\} \in [0, \frac{1}{2}]$, then $\tau = \tau_p$ and we have

$$\frac{t}{t+|x|} \cdot \frac{s}{s+|y|} \leq \frac{t+s}{t+s+|x|} \cdot 1 = \frac{t+s}{t+s+|x|},$$

therefore (2.14) holds. If $\max\{F_{x,y}(t), F_{y,z}(s)\} \in (\frac{1}{2}, 1]$, then $\tau = \tau_m$. If $\min\{F_{x,y}(t), F_{y,z}(s)\} = F_{x,y}(t)$, then (2.14) holds and if $\min\{F_{x,y}(t), F_{y,z}(s)\} = F_{y,z}(s)$, then

$$\frac{s}{s+|y|} \leq \frac{t}{t+|x|} \leq \frac{t+s}{t+s+|x|},$$

therefore (2.14) holds. Thus (X, F, τ) is a bi-complete PL_qM-space. Let $\varphi : [0,\infty) \to [0,\infty)$ and $T : X \to X$ be two mappings defined by

$$\varphi(t) = \frac{1}{2}t, \qquad T(x) = \frac{1}{2}\sin x.$$

Hence

$$\begin{aligned} F_{Tx,Ty}(\varphi(t)) &= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{1}{2}\sin x|} = \frac{t}{t + |\sin x|} \\ &\geq \frac{t}{t+|x|} = F_{x,y}(t). \end{aligned}$$

Now we show that T is a continuous mapping, to do this, suppose that (x_n) is a sequence in X such that bi-converges to x, then $\lim_{n\to\infty} F_{x_n,x}(t) = \lim_{n\to\infty} \frac{t}{t+|x_n|} = 1$ and $\lim_{n\to\infty} F_{x,x_n}(t) = \frac{t}{t+|x|} = 1$, hence $|x_n| \to 0$ and $x = 0$. Also we have

$$F_{Tx_n,T0}(t) = \frac{t}{t + |\frac{\sin x_n}{2}|} \geq \frac{t}{t + |\frac{x_n}{2}|}.$$

Now taking limit as $n \to \infty$, then we get $\lim_{n\to\infty} F_{Tx_n,T0}(t) = 1$. Also clearly $F_{T0,Tx_n}(t) = 1$, therefore T is continuous. If $\frac{-\pi}{2} \leq x \leq y \leq \frac{\pi}{2}$, then $Tx = \frac{\sin x}{2} \leq \frac{\sin y}{2} = Ty$. Therefore T is nondecreasing mapping and $0 \leq T(0)$. Hence, all the conditions of Theorem 2.3 hold and 0 is the unique fixed point of T.

Example 2.16. Let $X = [0,\infty)$ with the usual order and $F_{x,y}(t) = \frac{t}{t+|x|}$ for all $x, y \in X$ and $t > 0$. Similar to Example 2.15 we can see that (X, F, τ_m) is a bi-complete PL_qM-space. Let $\varphi : [0,\infty) \to [0,\infty)$ and $T : X \to X$ be two mappings defined by

$$\varphi(t) = \frac{t}{2}, \qquad T(x) = \begin{cases} \frac{x^2}{2+x}, & x \in [0,1], \\ \frac{1}{3}x, & x \in (1,\infty). \end{cases}$$

If $x \in [0,1]$ and $x \leq y$, then we have

$$\begin{aligned} F_{Tx,Ty}(\varphi(t)) &= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{x^2}{2+x}|} \geq \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{x}{2}|} \\ &= \frac{t}{t+x} = F_{x,y}(t). \end{aligned}$$

Also if $x \in (1,\infty)$ and $x \leq y$, then

$$\begin{aligned} F_{Tx,Ty}(\varphi(t)) &= \frac{\frac{t}{2}}{\frac{t}{2} + |\frac{1}{3}x|} = \frac{t}{t + |\frac{2}{3}x|} \\ &\geq \frac{t}{t+|x|} = F_{x,y}(t). \end{aligned}$$

