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Quadratic α-functional equations

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Abstract

In this paper, we solve the quadratic α-functional equations

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)), \tag{0.1}$$

where α is a fixed non-Archimedean number with $\alpha^{-2} \neq 3$. Using the fixed point method and the direct method, we prove the Hyers-Ulam stability of the functional equation (0.1) in non-Archimedean Banach spaces.

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1. Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam [32] concerning the stability of group homomorphisms.

The functional equation $f(x+y) = f(x) + f(y)$ is called the *Cauchy equation*. In particular, every solution of the Cauchy equation is said to be an *additive mapping*. Hyers [19] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers' Theorem was generalized by Aoki [2] for additive mappings and by Rassias [28] for linear mappings by considering an unbounded Cauchy difference. Gajda [15] following the same approach as in Rassias [28], gave an affirmative solution to this question for $p > 1$. It was shown by Gajda [15], as well as by Rassias and Šemrl [27] that one cannot prove a Rassias' type theorem when $p = 1$. The counterexamples of Gajda [15], as well as of Rassias and Šemrl [27] have stimulated several mathematicians to invent new definitions

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of *approximately additive* or *approximately linear* mappings, cf. Găvruta [16], who among others studied the Hyers-Ulam stability of functional equations (cf. the books of Czerwik [12, 13], Hyers, Isac and Th.M. Rassias [20]). The hyperstability of the Cauchy equation was proved by Brzdek [4].

The functional equation $f(x+y)+f(x-y)=2f(x)+2f(y)$ is called the quadratic functional equation. In particular, every solution of the quadratic functional equation is said to be a *quadratic mapping*. The stability of quadratic functional equation was proved by Skof [29] for mappings $f: E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. Cholewa [11] noticed that the theorem of Skof is still true if the relevant domain E_1 is replaced by an Abelian group. See [9, 10, 17, 18, 24, 25, 30, 31] for more functional equations. The survey on the Hyers-Ulam stability of functional equations was given by Brillouet-Bulluot, Brzdek and Cieplinski [3].

The functional equation $2f\left(\frac{x+y}{2}\right)+2\left(\frac{x-y}{2}\right) = f(x)+f(y)$ is called a *Jensen type quadratic equation.*

A *valuation* is a function $|\cdot|$ from a field K into $[0,\infty)$ such that 0 is the unique element having the 0 valuation, $|rs| = |r|\cdot|s|$ and the triangle inequality holds, i.e.,

$$|r+s| \le |r|+|s|, \qquad \forall r,s\in K.$$

A field K is called a *valued field* if K carries a valuation. The usual absolute values of $\mathbb{R}$ and $\mathbb{C}$ are examples of valuations.

Let us consider a valuation which satisfies a stronger condition than the triangle inequality. If the triangle inequality is replaced by

$$|r+s| \le \max\{|r|,|s|\}, \qquad \forall r,s\in K,$$

then the function $|\cdot|$ is called a *non-Archimedean valuation*, and the field is called a *non-Archimedean field*. Clearly $|1| = |-1| = 1$ and $|n| \le 1$ for all $n\in\mathbb{N}$. A trivial example of a non-Archimedean valuation is the function $|\cdot|$ taking everything except for 0 into 1 and $|0| = 0$.

Throughout this paper, we assume that the base field is a non-Archimedean field, hence call it simply a field.

Definition 1.1. ([23]) Let X be a vector space over a field K with a non-Archimedean valuation $|\cdot|$. A function $\|\cdot\| : X\to[0,\infty)$ is said to be a *non-Archimedean norm* if it satisfies the following conditions:
(i) $\|x\| = 0$ if and only if $x = 0$;
(ii) $\|rx\| = |r|\|x\| \qquad (r\in K, x\in X)$;
(iii) the strong triangle inequality

$$\|x+y\| \le \max\{\|x\|,\|y\|\}, \qquad \forall x,y\in X$$

holds. Then $(X,\|\cdot\|)$ is called a *non-Archimedean normed space.*

We recall a fundamental result in fixed point theory.

