Convergence Theorems of Implicit Iterates with Errors for Generalized Asymptotically Quasi-Nonexpansive Mappings in Banach Spaces

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Abstract

In this paper, we prove that an implicit iterative process with errors converges strongly to a common fixed point for a finite family of generalized asymptotically quasi-nonexpansive mappings on unbounded sets in a uniformly convex Banach space. Our results unify, improve and generalize the corresponding results of Ud-din and Khan [4], Sun [21], Wittman [23], Xu and Ori [26] and many others

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1. Introduction

In 1967, Browder [1] studied the iterative construction for fixed points of nonexpansive mappings on closed and convex subsets of a Hilbert space (see also [2]). The Ishikawa iteration process in the context of nonexpansive mappings on bounded closed convex subsets of a Banach space has been considered by a number of authors (see, e.g., Tan and Xu [22] and the references therein).

In 1973, Petryshyn and Williamson [17] established a necessary and sufficient condition for a Mann [15] iterative sequence to converge strongly to a fixed point of a quasi-nonexpansive mapping.

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Subsequently, Ghosh and Debnath [5] extended the results of [17] and obtained some necessary and sufficient condition for an Ishikawa-type iterative sequence to converge to a fixed point of a quasi-nonexpansive mapping. In 2001, Liu in [13, 14] extended the results of Ghosh and Debnath [5] to the more general asymptotically quasi-nonexpansive mappings. In 2003, Sahu and Jung [18] studied Ishikawa and Mann iteration process in Banach spaces and they proved some weak and strong convergence theorems for asymptotically quasi-nonexpansive type mapping. In 2006, Shahzad and Udomene [19] gave the necessary and sufficient condition for convergence of common fixed point of two-step modified Ishikawa iterative sequence for two asymptotically quasi-nonexpansive mappings in real Banach space.

Fixed point results for asymptotically nonexpansive mappings have been obtained by Kirk and Ray [11] on unbounded sets in a Banach space. Recently, Hussain and Khan [8] have constructed approximating sequences to fixed points of a class of mappings, containing nonexpansive mappings as a subclass, on closed convex unbounded subsets of a Hilbert space (see [16, 20] as well).

In 2003, Sun [21] has recently extended an implicit iteration process for a finite family of nonexpansive mapping due to Xu and Ori [26] to the case of asymptotically quasi-nonexpansive mappings.

Recently, Ud-din and Khan [4] studied an implicit iteration process with errors for a finite family of asymptotically quasi-nonexpansive mappings on unbounded sets in a uniformly convex Banach space and proved some weak and strong convergence theorems for said mappings.

The aim of this paper is to prove the strong convergence of an implicit iterative process with errors, in the sense of Xu [25], for a finite family of generalized asymptotically quasi-nonexpansive mappings on a closed convex unbounded set in the framework of real uniformly convex Banach space. Our results unify, improve and generalize the corresponding results of [4, 21, 23, 26] and many others.

2. Preliminaries and Lemmas

Let C be a nonempty subset of a normed space E and $T\colon C \to C$ be a given mapping. We assume that the set of fixed points of T, $F(T) = \{x \in C : T(x) = x\}$ is nonempty. The mapping T is said to be

(1) nonexpansive if

$$\|Tx - Ty\| \;\; \leq \;\; \|x - y\| \tag{2.1}$$

for all $x, y \in C$.

(2) quasi-nonexpansive [2] if

$$\|Tx - p\| \;\; \leq \;\; \|x - p\| \tag{2.2}$$

for all $x \in C$, $p \in F(T)$.

(3) asymptotically nonexpansive [6] if there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} u_n = 0$ such that

$$\|T^n x - T^n y\| \leq (1 + u_n)\|x - y\| \tag{2.3}$$

for all $x, y \in C$ and $n \geq 1$.

(4) asymptotically quasi-nonexpansive if $F(T) \neq \emptyset$ and there exists a sequence $\{u_n\}$ in $[0, \infty)$ with $\lim_{n \to \infty} u_n = 0$ such that

$$\|T^n x - p\| \leq (1 + u_n)\|x - p\| \tag{2.4}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(5) generalized asymptotically quasi-nonexpansive [9] if $F(T) \neq \emptyset$ and there exist two sequences of real numbers $\{u_n\}$ and $\{c_n\}$ with $\lim_{n \to \infty} u_n = 0 = \lim_{n \to \infty} c_n$ such that

$$\|T^n x - p\| \leq (1 + u_n)\|x - p\| + c_n, \tag{2.5}$$

for all $x \in C$, $p \in F(T)$ and $n \geq 1$.

(6) uniformly L-Lipschitzian if there exists a positive constant L such that

$$\|T^n x - T^n y\| \leq L\|x - y\| \tag{2.6}$$

for all $x, y \in C$ and $n \geq 1$.

If in definition (5), $c_n = 0$ for all $n \geq 1$, then T becomes asymptotically quasi-nonexpansive, and hence the class of generalized asymptotically quasi-nonexpansive maps includes the class of asymptotically quasi-nonexpansive maps.