Now we show that T is a continuous mapping, to do this, suppose that (x_n) is a sequence in X such that bi-converges to x, then $\lim_{n\to\infty} F_{x_n,x}(t) = \lim_{n\to\infty} \frac{t}{t+|x_n|} = 1$ and $\lim_{n\to\infty} F_{x,x_n}(t) = \frac{t}{t+|x|} = 1$, hence $|x_n| \to 0$ and $x = 0$. So there exists $N_0 \in \mathbb{N}$ such that $0 \le x_n \le 1$ for all $n \ge N_0$. For all $n \ge N_0$, we have

$$F_{Tx_n,T0}(t) = \frac{t}{t+|\frac{x_n^2}{2+x_n}|} \ge \frac{t}{t+|\frac{x_n}{2}|}.$$

Now taking limit as $n \to \infty$, then we get $\lim_{n\to\infty} F_{Tx_n,T0}(t) = 1$. Also clearly $F_{T0,Tx_n}(t) = 1$, therefore T is continuous. Also we show that T is nondecreasing mapping. To see this, let $x, y \in X$, if $x, y \in [0,1]$ and $x \le y$, then $Tx = \frac{x^2}{2+x} \le \frac{y^2}{2+y} = Ty$ since if $\frac{x^2}{2+x} > \frac{y^2}{2+y}$, then with a simple calculation we conclude that $(x-y)(2x+2y+xy) > 0$ which is impossible since $x \le y$. So $Tx \le Ty$. If $x \in [0,1]$ and $y \in (1,\infty)$, then $Tx = \frac{x^2}{2+x} \le \frac{x}{3} \le \frac{y}{3} = Ty$. If $x, y \in (1,\infty)$ and $x \le y$, then $Tx = \frac{x}{3} \le \frac{y}{3} = Ty$. Therefore T is nondecreasing mapping and $0 \le T(0)$. Hence, all the conditions of Theorem 2.3 hold and 0 is the unique fixed point of T.

Definition 2.17. [49] A Menger PQM-space is a PL_qM-space (X, F, τ), such that τ is a continuous t-norm and for all $p, q, \in X$ and $t > 0$ if $p = q$, then $F_{p,q}(t) = \epsilon_0(t)$.

Every Menger PQM-space is a PL_qM-space, but the following example shows that the converse is not true, in general.

Example 2.18. Let $X = \mathbb{R}$ and $F_{x,y}(t) = \frac{t}{t+|x|}$ for all $x, y \in X$ and $t > 0$. Similar to Example 2.15 we can see that (X, F, τ_m) is a PL_qM-space, but is not a Menger PQM-space.

Remark 2.19. Since every Menger PQM-space is a PL_qM-space, if in Theorem 2.1 of [46], we take B and L as identity mappings, then Theorem 2.1 in [46] is a special case of Theorem 2.3, but clearly converse is not true.

Example 2.20. Let $X = [0,\infty)$ with the usual order and for all $x, y \in X$ and $t > 0$ define

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x|} & t \le |x|, \\ 1 & t > |x|. \end{cases}$$

Similar to Example 2.15 we can see that (X, F, τ_m) is a bi-complete PL_qM-space. Let $\varphi : [0,\infty) \to [0,\infty)$ and $T : X \to X$ be two mappings defined by

$$\varphi(t) = \frac{t}{t+1}, \qquad T(x) = \frac{x}{x+1}.$$

If $\varphi(t) > |Tx|$, then $F_{Tx,Ty}(\varphi(t)) = 1$ so (2.5) holds. If $\varphi(t) \le |Tx|$, then $t \le x$ and by defnition of F we get

$$\begin{aligned} F_{Tx,Ty}(\varphi(t)) = \frac{\varphi(t)}{\varphi(t)+|Tx|} &= \frac{t}{t+(1+t)|Tx|} \\ &= \frac{t}{t+(1+t)\frac{x}{x+1}} \\ &\ge \frac{t}{t+(1+x)\frac{x}{x+1}} \\ &= \frac{t}{t+|x|} = F_{x,y}(t). \end{aligned}$$

Thus, T satisfies (2.5). Now we show that T is a continuous mapping, to do this, suppose that (x_n) is a sequence in X such that bi-converges to x. Then $\lim_{n\to\infty} F_{x_n,x}(t) = \lim_{n\to\infty} \frac{t}{t+|x_n|} = 1$ and $\lim_{n\to\infty} F_{x,x_n}(t) = \frac{t}{t+|x|} = 1$, hence $|x_n| \to 0$ and $x = 0$. So there exists $N_0 \in \mathbb{N}$ such that $0 \le x_n \le 1$ for all $n \ge N_0$. For all $n \ge N_0$, we have