Theorem 1.2. ([6, 14]) Let (X,d) be a complete generalized metric space and let $J: X\to X$ be a strictly contractive mapping with Lipschitz constant $L<1$. Then for each given element $x\in X$, either

$$d(J^n x, J^{n+1}x) = \infty$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

In 1996, Isac and Rassias [21] were the first to provide applications of stability theory of functional equations for the proof of new fixed point theorems with applications. By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [1, 7, 8, 26]). Recently, Brzdek, Cădariu and Cieplinski [5] gave a survey on the fixed point method and the direct method to prove the Hyers-Ulam stability of functional equations and functional inequalities.

In Section 2, we solve the quadratic α-functional equation (0.1) in vector spaces and prove the Hyers-Ulam stability of the quadratic α-functional equation (0.1) in non-Archimedean Banach spaces by using the fixed point method.

In Section 3, we prove the Hyers-Ulam stability of the quadratic α-functional equation (0.1) in non-Archimedean Banach spaces by using the direct method.

Throughout this paper, assume that X is a non-Archimedean normed space and that Y is a non-Archimedean Banach space. Let $|2| \neq 1$ and let α be a fixed non-Archimedean number with $\alpha^{-2} \neq 3$.

2. Quadratic α-functional equation (0.1) in non-Archimedean Banach spaces: a fixed point method

We solve the quadratic α-functional equation (0.1) in vector spaces.

Lemma 2.1. *Let X and Y be vector spaces. If a mapping $f : X \to Y$ satisfies*

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)) \tag{2.1}$$

for all $x, y \in X$, then $f : X \to Y$ is quadratic.

Proof . Assume that $f : X \to Y$ satisfies (2.1).

Letting $x = y = 0$ in (2.1), we get $3f(0) = \alpha^{-2} f(0)$. So $f(0) = 0$.

Letting $y = 0$ in (2.1), we get $f(x) = \alpha^{-2} f(\alpha x)$ and so $f(\alpha x) = \alpha^2 f(x)$ for all $x \in X$. Thus

$$2f(x) + 2f(y) = f(x+y) + \alpha^{-2} f(\alpha(x-y)) = f(x+y) + f(x-y)$$

for all $x, y \in X$, as desired. □

Using the fixed point method, we prove the Hyers-Ulam stability of the quadratic α-functional equation (2.1) in non-Archimedean Banach spaces.

Theorem 2.2. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{|4|} \varphi(x, y) \tag{2.2}$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|2f(x) + 2f(y) - f(x+y) - \alpha^{-2} f(\alpha(x-y))\| \leq \varphi(x, y) \tag{2.3}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{L}{|4|(1-L)} \varphi(x, x) \tag{2.4}$$

for all $x \in X$.

Proof . Letting $y = x$ in (2.3), we get

$$\|f(2x) - 4f(x)\| \le \varphi(x,x) \tag{2.5}$$

for all $x \in X$.

Consider the set

$$S := \{h : X \to Y, \quad h(0) = 0\}$$

and introduce the generalized metric on S:

$$d(g,h) = \inf\left\{\mu \in \mathbb{R}_+ : \|g(x) - h(x)\| \le \mu\varphi\left(x,x\right), \quad \forall x \in X\right\},$$

where, as usual, $\inf \phi = +\infty$. It is easy to show that (S,d) is complete (see [22]).

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := 4g\left(\frac{x}{2}\right)$$

for all $x \in X$.