The Mann and Ishikawa iteration processes have been used by a number of authors to approximate the fixed points of nonexpansive, asymptotically nonexpansive mappings, and quasi-nonexpansive mappings on Banach spaces (see, e.g., [5, 10, 12, 13, 14, 17, 20, 25]).

For a nonempty subset C of a normed space E and $T: C \to E$, Liu [12] introduced in 1995 the concept of Ishikawa iteration process with errors by the iterative sequence $\{x_n\}$ defined as follows:

$$\begin{aligned}
x_1 &= x \in C, \\
x_{n+1} &= (1 - \alpha_n)x_n + \alpha_n T^n y_n + u_n, \\
y_n &= (1 - \beta_n)x_n + \beta_n T^n x_n + v_n, \ n \geq 1,
\end{aligned} \tag{2.7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ are real sequences in $[0, 1]$ satisfying appropriate conditions and $\sum_{n=1}^{\infty} \|u_n\| < \infty$, $\sum_{n=1}^{\infty} \|v_n\| < \infty$. If $\beta_n = 0$, $v_n = 0$ for all $n \geq 1$, then this process becomes the Mann iteration process with errors.

The above definition of Liu depend on the convergence of the error terms u_n and v_n. The occurrence of errors is random and so the conditions imposed on the error terms are unreasonable.

Moreover, there is no assurance that the iterates defined by Liu will fall within the domain under consideration.

In 1998, Xu [25] gave the following new definitions in place of these noncompatible ones.

For a nonempty convex subset C of a normed space E and $T: C \to C$, the Ishikawa iteration process with errors is the iterative sequence $\{x_n\}$ defined by

$$
\begin{aligned}
x_1 &= x \in C, \\
x_{n+1} &= \alpha_n x_n + \beta_n T^n y_n + \gamma_n u_n, \\
y_n &= \alpha'_n x_n + \beta'_n T^n x_n + \gamma'_n v_n, \ n \geq 1,
\end{aligned}
\tag{2.8}
$$

where $\{u_n\}$, $\{v_n\}$ are bounded sequences in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$, $\{\alpha'_n\}$, $\{\beta'_n\}$, $\{\gamma'_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1 = \alpha'_n + \beta'_n + \gamma'_n$ for all $n \geq 1$.

It reduces to the Mann iteration process with errors when $\beta'_n = 0 = \gamma'_n$ for all $n \geq 1$.

Clearly, the normal Ishikawa and Mann iteration processes are special cases of the Ishikawa iteration process with errors.

Huang [7] has computed fixed points of asymptotically nonexpansive mappings while ud-din and Khan [3] have approximated common fixed points of two asymptotically nonexpansive mappings, using iteration process with errors in the sense of Liu [12].

Denote the indexing set $\{1, 2, \ldots, N\}$ by I. Let $\{T_i : i \in I\}$ be a finite family of generalized asymptotically quasi-nonexpansive self-mappings on a convex subset C of a normed space E. The implicit iterative process with an error term, in the sense of Xu [25], and with an initial value $x_0 \in C$, is defined as follows:

$$
\begin{aligned}
x_1 &= \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1, \\
x_2 &= \alpha_2 x_1 + \beta_2 T_2 x_2 + \gamma_2 u_2, \\
&\vdots \\
x_N &= \alpha_N x_{N-1} + \beta_N T_N x_N + \gamma_N u_N, \\
x_{N+1} &= \alpha_{N+1} x_N + \beta_{N+1} T_1^2 x_{N+1} + \gamma_{N+1} u_{N+1}, \\
&\vdots \\
x_{2N} &= \alpha_{2N} x_{2N-1} + \beta_{2N} T_N^2 x_{2N} + \gamma_{2N} u_{2N}, \\
x_{2N+1} &= \alpha_{2N+1} x_{2N} + \beta_{2N+1} T_1^3 x_{2N+1} + \gamma_{2N+1} u_{2N+1}, \\
&\vdots
\end{aligned}
\tag{2.9}
$$

where $\{u_n\}$ is a bounded sequence in C and $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ are sequences in $[0,1]$ such that $\alpha_n + \beta_n + \gamma_n = 1$.

The above sequence can be written in compact form as

$$x_n = \alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n \tag{2.10}$$

with $n \geq 1$, $n = (k-1)N + i$, $i \in I$ and $T_n = T_i (mod\ N) = T_i$.

Denote the indexing set $\{1, 2, \ldots, N\}$ by I. Let $\{T_i : i \in I\}$ be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive self-mappings of C. We show that (2.10) exists. Let $x_0 \in C$ and $x_1 = \alpha_1 x_0 + \beta_1 T_1 x_1 + \gamma_1 u_1$. Define $S: C \to C$ by: $Sx = \alpha_1 x_0 + \beta_1 T_1 x + \gamma_1 u_1$ for all $x \in C$. The existence of x_1 is guaranteed if W has a fixed point. For any $x, y \in C$, we have

$$\|Sx - Sy\| \leq \beta_1 \|T_1 x - T_1 y\| \leq \beta_1 L \|x - y\|. \tag{2.11}$$

Now, S is a contraction if $\beta_1 L < 1$ or $L < 1/\beta_1$. As $\beta_1 \in (0, 1)$, therefore W is a contraction even if $L > 1$. By the Banach contraction principle, S has a unique fixed point. Thus, the existence of x_1 is established. Similarly, we can establish the existence of $x_2, x_3, x_4, \ldots$. Thus, the implicit algorithm (2.10) is well defined.