$$F_{Tx_n,T0}(t) = \frac{t}{t + |\frac{x_n}{x_n+1}|} \ge \frac{t}{t+|x_n|}.$$

Now taking limit as $n \to \infty$, then we get $\lim_{n\to\infty} F_{Tx_n,T0}(t) = 1$. Also clearly $F_{T0,Tx_n}(t) = 1$, therefore T is continuous. Also we show that T is nondecreasing mapping. To see this, let $x, y \in X$, and $x \le y$, then $Tx = \frac{x}{1+x} \le \frac{y}{1+y} = Ty$, since if $\frac{x}{1+x} > \frac{y}{1+y}$, then with a simple calculation we conclude that $x > y$ which is impossible, so $Tx \le Ty$. Clearly $0 \le T(0)$, hence, all the conditions of Theorem 2.3 hold and 0 is the fixed point of T.

Remark 2.21. If in Theorem 3.1 of [45], we take f and g as identity mapping and single-valued mapping respectively, then Theorem 3.1 of [45] is a special case of Theorem 2.3, but converse is not true. The example discussed above cannot be covered by Theorem 3.1 of [45], because $\sum_{n=1}^{\infty} \varphi^n(t) = \sum_{n=1}^{\infty} \frac{1}{1+n} = \infty$.

Example 2.22. Let $X = [0, 4]$ with the usual order and $F_{x,y}(t) = \frac{t}{t+d(x,y)}$, where $d(x, y) = |x - y| + |x| + |y|$, for all $x, y \in X$ and $t > 0$. It is clear that $F_{x,y}(t) = F_{y,x}(t)$ and if $F_{x,y}(t) = \epsilon_0(t)$, for every $x, y \in X$ and $t > 0$, then $x = y$. Now we show that for all $x, y, z \in X$ and $t, s > 0$

$$F_{x,z}(t + s) \ge \tau(F_{x,y}(t), F_{y,z}(s)), \tag{2.15}$$

where τ is defined in Example 2.14. If $\max\{F_{x,y}(t), F_{y,z}(s)\} \in [0, \frac{1}{2}]$, then $\tau = \tau_p$ and since

$$\begin{aligned} ts(t + s + d(x, z)) =& t^2 s + ts^2 + tsd(x, z) \\ \le& t^2 s + ts^2 + ts(d(x, y) + d(y, z)) \\ \le& (t + s)(t + d(x, y))(s + d(y, z)), \end{aligned}$$

then (2.15) holds. Now if $\max\{F_{x,y}(t), F_{y,z}(s)\} \in (\frac{1}{2}, 1]$, then $\tau = \tau_m$. If $\min\{F_{x,y}(t), F_{y,z}(s)\} = F_{y,z}(s)$, then

$$\frac{s}{s + d(z, y)} \le \frac{t}{t + d(x, y)},$$

so $sd(x, y) \le td(z, y)$. By using the triangle inequality and the recent relation we have

$$sd(x, z) \le sd(x, y) + sd(z, y) \le (t + s)d(z, y). \tag{2.16}$$

By adding $s^2 + st$ to (2.16), we have

$$s^2 + st + sd(x, z) \le s^2 + st + (t + s)d(z, y),$$

therefore

$$F_{y,z}(s) = \frac{s}{s + d(z, y)} \le \frac{t + s}{t + s + d(x, z)} = F_{x,z}(t + s),$$

therefore (2.15) holds. Thus (X, F, τ) is a complete PLM-space. Let $\varphi : [0, \infty) \to [0, \infty)$ and $T : X \to X$ be two mappings defined by

$$\varphi(t) = \frac{2}{3}t, \qquad T(x) = \begin{cases} \frac{x}{3} & x \in [0, 3], \\ 1 & x \in (3, 4]. \end{cases}$$