Let $g, h \in S$ be given such that $d(g,h) = \varepsilon$. Then

$$\|g(x) - h(x)\| \le \varepsilon\varphi\left(x,x\right)$$

for all $x \in X$. Hence

$$\begin{aligned} \|Jg(x) - Jh(x)\| &= \left\|4g\left(\frac{x}{2}\right) - 4h\left(\frac{x}{2}\right)\right\| \le |4|\varepsilon\varphi\left(\frac{x}{2}, \frac{x}{2}\right) \\ &\le |4|\varepsilon\frac{L}{|4|}\varphi\left(x,x\right) \le L\varepsilon\varphi\left(x,x\right) \end{aligned}$$

for all $x \in X$. So $d(g,h) = \varepsilon$ implies that $d(Jg, Jh) \le L\varepsilon$. This means that

$$d(Jg, Jh) \le Ld(g,h)$$

for all $g, h \in S$.

It follows from (2.5) that

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \le \varphi\left(\frac{x}{2}, \frac{x}{2}\right) \le \frac{L}{|4|}\varphi(x,x)$$

for all $x \in X$. So $d(f, Jf) \le \frac{L}{|4|}$.

By Theorem 1.2, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, i.e.,

$$Q\left(x\right) = 4Q\left(\frac{x}{2}\right) \tag{2.6}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set

$$M = \{g \in S : d(f,g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (2.6) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\|f(x) - Q(x)\| \le \mu\varphi\left(x,x\right)$$

for all $x \in X$;

(2) $d(J^l f, Q) \to 0$ as $l \to \infty$. This implies the equality

$$\lim_{l\to\infty} 4^n f\left(\frac{x}{2^n}\right) = Q(x)$$

for all $x \in X$;

(3) $d(f, Q) \leq \frac{1}{1-L} d(f, Jf)$, which implies

$$\|f(x) - Q(x)\| \leq \frac{L}{|4|(1-L)}\varphi(x,x)$$

for all $x \in X$.

It follows from (2.2) and (2.3) that

$$\begin{aligned}
&\|2Q(x) + 2Q(y) - Q(x+y) - \alpha^{-2}Q(\alpha(x-y))\| \\
&= \lim_{n\to\infty} |4|^n \left\|2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{y}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - \alpha^{-2} f\left(\alpha\frac{x-y}{2^n}\right)\right\| \\
&\leq \lim_{n\to\infty} |4|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0
\end{aligned}$$

for all $x, y \in X$. So

$$2Q(x) + 2Q(y) - Q(x+y) - \alpha^{-2}Q(\alpha(x-y)) = 0$$

for all $x, y \in X$. By Lemma 2.1, the mapping $Q : X \to Y$ is quadratic. $\square$

Corollary 2.3. Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and

$$\|2f(x) + 2f(y) - f(x+y) - \alpha^{-2} f(\alpha(x-y))\| \leq \theta(\|x\|^r + \|y\|^r) \tag{2.7}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{2\theta}{|2|^r - |4|}\|x\|^r$$

for all $x \in X$.

Proof . The proof follows from Theorem 2.2 by taking $\varphi(x, y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = |2|^{2-r}$ and we get the desired result. $\square$

Theorem 2.4. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq |4| L \varphi\left(\frac{x}{2}, \frac{y}{2}\right)$$

for all $x, y \in X$ Let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.3). Then there exists a unique quadratic mapping $Q : X \to Y$ such that

$$\|f(x) - Q(x)\| \leq \frac{1}{|4|(1-L)}\varphi(x,x)$$

for all $x \in X$.

Proof . It follows from (2.5) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \le \frac{1}{|4|} \varphi(x,x) \tag{2.8}$$

for all $x \in X$.

Let (S,d) be the generalized metric space defined in the proof of Theorem 2.2.

Now we consider the linear mapping $J : S \to S$ such that

$$Jg(x) := \frac{1}{4} g(2x)$$

for all $x \in X$.

It follows from (2.8) that $d(f, Jf) \le \frac{1}{|4|}$. So

$$\|f(x) - Q(x)\| \le \frac{1}{|4|(1-L)} \varphi(x,x)$$

for all $x \in X$.