The distance between a point x and a set C and closed ball with center zero and radius r in E are, respectively, defined by

$$d(x, C) = \inf_{y \in C} \|x - y\|, \quad B_r(0) = \{x \in E : \|x\| \leq r\}. \tag{2.12}$$

Definition 2.1. *([21]) Let C be a closed subset of a normed space E and let $T: C \to C$ be a mapping. Then T is said to be semi-compact if for any bounded sequence $\{x_n\}$ in C with $\|x_n - Tx_n\| \to 0$ as $n \to \infty$, there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to x^* \in C$ as $n_k \to \infty$.*

Lemma 2.2. *([22]) Let $\{a_n\}$ and $\{b_n\}$ be sequences of nonnegative real numbers satisfying the inequality*

$$a_{n+1} \leq a_n + b_n, \ n \geq 1. \tag{2.13}$$

If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists. In particular, if $\{a_n\}$ has a subsequence converging to zero, then $\lim_{n \to \infty} a_n = 0$.

Lemma 2.3. *([24]) Let $p > 1$ and $r > 0$ be two fixed real numbers and E a Banach space. Then E is uniformly convex if and only if there exists a continuous, strictly increasing and convex function $g: [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + (1 - \lambda)y\|^p \leq \lambda \|x\|^p + (1 - \lambda)\|y\|^p - W_p(\lambda)g(\|x - y\|) \tag{2.14}$$

for all $x, y \in B_r(0)$ where $0 \leq \lambda \leq 1$ and $W_p(\lambda) = \lambda(1 - \lambda)^p + \lambda^p(1 - \lambda)$.

3. Main Results

We begin with a necessary and sufficient condition for convergence of $\{x_n\}$ generated by the implicit iteration process with errors (2.10) to prove the following result.

Theorem 3.1. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_i : C \to C$ ($i \in I = \{1, 2, \ldots, N\}$) be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.10) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i : i \in I\}$ if and only if*

$$\liminf_{n \to \infty} d(x_n, F) = 0.$$

Proof . The necessity is obvious and so it is omitted. Now, we prove the sufficiency. For any $p \in F = \cap_{i=1}^{N} F(T_i)$, using (2.5) and (2.10), we have

$$
\begin{aligned}
\|x_n - p\| &= \|\alpha_n x_{n-1} + \beta_n T_i^k x_n + \gamma_n u_n - p\| \\
&= \|\alpha_n (x_{n-1} - p) + \beta_n (T_i^k x_n - p) + \gamma_n (u_n - p)\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n T_i^k x_n - p + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + \beta_n [(1 + u_{ik}) \|x_n - p\| \\
&\quad + c_{ik}] + \gamma_n \|u_n - p\| \\
&= \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n - \gamma_n)[(1 + u_{ik}) \|x_n - p\| \\
&\quad + c_{ik}] + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)[(1 + u_{ik}) \|x_n - p\| \\
&\quad + c_{ik}] + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n)(1 + u_{ik}) \|x_n - p\| \\
&\quad + (1 - \alpha_n) c_{ik} + \gamma_n \|u_n - p\| \\
&\leq \alpha_n \|x_{n-1} - p\| + (1 - \alpha_n + u_{ik}) \|x_n - p\| \\
&\quad + (1 - \alpha_n) c_{ik} + \gamma_n \|u_n - p\|. \tag{3.1}
\end{aligned}
$$

Since $\lim_{n \to \infty} \gamma_n = 0$, there exists a natural number n_1 such that for $n > n_1$, $\gamma_n \leq \frac{s}{2}$. Hence

$$\alpha_n = 1 - \beta_n - \gamma_n \geq 1 - (1 - s) - \frac{s}{2} = \frac{s}{2}, \qquad (*)$$

for $n > n_1$. Thus, we have from (3.1) that

$$
\begin{aligned}
\alpha_n \|x_n - p\| &\leq \alpha_n \|x_{n-1} - p\| + u_{ik} \|x_n - p\| \\
&\quad + (1 - \alpha_n) c_{ik} + \gamma_n \|u_n - p\|, \tag{3.2}
\end{aligned}
$$

and

$$
\begin{aligned}
\|x_n - p\| &\leq \|x_{n-1} - p\| + \frac{u_{ik}}{\alpha_n}\|x_n - p\| \\
&\quad + \left(\frac{1}{\alpha_n} - 1\right)c_{ik} + \frac{\gamma_n}{\alpha_n}\|u_n - p\| \\
&\leq \|x_{n-1} - p\| + \frac{2u_{ik}}{s}\|x_n - p\| \\
&\quad + \left(\frac{2}{s} - 1\right)c_{ik} + \frac{2\gamma_n}{s}\|u_n - p\|.
\end{aligned}
\tag{3.3}
$$