If $x, y \in [0, 3]$, then we have

$$\begin{aligned}
F_{Tx,Ty}(\varphi(t)) &= \frac{\frac{2}{3}t}{\frac{2}{3}t + |Tx - Ty| + |Tx| + |Ty|} \\
&= \frac{\frac{2}{3}t}{\frac{2}{3}t + |\frac{x}{3} - \frac{y}{3}| + |\frac{x}{3}| + |\frac{y}{3}|} \\
&= \frac{\frac{2}{3}t}{\frac{2}{3}t + \frac{1}{3}|x - Tx + Tx - Ty + Ty - y| + \frac{1}{3}|x| + \frac{1}{3}|y|} \\
&\geq \frac{\frac{2}{3}t}{\frac{2}{3}t + \frac{1}{3}|x - Tx| + \frac{1}{3}|Tx - Ty| + \frac{1}{3}|Ty - y| + \frac{1}{3}|x| + \frac{1}{3}|y|} \\
&\geq \frac{\frac{2}{3}t}{\frac{2}{3}t + \frac{1}{3}|x - Tx| + \frac{1}{3}|Tx| + \frac{1}{3}|Ty| + \frac{1}{3}|Ty - y| + \frac{1}{3}|x| + \frac{1}{3}|y|} \\
&= \frac{\frac{2}{3}t}{\frac{2}{3}t + \frac{1}{3}(|x - Tx| + |Tx| + |Ty| + |Ty - y| + |x| + |y|)} \\
&= \frac{2t}{2t + (|x - Tx| + |Tx| + |x| + |Ty - y| + |y| + |Ty|)} \\
&\geq \min\{\frac{t}{t + |x - Tx| + |Tx| + |x|}, \frac{t}{t + |Ty - y| + |y| + |Ty|}\} \\
&= \min\{F_{x,Tx}(t), F_{y,Ty}(t)\}.
\end{aligned}$$

If $x, y \in (3, 4]$, then we have

$$\begin{aligned}
F_{Tx,Ty}(\varphi(t)) \geq F_{Tx,Ty}(\frac{1}{2}t) &= \frac{\frac{1}{2}t}{\frac{1}{2}t + |Tx - Ty| + |Tx| + |Ty|} \\
&= \frac{\frac{1}{2}t}{\frac{1}{2}t + 2} \\
&\geq \frac{t}{t + 2\max\{x, y\}} = \frac{t}{t + |x - y| + |x| + |y|} \\
&= F_{x,y}(t).
\end{aligned}$$

Now, let $x \in [0, 3]$, $y \in (3, 4]$, then we have

$$\begin{aligned}
F_{Tx,Ty}(\varphi(t)) \geq F_{Tx,Ty}(\frac{1}{2}t) &= \frac{\frac{1}{2}t}{\frac{1}{2}t + |Tx - Ty| + |Tx| + |Ty|} \\
&= \frac{\frac{1}{2}t}{\frac{1}{2}t + |\frac{x}{3} - 1| + |\frac{x}{3}| + |1|} = \frac{\frac{1}{2}t}{\frac{1}{2}t + 2} \\
&\geq \frac{\frac{1}{2}t}{\frac{1}{2}t + y} = \frac{\frac{1}{2}t}{\frac{1}{2}t + \frac{1}{2}|y - \frac{y}{4}| + \frac{1}{2}|y| + \frac{1}{2}|\frac{y}{4}|} \\
&= \frac{t}{t + |y - \frac{y}{4}| + |y| + |\frac{y}{4}|} \\
&= F_{y,Ty}(t).
\end{aligned}$$

Similarly, if $y \in [0, 3]$, $x \in (3, 4]$, then we have

$$F_{Tx,Ty}(\varphi(t)) \geq F_{x,Tx}(t).$$

Thus, T satisfies (2.8). Now we show that T is a continuous mapping, to do this, suppose that (x_n) is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} d(x_n, x) = 0$ and hence $\lim_{n\to\infty} |x_n| = 0$ and $x = 0$. So there exists $N_0 \in \mathbb{N}$ such that $0 \le x_n \le 3$ for all $n \ge N_0$. For all $n \ge N_0$, we have

$$\begin{aligned} F_{Tx_n,Tx}(t) = \frac{t}{t + |Tx_n - Tx| + |Tx_n| + |Tx|} &= \frac{t}{t + |\frac{x_n}{3} - \frac{x}{3}| + |\frac{x_n}{3}| + |\frac{x}{3}|} \\ &= \frac{t}{t + \frac{1}{3}(|x_n - x| + |x_n| + |x|)} \\ &= \frac{t}{t + \frac{1}{3}d(x_n, x)} \to 1 \end{aligned}$$

so T is continuous. Now let $x, y \in [0, 3]$ and $x \le y$, then $Tx = \frac{x}{3} \le \frac{y}{3} = Ty$. If $x, y \in (3, 4]$ and $x \le y$, then $Tx = 1 = Ty$. Therefore T is nondecreasing mapping and $0 \le T(0)$. Hence, all the conditions of Theorem 2.7 hold and 0 is the fixed point of T.