The rest of the proof is similar to the proof of Theorem 2.2. □

Corollary 2.5. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and (2.7). Then there exists a unique quadratic mapping $Q : X \to Y$ such that*

$$\|f(x) - Q(x)\| \le \frac{2\theta}{|4| - |2|^r} \|x\|^r$$

for all $x \in X$.

Proof . The proof follows from Theorem 2.4 by taking $\varphi(x,y) = \theta(\|x\|^r + \|y\|^r)$ for all $x, y \in X$. Then we can choose $L = |2|^{r-2}$ and we get the desired result. □

3. Quadratic α-functional equation (0.1) in non-Archimedean Banach spaces: a direct method

In this section, using the direct method, we prove the Hyers-Ulam stability of the quadratic ρ-functional inequality (2.1) in non-Archimedean Banach spaces.

Theorem 3.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying $f(0) = 0$ and*

$$\lim_{j\to\infty} |4|^j \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) = 0, \tag{3.1}$$

$$\|2f(x) + 2f(y) - f(x+y) - \alpha^{-2} f(\alpha(x-y))\| \le \varphi(x,y) \tag{3.2}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ |4|^{j-1} \varphi\left(\frac{x}{2^j}, \frac{x}{2^j}\right) \right\} \tag{3.3}$$

for all $x \in X$.

Proof . Letting $y = x$ in (3.2), we get

$$\|f(2x) - 4f(x)\| \leq \varphi(x,x) \tag{3.4}$$

for all $x \in X$. So

$$\left\|f(x) - 4f\left(\frac{x}{2}\right)\right\| \leq \varphi\left(\frac{x}{2}, \frac{x}{2}\right)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\|4^l f\left(\frac{x}{2^l}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\| \\ &\qquad \leq \max\left\{\left\|4^l f\left(\frac{x}{2^l}\right) - 4^{l+1} f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, \left\|4^{m-1} f\left(\frac{x}{2^{m-1}}\right) - 4^m f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\qquad \leq \max\left\{|4|^l \left\|f\left(\frac{x}{2^l}\right) - 4f\left(\frac{x}{2^{l+1}}\right)\right\|, \cdots, |4|^{m-1}\left\|f\left(\frac{x}{2^{m-1}}\right) - 4f\left(\frac{x}{2^m}\right)\right\|\right\} \\ &\qquad \leq \sup_{j\in\{l,l+1,\cdots\}} \left\{|4|^j \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right)\right\} \end{aligned} \tag{3.5}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.5) that the sequence $\{4^n f(\frac{x}{2^n})\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{4^n f(\frac{x}{2^n})\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} 4^n f(\frac{x}{2^n})$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.5), we get (3.3).

It follows from (3.1) and (3.2) that

$$\begin{aligned} &\|2h(x) + 2h(y) - h(x+y) - \alpha^{-2}h(\alpha(x-y))\| \\ &= \lim_{n\to\infty} |4|^n \left\|2f\left(\frac{x}{2^n}\right) + 2f\left(\frac{y}{2^n}\right) - f\left(\frac{x+y}{2^n}\right) - \alpha^{-2} f\left(\alpha\frac{x-y}{2^n}\right)\right\| \\ &\leq \lim_{n\to\infty} |4|^n \varphi\left(\frac{x}{2^n}, \frac{y}{2^n}\right) = 0 \end{aligned}$$

for all $x, y \in X$. So

$$2h(x) + 2h(y) - h(x+y) - \alpha^{-2}h(\alpha(x-y)) = 0$$

for all $x, y \in X$. By Lemma 2.1, the mapping $h : X \to Y$ is quadratic.