That is,

$$
\begin{aligned}
\|x_n - p\| &\leq \left(\frac{s}{s - 2u_{ik}}\right)\|x_{n-1} - p\| + \left(\frac{2}{s} - 1\right)\left(\frac{s}{s - 2u_{ik}}\right)c_{ik} \\
&\quad + \left(\frac{s}{s - 2u_{ik}}\right)\frac{2M}{s}\gamma_n \\
&= \left(1 + \frac{2u_{ik}}{s - 2u_{ik}}\right)\|x_{n-1} - p\| + \left(\frac{2}{s} - 1\right)\left(1 + \frac{2u_{ik}}{s - 2u_{ik}}\right)c_{ik} \\
&\quad + \left(1 + \frac{2u_{ik}}{s - 2u_{ik}}\right)\frac{2M}{s}\gamma_n
\end{aligned}
\tag{3.4}
$$

where $M = \sup_{n \geq 1}\{\|u_n - p\|\}$, since $\{u_n\}$ is a bounded sequence in C. Since $\sum_{k=1}^{\infty} u_{ik} < \infty$ for all $i \in I$. This gives that there exists a natural number n_0 (as $k > (n_0/N) + 1$) such that $s - 2u_{ik} > 0$ and $u_{ik} < s/4$ for all $n > n_0$. Let $v_{ik} = \frac{2u_{ik}}{s - 2u_{ik}}$ and $t_{ik} = \left(\frac{2}{s} - 1\right)\left(1 + \frac{2u_{ik}}{s - 2u_{ik}}\right)c_{ik}$. Since $\sum_{k=1}^{\infty} u_{ik} < \infty$ and $\sum_{k=1}^{\infty} c_{ik} < \infty$ for all $i \in I$, it follows that $\sum_{k=1}^{\infty} v_{ik} < \infty$ and $\sum_{k=1}^{\infty} t_{ik} < \infty$. Therefore from (3.4), we get

$$
\|x_n - p\| \leq (1 + v_{ik})\|x_{n-1} - p\| + t_{ik} + \frac{2M}{s}(1 + v_{ik})\gamma_n.
\tag{3.5}
$$

This further implies that

$$
d(x_n, F) \leq (1 + v_{ik})d(x_{n-1}, F) + t_{ik} + \frac{2M}{s}(1 + v_{ik})\gamma_n.
\tag{3.6}
$$

Since by assumptions, $\sum_{k=1}^{\infty} v_{ik} < \infty$, $\sum_{k=1}^{\infty} t_{ik} < \infty$ and $\sum_{n=1}^{\infty} \gamma_n < \infty$. Therefore, applying Lemma 2.2 to the inequalities (3.5) and (3.6), we conclude that both $\lim_{n\to\infty}\|x_n - p\|$ and $\lim_{n\to\infty} d(x_n, F)$ exists. Since by hypothesis $\liminf_{n\to\infty} d(x_n, F) = 0$, so by Lemma 2.2, we have

$$
\lim_{n\to\infty} d(x_n, F) = 0.
\tag{3.7}
$$

Next we prove that $\{x_n\}$ is a Cauchy sequence in C. Note that when $x > 0$, $1 + x \leq e^x$. It follows from (3.5) that for any $m \geq 1$, for all $n \geq n_0$ and for any $p \in F$, we have

$$\|x_{n+m} - p\| \leq exp\Big[\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{ik}\Big]\|x_n - p\| + \sum_{i=1}^{N}\sum_{k=1}^{\infty} t_{ik}$$

$$+\frac{2M}{s} exp\Big[\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{ik}\Big]\sum_{n=1}^{\infty} \gamma_n$$

$$< Q\|x_n - p\| + \sum_{i=1}^{N}\sum_{k=1}^{\infty} t_{ik} + \frac{2QM}{s}\sum_{n=1}^{\infty} \gamma_n, \tag{3.8}$$

where,

$$Q = exp\Big[\sum_{i=1}^{N}\sum_{k=1}^{\infty} v_{ik}\Big] + 1 < \infty.$$

Let $\varepsilon > 0$. As $\lim_{n\to\infty} \|x_n - p\|$ exists, therefore for $\varepsilon > 0$, there exists a natural number n_1 such that $\|x_n - p\| < \varepsilon/6(1+Q)$, $\sum_{i=1}^{N}\sum_{k=1}^{\infty} t_{ik} < \varepsilon/3$ and $\sum_{n=1}^{\infty} \gamma_n < s\varepsilon/6QM$ for all $n \geq n_1$. So we can find $p^* \in F$ such that $\|x_{n_1} - p^*\| < \varepsilon/3(1+Q)$. Hence, for all $n \geq n_1$ and $m \geq 1$, we have that