Example 2.23. Let $X = \mathbb{R}$ and consider a relation $\preceq$ on X as follows:

$$x \preceq y \;\Leftrightarrow\; x = y \text{ or } (x, y \in [0, 1] \text{ with } x \le y).$$

It is easy to see that $\preceq$ is a partial order on X. Let $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for every $x, y \in X, t > 0$, where $d(x, y) = |x - y| + |x| + |y|$. Similar to Example 2.22 we can see that (X, F, τ_m) is a complete PLM-space. Now, define a self-map T on X as follows:

$$T(x) = \begin{cases} 0, & x < 0, \\ \frac{x}{4}, & 0 \le x \le 1, \\ \frac{1}{2}x - \frac{1}{4}, & 1 < x. \end{cases}$$

Now, we claim that the condition (2.10) of Corollary 2.8 is satisfied with $k = \frac{1}{2}$. Indeed, if $x, y \notin [0, 1]$, then $x \preceq y \Leftrightarrow x = y$. Therefore, if $x = y < 0$, since $F_{Tx,Ty}(kt) = 1$, then the condition (2.10) is satisfied. If $x = y > 1$, then for $k = \frac{1}{2}$ we have

$$\begin{aligned} F_{Tx,Ty}(kt) = \frac{\frac{1}{2}t}{\frac{1}{2}t + 2|\frac{1}{2}x - \frac{1}{4}|} &\ge \frac{\frac{1}{2}t}{\frac{1}{2}t + 2|\frac{1}{2}x|} \\ &= \frac{t}{t + 2|x|} = F_{x,y}(t). \end{aligned}$$

then the condition (2.10) is satisfied. Again, if $x \in [0, 1]$ and $y \notin [0, 1]$, then x and y are not comparative. Now, if $x, y \in [0, 1]$, then $x \preceq y \Leftrightarrow x \le y$ and

$$\begin{aligned} F_{Tx,Ty}(kt) &= \frac{\frac{1}{2}t}{\frac{1}{2}t + \frac{1}{4}|x - y| + \frac{1}{4}|x| + \frac{1}{4}|y|} \\ &\ge \frac{\frac{1}{2}t}{\frac{1}{2}t + \frac{1}{2}|x - y| + \frac{1}{2}|x| + \frac{1}{2}|y|} \\ &= F_{x,y}(t). \end{aligned}$$

Now we show that T is a continuous mapping, to do this, suppose that (x_n) is a sequence in X such that $x_n \to x$, then $\lim_{n\to\infty} d(x_n, x) = 0$. By definition d we conclude $x = 0$ and $|x_n| \to 0$, so there exists

$N_0 \in \mathbb{N}$ such that $|x_n| \leq 1$ for all $n \geq N_0$. For each $n \geq N_0$, if $x_n \leq 0$, then $F_{Tx_n,T0}(t) = 1$. If $0 < x_n \leq 1$, then

$$d(Tx_n, T0) = |\frac{x_n}{4} - \frac{0}{4}| + |\frac{x_n}{4}| + |\frac{0}{4}| = \frac{1}{2}|x_n| \to 0$$

so T is continuous. Let $x \preceq y$, if $x = y$, then $Tx = Ty$, if $x, y \in [0, 1]$ and $x \leq y$, then $Tx = \frac{x}{4} \leq \frac{y}{4} = Ty$. Therefore T is nondecreasing mapping and $0 \leq T(0)$. Hence, all the conditions of Corollary 2.8 are satisfied and so T has a fixed point 0 in X.

Example 2.24. Let $X = \{0, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots\}$ with the usual order and $F_{x,y}(t) = \frac{t}{t+d(x,y)}$ for every $x, y \in X, t > 0$, where $d(x, y) = |x-y|+|x|+|y|$. Similar to Example 2.22 we can see that (X, F, τ_m) is a complete PLM-space. Let $k = \frac{4}{5}$, $T : X^2 \to X$ be defined by $T(x, y) = \frac{x-2y}{5}$. Then we have