Now, let $T : X \to Y$ be another quadratic mapping satisfying (3.3). Then we have

$$\begin{aligned} \|h(x) - T(x)\| &= \left\|4^q h\left(\frac{x}{2^q}\right) - 4^q T\left(\frac{x}{2^q}\right)\right\| \\ &\leq \max\left\{\left\|4^q h\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|, \left\|4^q T\left(\frac{x}{2^q}\right) - 4^q f\left(\frac{x}{2^q}\right)\right\|\right\} \\ &\leq \sup_{j\in\mathbb{N}} \left\{|4|^{q+j-1} \varphi\left(\frac{x}{2^{q+j}}, \frac{x}{2^{q+j}}\right)\right\}, \end{aligned}$$

which tends to zero as $q \to \infty$ for all $x \in X$. So we can conclude that $h(x) = T(x)$ for all $x \in X$. This proves the uniqueness of h. Thus the mapping $h : X \to Y$ is a unique quadratic mapping satisfying (3.3). $\square$

Corollary 3.2. *Let $r < 2$ and θ be nonnegative real numbers, and let $f : X \to Y$ be a mapping such that*

$$\|2f(x) + 2f(y) - f(x+y) - \alpha^{-2} f(\alpha(x-y))\| \le \theta(\|x\|^r + \|y\|^r) \tag{3.6}$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \frac{2\theta}{|2|^r} \|x\|^r$$

for all $x \in X$.

Theorem 3.3. *Let $\varphi : X^2 \to [0, \infty)$ be a function and let $f : X \to Y$ be a mapping satisfying* (3.2) *and*

$$\lim_{j\to\infty} \left\{ \frac{1}{|4|^j} \varphi(2^{j-1}x, 2^{j-1}y) \right\} = 0$$

for all $x, y \in X$. Then there exists a unique quadratic mapping $h : X \to Y$ such that

$$\|f(x) - h(x)\| \le \sup_{j\in\mathbb{N}} \left\{ \frac{1}{|4|^j} \varphi(2^{j-1}x, 2^{j-1}x) \right\} \tag{3.7}$$

for all $x \in X$.

Proof . It follows from (3.4) that

$$\left\| f(x) - \frac{1}{4} f(2x) \right\| \le \frac{1}{|4|} \varphi(x, x)$$

for all $x \in X$. Hence

$$\begin{aligned} &\left\| \frac{1}{4^l} f(2^l x) - \frac{1}{4^m} f(2^m x) \right\| \\ &\le \max\left\{ \left\| \frac{1}{4^l} f\left(2^l x\right) - \frac{1}{4^{l+1}} f\left(2^{l+1}x\right) \right\|, \cdots, \left\| \frac{1}{4^{m-1}} f\left(2^{m-1}x\right) - \frac{1}{4^m} f\left(2^m x\right) \right\| \right\} \\ &\le \max\left\{ \frac{1}{|4|^l} \left\| f\left(2^l x\right) - \frac{1}{4} f\left(2^{l+1}x\right) \right\|, \cdots, \frac{1}{|4|^{m-1}} \left\| f\left(2^{m-1}x\right) - \frac{1}{4} f\left(2^m x\right) \right\| \right\} \\ &\le \sup_{j\in\{l, l+1, \cdots\}} \left\{ \frac{1}{|4|^{j+1}} \varphi(2^j x, 2^j x) \right\} \end{aligned} \tag{3.8}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It follows from (3.8) that the sequence $\{\frac{1}{4^n} f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{\frac{1}{4^n} f(2^n x)\}$ converges. So one can define the mapping $h : X \to Y$ by

$$h(x) := \lim_{n\to\infty} \frac{1}{4^n} f(2^n x)$$

for all $x \in X$. Moreover, letting $l = 0$ and passing the limit $m \to \infty$ in (3.8), we get (3.7).

The rest of the proof is similar to the proof of Theorem 3.1. □

Corollary 3.4. *Let $r > 2$ and θ be positive real numbers, and let $f : X \to Y$ be a mapping satisfying* (3.6)*. Then there exists a unique quadratic mapping $h : X \to Y$ such that*

$$\|f(x) - h(x)\| \le \frac{2\theta}{|4|} \|x\|^r$$

for all $x \in X$.

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