$$\|x_{n+m} - x_n\| \leq \|x_{n+m} - p^*\| + \|x_n - p^*\|$$

$$\leq Q\|x_{n_1} - p^*\| + \sum_{i=1}^{N}\sum_{k=n_1}^{\infty} t_{ik}$$

$$+\frac{2QM}{s}\sum_{n=n_1}^{\infty} \gamma_n + \|x_{n_1} - p^*\|$$

$$= (1+Q)\|x_{n_1} - p^*\| + \sum_{i=1}^{N}\sum_{k=n_1}^{\infty} t_{ik}$$

$$+\frac{2QM}{s}\sum_{n=n_1}^{\infty} \gamma_n$$

$$< (1+Q).\frac{\varepsilon}{3(1+Q)} + \frac{\varepsilon}{3}$$

$$+\frac{2QM}{s}.\frac{s\varepsilon}{6QM} = \varepsilon. \tag{3.9}$$

This proves that $\{x_n\}$ is a Cauchy sequence in C. Thus, the completeness of E implies that $\{x_n\}$ must be convergent. Assume that $\lim_{n\to\infty} x_n = p$. Now, we have to show that p is a common fixed point of the mappings $\{T_i : i \in I\}$. Indeed, we know that the set $F = \cap_{i=1}^{N} F(T_i)$ is closed. From the continuity of $d(x, F) = 0$ with $\lim_{n\to\infty} d(x_n, F) = 0$ and $\lim_{n\to\infty} x_n = p$, we get

$$d(p, F) = 0, \tag{3.10}$$

and so $p \in F$, that is, p is a common fixed point of the mappings $\{T_i\}_{i=1}^{N}$. This completes the proof.
$\square$

Theorem 3.2. *Let C be a nonempty closed convex subset of a Banach space E. Let $T_i : C \to C$ ($i \in I = \{1, 2, \dots, N\}$) be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.10) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point p of the mappings $\{T_i : i \in I\}$ if and only if there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges to p.*

Proof . The proof of Theorem 3.2 follows from Lemma 2.2 and Theorem 3.1. $\square$

We prove a lemma which plays an important role in establishing strong convergence of the implicit iteration process with errors in a uniformly convex Banach space.

Lemma 3.3. *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_i : C \to C$ ($i \in I = \{1, 2, \dots, N\}$) be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.10) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then $\lim_{n \to \infty} x_n - T_l x_n = 0$ for all $l \in I$.*

Proof . Set $\sigma_n = T_n^k x_n - x_{n-1}$, $n = (k-1)N + i$, $i \in I$. As in the proof of Theorem 3.1, $\lim_{n \to \infty} x_n - q$ exists for all $q \in F$, so $\{x_n - q, T_i^k x_n - q\}$ is a bounded set. Hence, we can obtain a closed ball $B_r(0) \supset \{x_n - q, T_i^k x_n - q\}$ for some $r > 0$. By Lemma 2.3, (2.5) and the scheme (2.10), we get

$$
\begin{aligned}
\|x_n - q\|^2 &= \|\alpha_n(x_{n-1} - q) + (1 - \alpha_n)(T_i^k x_n - q) + \gamma_n(u_n - T_i^k x_n)\|^2 \\
&\leq \|\alpha_n(x_{n-1} - q) + (1 - \alpha_n)(T_i^k x_n - q)\|^2 + \gamma_n K \quad for some \ K > 0 \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)\|T_i^k x_n - q\|^2 \\
&\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n K \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)[(1 + u_{ik})\|x_n - q\| + c_{ik}]^2 \qquad (3.11) \\
&\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n K \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)[(1 + u_{ik})^2\|x_n - q\|^2 + \rho_{ik}] \\
&\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n K \\
&= \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n)[(1 + w_{ik})\|x_n - q\|^2 + \rho_{ik}] \\
&\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n K, \\
&\leq \alpha_n \|x_{n-1} - q\|^2 + (1 - \alpha_n + w_{ik})\|x_n - q\|^2 + (1 - \alpha_n)\rho_{ik}] \\
&\quad - W_2(\alpha_n)g(\sigma_n) + \gamma_n K,
\end{aligned}
$$

where $\rho_{ik} = 2(1 + u_{ik})\|x_n - q\|c_{ik} + c_{ik}^2$ and $w_{ik} = 2u_{ik} + u_{ik}^2$. Since $\sum_{k=1}^{\infty} u_{ik} < \infty$ and $\sum_{k=1}^{\infty} c_{ik} < \infty$, it follows that $\sum_{k=1}^{\infty} \rho_{ik} < \infty$ and $\sum_{k=1}^{\infty} w_{ik} < \infty$ for all $i \in I$. Thus from the above inequality and ($*$), we have that

$$\|x_n - q\|^2 \leq \|x_{n-1} - q\|^2 + \frac{2w_{ik}}{s}\|x_n - q\|^2 + \left(\frac{2}{s} - 1\right)\rho_{ik}$$
$$- (1 - \alpha_n)g(\sigma_n) + \frac{2\gamma_n}{s}K. \tag{3.12}$$