$$\begin{aligned}
\min\{F_{T(x,y),T(u,v)}(kt), F_{T(y,x),T(v,u)}(kt)\} &= \min\{\frac{kt}{kt + |\frac{x-2y}{5} - \frac{u-2v}{5}| + |\frac{x-2y}{5}| + |\frac{u-2v}{5}|}, \frac{kt}{kt + |\frac{y-2x}{5} - \frac{v-2u}{5}| + |\frac{y-2x}{5}| + |\frac{v-2u}{5}|}\} \\
&\geq \min\{\frac{kt}{kt + \frac{1}{5}|x-u| + \frac{1}{5}|x| + \frac{1}{5}|u| + \frac{2}{5}|y-v| + \frac{2}{5}|y| + \frac{2}{5}|v|}, \\
&\qquad \frac{kt}{kt + \frac{1}{5}|y-v| + \frac{1}{5}|y| + \frac{1}{5}|v| + \frac{2}{5}|x-u| + \frac{2}{5}|x| + \frac{2}{5}|u|}\} \\
&\geq \min\{\frac{kt}{kt + \frac{2}{5}|x-u| + \frac{2}{5}|x| + \frac{2}{5}|u| + \frac{2}{5}|y-v| + \frac{2}{5}|y| + \frac{2}{5}|v|}, \\
&\qquad \frac{kt}{kt + \frac{2}{5}|y-v| + \frac{2}{5}|y| + \frac{2}{5}|v| + \frac{2}{5}|x-u| + \frac{2}{5}|x| + \frac{2}{5}|u|}\} \\
&= \frac{kt}{kt + \frac{2}{5}|x-u| + \frac{2}{5}|x| + \frac{2}{5}|u| + \frac{2}{5}|y-v| + \frac{2}{5}|y| + \frac{2}{5}|v|}, \ (k = \frac{4}{5}) \\
&= \frac{t}{t + \frac{1}{2}|x-u| + \frac{1}{2}|x| + \frac{1}{2}|u| + \frac{1}{2}|y-v| + \frac{1}{2}|y| + \frac{1}{2}|v|} \\
&\geq \min\{\frac{t}{t + |x-u| + |x| + |u|}, \frac{t}{t + |y-v| + |y| + |v|}\} \\
&= \min\{F_{x,u}(t), F_{y,v}(t)\}.
\end{aligned}$$

Now we show that T is a continuous mapping, to do this, suppose that $((x_n, y_n))$ is a sequence in $X \times X$ such that $(x_n, y_n) \to (x, y)$. Then $\lim_{n\to\infty} d(x_n, x) = 0 = \lim_{n\to\infty} d(y_n, y)$, and we have

$$\begin{aligned}
d(T(x_n, y_n), T(x, y)) &= |\frac{x_n - 2y_n}{5} - \frac{x - 2y}{5}| + |\frac{x_n - 2y_n}{5}| + |\frac{x - 2y}{5}| \\
&\leq \frac{1}{5}|x_n - x| + \frac{2}{5}|y_n - y| + \frac{1}{5}|x_n| + \frac{2}{5}|y_n| + \frac{1}{5}|x| + \frac{2}{5}|y| \\
&= \frac{1}{5}(|x_n - x| + |x_n| + |x|) + \frac{2}{5}(|y_n - y| + |y_n| + |y|) \\
&= \frac{1}{5}d(x_n, x) + \frac{2}{5}d(y_n, y) \to 0,
\end{aligned}$$

so T is continuous. Also we show that T has mixed monotone property. To see this, let $x_1, x_2, y_1, y_2 \in X$ such that $x_1 \leq x_2$ and $y_1 \leq y_2$, then for any $x, y \in X$ we have

$$T(x_1, y) = \frac{x_1 - 2y}{5} \leq \frac{x_2 - 2y}{5} = T(x_2, y),$$

and

$$T(x, y_1) = \frac{x - 2y_1}{5} \geq \frac{x - 2y_2}{5} = T(x, y_2),$$

therefore T has mixed monotone property. Clearly, $0 \leq T(0, 0)$ and $0 \geq T(0, 0)$. Hence we conclude that all the conditions of Corollary 2.12 hold and $(0, 0)$ is a coupled fixed point of the mapping T.