Therefore, as in Theorem 3.1, it can be shown that $\lim_{n\to\infty}\|x_n - q\|^2 = d$ exists. From (3.12), it follows that

$$(1 - \alpha_n)g(\sigma_n) \leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \frac{2w_{ik}}{s}\|x_n - q\|^2$$
$$+ \left(\frac{2}{s} - 1\right)\rho_{ik} + \frac{2\gamma_n}{s}K. \tag{3.13}$$

From $(1 - \alpha_n) \geq (1 - s/2)$, we have

$$\left(\frac{2 - s}{2}\right)g(\sigma_n) \leq \|x_{n-1} - q\|^2 - \|x_n - q\|^2 + \frac{2w_{ik}}{s}\|x_n - q\|^2$$
$$+ \left(\frac{2}{s} - 1\right)\rho_{ik} + \frac{2\gamma_n}{s}K. \tag{3.14}$$

Let m be a positive integer such that $m \geq n$. Then

$$\sum_{n=1}^{m}g(\sigma_n) \leq \left(\frac{2}{2 - s}\right)[\|x_0 - q\|^2 - \|x_m - q\|^2] + \frac{4}{s(2 - s)}\|x_n - q\|^2\sum_{k=1}^{m}w_{ik}$$
$$+ \frac{4}{s}\sum_{k=1}^{m}\rho_{ik} + \frac{4K}{s(2 - s)}\sum_{n=1}^{m}\gamma_n$$
$$\leq \left(\frac{2}{2 - s}\right)\|x_0 - q\|^2 + \frac{4}{s(2 - s)}\|x_n - q\|^2\sum_{k=1}^{m}w_{ik}$$
$$+ \frac{4}{s}\sum_{k=1}^{m}\rho_{ik} + \frac{4K}{s(2 - s)}\sum_{n=1}^{m}\gamma_n$$
$$\leq \left(\frac{2}{2 - s}\right)\|x_0 - q\|^2 + Q'\sum_{k=1}^{m}w_{ik}$$
$$+ \frac{4}{s}\sum_{k=1}^{m}\rho_{ik} + \frac{4K}{s(2 - s)}\sum_{n=1}^{m}\gamma_n \qquad for some \ Q' > 0. \tag{3.15}$$

When $m \to \infty$ in (3.15), we have that $\lim_{n\to\infty}g(\sigma_n) = 0$. Since g is strictly increasing and continuous with $g(0) = 0$, it follows that $\lim_{n\to\infty}\sigma_n = 0$. Hence

$$\|x_n - x_{n-1}\| \leq \beta_n\|T_n^k x_n - x_{n-1}\| + \gamma_n u_n - x_{n-1}$$
$$\leq (1 - \alpha_n)\|T_n^k x_n - x_{n-1}\| + \gamma_n R', \ for some \ R' > 0$$
$$\leq (1 - s/2)\|T_n^k x_n - x_{n-1}\| + \gamma_n R', \tag{3.16}$$

which implies that $\lim_{n\to\infty} x_n - x_{n-1} = 0$. That is, $\lim_{n\to\infty} \|x_n - x_{n+l}\| = 0$ for all $l > 2N$. For $n > N$, we have

$$
\begin{aligned}
\|x_{n-1} - T_n x_n\| &\leq \|x_{n-1} - T_n^k x_n\| + \|T_n^k x_n - T_n x_n\| \\
&\leq \sigma_n + L\|T_n^{k-1} x_n - x_n\| \\
&\leq \sigma_n + L\Big[\|T_n^{k-1} x_n - T_{n-N}^{k-1} x_{n-N}\| \\
&\quad + \|T_{n-N}^{k-1} x_{n-N} - x_{(n-N)-1}\|\Big] \\
&\quad + L\|x_{(n-N)-1} - x_n\|.
\end{aligned}
\tag{3.17}
$$

By $n \equiv (n-N)(mod\ N)$, we get $T_n = T_{n-N}$. Now the above inequality becomes

$$
\begin{aligned}
\|x_{n-1} - T_n x_n\| &\leq \sigma_n + L^2\|x_n - x_{n-N}\| + L\sigma_{n-N} \\
&\quad + L\|x_{(n-N)-1} - x_n\|,
\end{aligned}
\tag{3.18}
$$

which yields that $\lim_{n\to\infty} \|x_{n-1} - T_n x_n\| = 0$. Since

$$
\|x_n - T_n x_n\| \leq \|x_n - x_{n-1}\| + \|x_{n-1} - T_n x_n\|,
\tag{3.19}
$$

so we have that

$$
\lim_{n\to\infty} \|x_n - T_n x_n\| = 0.
\tag{3.20}
$$

Hence, for all $l \in I$, we have

$$
\begin{aligned}
\|x_n - T_{n+l} x_n\| &\leq \|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_{n+l}\| \\
&\quad + \|T_{n+l} x_{n+l} - T_{n+l} x_n\| \\
&\leq (1+L)\|x_n - x_{n+l}\| + \|x_{n+l} - T_{n+l} x_n\|,
\end{aligned}
\tag{3.21}
$$

which implies that

$$
\lim_{n\to\infty} \|x_n - T_{n+l} x_n\| = 0 \quad \forall\, l \in I.
\tag{3.22}
$$

Thus

$$
\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \quad \forall\, l \in I.
\tag{3.23}
$$

$\square$

Now, we are in a position to prove our strong convergence theorems.

Theorem 3.4. *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_i \colon C \to C$ ($i \in I = \{1, 2, \ldots, N\}$) be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$ and $\sum_{n=1}^{\infty} c_{in} < \infty$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.10) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. If at least one member T in $\{T_i : i \in I\}$ is semi-compact, then the implicitly defined sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Proof . By Lemma 3.3, it follows that

$$\lim_{n\to\infty} \|x_n - T_l x_n\| = 0 \quad \forall \, l \in I. \tag{3.24}$$

Without any loss of generality, assume that T_1 is semi-compact. Therefore, by (3.24), it follows that $\lim_{n\to\infty} \|x_n - T_1 x_n\| = 0$. Since T_1 is semi-compact, therefore there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in C$. Now consider

$$\|x^* - T_l x^*\| = \lim_{n_j\to\infty} \|x_{n_j} - T_l x_{n_j}\| = 0 \quad \forall \, l \in I. \tag{3.25}$$

This proves that $x^* \in F$. As $\lim_{n\to\infty} \|x_n - q\|$ exists for all $q \in F$, therefore $\{x_n\}$ converges to $x^* \in F$, and hence the result. $\square$

Remark 3.5. *Theorem 3.4 extends and improves Theorem 3.3 due to Sun [21] to the case of more general class of asymptotically quasi-nonexpansive mapping and implicit iteration process with errors and without the boundedness of C which in turn generalizes Theorem 2 by Wittmann [23] from Hilbert spaces to uniformly convex Banach spaces.*

Definition 3.6. *(condition $(*)$)(see [4]) The family $\{T_i : i \in I\}$ of N-self mappings on a subset C of a normed space E satisfies condition $(*)$ if there exists a nondecreasing function $f \colon [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that $\frac{1}{N} \sum_{i=1}^{N} \|x - T_i x\| \geq f(d(x, F))$ for all $x \in C$ where $d(x, F) = \inf\{\|x - p\| : p \in F\}$.*

Note that condition $(*)$ defined above reduces to the condition (A) [22] if we choose $T_i = T$ (say) for all $i \in I$.

Finally, an application of the convergence criteria established in Theorem 3.1 is given below to obtain yet another strong convergence result in our setting.

Theorem 3.7. *Let C be a nonempty closed convex subset of a real uniformly convex Banach space E. Let $T_i \colon C \to C$ ($i \in I = \{1, 2, \ldots, N\}$) be N uniformly L-Lipschitzian generalized asymptotically quasi-nonexpansive mappings with $\{u_{in}\}, \{c_{in}\} \subset [0, \infty)$ such that $\sum_{n=1}^{\infty} u_{in} < \infty$, $\sum_{n=1}^{\infty} c_{in} < \infty$ and satisfy the condition $(*)$. Suppose that $F = \cap_{i=1}^{N} F(T_i) \neq \emptyset$. Let $\{x_n\}$ be the implicit iteration process with errors defined by (2.10) with the restrictions $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\{\beta_n\} \subset (s, 1-s)$ for some $s \in (0, \frac{1}{2})$. Then the iterative sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$.*

Proof . As in the proof of Theorem 3.4, (3.24) holds. Taking $\liminf$ on both sides of condition $(*)$ and using (3.24), we have that $\liminf_{n\to\infty} f(d(x_n, F)) = 0$. Since f is a nondecreasing function with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$, it follows that $\liminf_{n\to\infty} d(x_n, F) = 0$. Now by Theorem 3.1, $x_n \to p \in F$. Thus the sequence $\{x_n\}$ converges strongly to a common fixed point of the mappings $\{T_i : i \in I\}$. This completes the proof. $\square$

Remark 3.8. *(i) Our results extend the corresponding results of Ud-din and Khan [4] to the case of more general class of asymptotically quasi-nonexpansive mappings considered in this paper.*

(ii) Our results also generalize and improve the corresponding results of Sun [21], Wittmann [23] and Xu and Ori [26] to the case of more general class of nonexpansive, asymptotically quasi-nonexpansive mappings and implicit iteration process with errors considered in this paper.

Example 3.9. *Let E be the real line with the usual norm $|.|$ and $K = [0, 1]$. Define $T : K \to K$ by*

$$T(x) = \sin x, \quad x \in [0, 1],$$

for $x \in K$. Obviously $T(0) = 0$, that is, 0 is a fixed point of T, that is, $F(T) = \{0\}$. Now we check that T is generalized asymptotically quasi-nonexpansive mapping. In fact, if $x \in [0, 1]$, $p = 0 \in [0, 1]$ and $c_n = 1/n^2$ then

$$|T(x) - p| = |T(x) - 0| = |\sin x - 0| = |\sin x| \leq |x| = |x - 0| = |x - p|,$$

that is

$$|T(x) - p| \leq |x - p|.$$

Thus, T is quasi-nonexpansive. It follows that T is asymptotically quasi-nonexpansive with the constant sequence $\{k_n\} = \{1\}$ for each $n \geq 1$ and hence it is generalized asymptotically quasi-nonexpansive mapping with the constant sequence $\{k_n\} = \{1\}$ and sequence $\{c_n\} = \{1/n^2\}$ for each $n \geq 1$. But the converse does not hold in general.