3. Application to integral equation

In this section our aim is to give an existence theorem for a solution of the following integral equation

$$T(t) = \int_0^a k(t,s,u(s))ds + g(t),\ t \in [0,a], \tag{3.1}$$

where $a > 0$. Let $X = C([0,a])$ be the set of all continuous functions defined on $[0,a]$. Define $d : X \times X \to \mathbb{R}^+$ by

$$d(x,y) = \sup_{t\in[0,a]} (|x(t) - y(t)| + |x(t)| + |y(t)|).$$

Then, (X, d) is a complete quasi metric space. Define an ordered relation $\leq$ on X by

$$x \leq y \ \Leftrightarrow \ x(t) \leq y(t), \qquad \forall\, t \in [0,a].$$

Then, $(X, \leq)$ is a partially ordered set. Next, we define the mapping $F : X \times X \to \Delta^+$ by $F_{x,y}(t) = \frac{t}{t+d(x,y)}$, for all $x, y \in X$ and $t \geq 0$. Then the space (X, F, τ_m) is the complete PLM-space.
Now, we discuss the existence of solution for integral equation (3.1).

Theorem 3.1. *Let $(X, F, \tau_m, \leq)$ be the partially ordered complete PLM-space as defined above and suppose that the following hypotheses hold:*

(1) $k : [0,a] \times [0,a] \times \mathbb{R}^+ \to \mathbb{R}^+$ *and* $g : \mathbb{R} \to \mathbb{R}$ *are continuous functions such that*

$$|g(t)| \leq \int_0^a k(t,s,x(s))ds, \quad \forall x \in X, t \in \mathbb{R}.$$

(2) *There exists a continuous function* $G : [0,a] \times [0,a] \to [0,\infty]$ *such that*

$$|k(t,s,u)| + |k(t,s,v)| \leq \frac{1}{3}G(t,s)(|u-v| + |u| + |v|),$$

for each comparable $u, v \in \mathbb{R}$ *and each* $t, s \in [0,a]$.

(3) $\sup_{t\in[0,a]} \int_0^a G(t,s)ds \leq r$, *for some* $r < 1$.

Then, the integral equation (3.1) *has a solution* $u \in C([0,a])$.

Proof . Define $T : C([0,a]) \to C([0,a])$ by

$$Tx(t) = \int_0^a k(t,s,x(s))ds + g(t),\ t \in [0,a].$$

For $x, y \in C([0,a])$ are comparable, we have

$$\begin{aligned}
d(Tx,Ty) &= \sup_{t\in[0,a]} (|Tx(t) - Ty(t)| + |Tx(t)| + |Ty(t)|) = \sup_{t\in[0,a]} (\left|\int_0^a (k(t,s,x(s)) - k(t,s,y(s)))ds\right| \\
&\quad + \left|\int_0^a k(t,s,x(s))ds + g(t)\right| + \left|\int_0^a k(t,s,y(s))ds + g(t)\right|) \\
&\leq \sup_{t\in[0,a]} (\int_0^a |(k(t,s,x(s)) - k(t,s,y(s)))|\, ds + \int_0^a |k(t,s,x(s))|\, ds + |g(t)| \\
&\quad + \int_0^a |k(t,s,y(s))| ds + |g(t)|)
\end{aligned}$$

and so

$$\begin{aligned}
d(Tx,Ty) \leq & \sup_{t\in[0,a]} (\int_0^a (|(k(t,s,x(s))| + |k(t,s,y(s))|)ds + 2\int_0^a (|k(t,s,x(s))| + |k(t,s,y(s))|)ds) \\
&= \sup_{t\in[0,a]} (3\int_0^a (|(k(t,s,x(s))| + |k(t,s,y(s))|)ds) \\
&\leq \sup_{t\in[0,a]} (3\int_0^a \frac{1}{3}G(t,s)(|x(s)-y(s)| + |x(s)| + |y(s)|ds) \\
&\leq \sup_{t\in[0,a]} (|x(t)-y(t)| + |x(t)| + |y(t)|) \sup_{t\in[0,a]} \int_0^a G(t,s)ds \\
&= d(x,y) \sup_{t\in[0,a]} \int_0^a G(t,s)ds \\
&\leq r d(x,y).
\end{aligned}$$

Hence

$$F_{Tx,Ty}(rt) \geq F_{x,y}(t) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t)\},$$

for every $t > 0$ and $x, y \in C([0,a])$. Moreover, if (f_n) is a nondecreasing sequence in $C([0,a])$ such that $f_n \to f$ as $n \to \infty$, then f_n, f are comparable for all $n \in \mathbb{N}$. Thus, all the required hypotheses of Corollary 2.8 are satisfied. Thus, there exist a solution $u \in C([0,a])$ of the integral equation (3.1). $\square$

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