References

[1] F. E. Browder, *Convergence of approximants to fixed points of nonexpansive non-linear mappings in Banach spaces*, Arch. Ration. Mech. Anal., 24 (1967) 82-90.
[2] J. B. Diaz and F. T. Metcalf, *On the structure of the set of subsequential limit points of successive approximations*, Bull. Amer. Math. Soc., 73 (1967) 51-519.
[3] H. Fukhar-ud-din and S. H. Khan, *Convergence of two-step iterative scheme with errors for two asymptotically nonexpansive mappings*, Int. J. Math. Math. Sci., 40-37 (2004) 1965-1971.
[4] H. Fukhar-ud-din and A. R. Khan, *Convergence of implicit iterates with errors for mappings with unbounded domain in Banach spaces*, Int. J. Math. Math. Sci., 10 (2005) 1643-1653.
[5] M. K. Ghosh and L. Debnath, *Convergence of Ishikawa iterates of quasi-nonexpansive mappings*, J. Math. Anal. Appl., 207 (1997) 96-103.
[6] K. Goebel and W. A. Kirk, *A fixed point theorem for asymptotically nonexpansive mappings*, Proc. Amer. Math. Soc., 35 (1972) 171-174.

[7] Z. Huang, *Mann and Ishikawa iterations with errors for asymptotically nonexpansive mappings*, Comput. Math. Appl., 37 (1999) 1-7.

[8] N. Hussain and A. R. Khan, *Applications of the best approximation operator to ∗-nonexpansive maps in Hilbert spaces*, Numer. Funct. Anal. Optim., 24 (2003) 327-338.

[9] S. Imnang and S. Suantai, *Common fixed points of multi-step Noor iterations with errors for a finite family of generalized asymptotically quasi-nonexpansive mappings*, Abstr. Appl. Anal. Vol. 2009, Article ID 728510, 14 pages.

[10] S. H. Khan and W. Takahashi, *Approximating common fixed points of two asymptotically nonexpansive mappings*, Sci. Math. Jpn., 53 (2001) 143-148.

[11] W. A. Kirk and W. O. Ray, *Fixed point theorems for mappings defined on unbounded sets in Banach spaces*, Studia Math., 64 (1979) 127-138.

[12] L. S. Liu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive mappings in Banach spaces*, J. Math. Anal. Appl., 194 (1995) 114-125.

[13] Q. H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl., 259 (2001) 1-7.

[14] Q. H. Liu, *Iterative sequences for asymptotically quasi-nonexpansive mappings with error member*, J. Math. Anal. Appl., 259 (2001) 18-24.

[15] W. R. Mann, *Mean value methods in iteration*, Proc. Amer. Math. Soc., 4 (1953) 506-510.

[16] G. Marino and G. Trombetta, *Best approximation and fixed point theorems for nonexpansive mappings in Hilbert spaces*, Atti. Sem. Mat. Fis. Univ. Modena, 40 (1992) 421-429.

[17] W. V. Petryshyn and T. E. Williamson, *Strong and weak convergence of the sequence of successive approximations for quasi-nonexpansive mappings*, J. Math. Anal. Appl., 43 (1973) 459-497.

[18] D. R. Sahu and J. S. Jung, *Fixed point iteration processes for non-Lipschitzian mappings of asymptotically quasi-nonexpansive type*, Int. J. Math. Math. Sci., 33 (2003) 2075-2081.

[19] N. Shahzad and A. Udomene, *Approximating common fixed points of two asymptotically quasi-nonexpansive mappings in Banach spaces*, Fixed Point Theory and Applications, Vol. 2006, Article ID 18909, Pages 1-10.

[20] T. Shimizu and W. Takahashi, *Strong convergence theorem for asymptotically nonexpansive mappings*, Nonlinear Anal., 26 (1996) 265-272.

[21] Z. Sun, *Strong convergence of an implicit iteration process for a finite family of asymptotically quasi-nonexpansive mappings*, J. Math. Anal. Appl., 286 (2003) 351-358.

[22] K. K. Tan and H. K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., 178 (1993) 301-308.

[23] R. Wittmann, *Approximation of fixed points of nonexpansive mappings*, Arch. Math., 58 (1992) 486-491.

[24] H. K. Xu, *Inequalities in Banach spaces with applications*, Nonlinear Anal., 16 (1991) 1127-1138.

[25] Y. Xu, *Ishikawa and Mann iterative process with errors for nonlinear strongly accretive operator equations*, J. Math. Anal. Appl., 224 (1998) 91-101.

[26] H. K. Xu and R. G. Ori, *An implicit iteration process for nonexpansive mappings*, Numer. Funct. Anal. Optim., 22 (2001) 767-773.