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properties of $M-$hyoellipticity for pseudo differential operators

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Abstract

In this paper we study properties of symbols such that these belong to class of symbols sitting inside $S^m_{\rho,\varphi}$ that we shall introduce as the following. So for because hypoelliptic pseudodifferential operators plays a key role in quantum mechanics we will investigate some properties of $M-$hypoelliptic pseudo differential operators for which define base on this class of symbols. Also we consider maximal and minimal operators of $M-$hypoelliptic pseudo differential operators and we express some results about these operators.

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1. Introduction

The theory of pseudodifferential operators was born in the early 1960 and, thereafter, it evolved with the theory of partial differential equations. Therefore, many topics in these two theories are closely related, like the hypoellipticity of operators, the sharp form of Gardings inequality, the parametrix of operators, and so on. In the theory of pseudodifferential operators, one of the most interesting topics is to investigate the behavior of pseudodifferential operators of Hormanders class, $S^m_{\rho,\delta}$ in $L^P(\mathbb{R}^n)$ and Sobolev spaces. The behavior of operators in $L^P(\mathbb{R}^n)$ spaces plays an essential role in the theory of linear and nonlinear partial differential equations. For example, one of the most important equations in mathematical physics is the Schrödinger equation, as it plays a key role in quantum mechanics. An important feature of this equation is the fact that it is instationary

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and therefore, one cannot use standard elliptic techniques for its resolution. One overcomes this problem by performing a standard regularization procedure which allows some degree of control over these singularities and, thus, enables (to some extend) the application of methods for hypoelliptic boundary value problems. Hypoelliptic theory has its roots in the work of Hörmander in [6], where a necessary and sufficient condition for a solution of a homogeneous boundary value problem to be C^∞ up to boundary of the domain was given. His condition, of an algebraic nature, was formulated in terms of behavior of the zeros of the so-called characteristic function of the boundary value problem near the infinity. So for because hypoelliptic pseudodifferential operators plays a key role in quantum mechanics we study in this paper some properties hypoelliptic theory for pseudodifferential operatots with the symbols type of $M-$hypoelliptic that introduce the following.

Let m, ρ and δ be real numbers; $0 \leq \delta, \rho \leq 1$. The class $S^m_{\delta,\rho}(X \times \mathbb{R}^n)$ consist of functions $\sigma(x,\xi) \in C^\infty(X \times \mathbb{R}^n)$ such that for any multi-indices α, β and any compact set $K \subset X$ constants $R, C_{\alpha,\beta,K}$ exist for which

$$|(\partial^\alpha_\xi \partial^\beta_x \sigma)(x,\xi)| \leq C_{\alpha,\beta,K} |\xi|^{m-\rho|\alpha|+\delta|\beta|}, \tag{1.1}$$

where $|\xi| \geq R$ and $x \in K$. We also take $S^{-\infty} = \cap_{m\in\mathbb{R}} S^m$. In order to a general class as such functions $\sigma \in C^\infty(X \times \mathbb{R}^n)$ suppose that $\varphi \in C^\infty_0(\mathbb{R}^n)$ is a positive function such that there exist positive constants μ_0, μ_1, c_0 and c_1 for which

$$c_0(1+|\xi|)^{\mu_0} \leq \varphi(\xi) \leq c_1(1+|\xi|)^{\mu_1} \quad \xi \in \mathbb{R}^n.$$

Such a function φ is said to be a weight with polynomial growth. Moreover,we assume that there exists a real number μ such that $\mu \geq \mu_1$ and for all multi-indices α and γ with $\gamma_j \in \{0,1\}, \quad j = 1,2,...,n$, there is a positive constant $C_{\alpha,\gamma}$ for which

$$|\xi^\gamma(\partial^{\alpha+\gamma})\varphi(\xi)| \leq C_{\alpha,\gamma}\varphi(\xi)^{1-\frac{1}{\mu}|\alpha|} \quad \xi \in \mathbb{R}^n.$$

Let $m \in \mathbb{R}$ and $\rho \in (0, \frac{1}{\mu}]$. Then we define $S^m_{\rho,\varphi}$ to be the set of all functions $\sigma \in C^\infty(X \times \mathbb{R}^n)$, where X is open set in $\mathbb{R}^n$, such that for all multi-indices α and β, there exists a positive constant $C_{\alpha,\beta}$ depending on α and β, only, for which

$$|\partial^\alpha_x \partial^\beta_\xi \sigma(x,\xi)| \leq C_{\alpha,\beta}\ \varphi(\xi)^{m-\rho|\beta|}$$

for $x \in X$ and $\xi \in \mathbb{R}^n$.

A function in $S^m_{\rho,\varphi}$ is said to be a symbol of order m and type ρ with weight φ. It should be noted that if we let φ be the weight defined by $\varphi(\xi) = \sqrt{1+|\xi|^2}$, for $\xi \in \mathbb{R}^n$, then $S^m_{\rho,\varphi}$ is the same as the Hörmander class $S^m_{\rho,0}$.

The L^p-boundedness for $p \neq 2$ is a completely different matter when $\rho < 1$. That L^p-boundedness is a much harder issue even for the Hörmander class is well documented by Fefferman. Due to this difficulty, the following class of symbols sitting inside $S^m_{\rho,\varphi}$ is introduced.
For $m \in \mathbb{R}$ and $\rho \in (0, \frac{1}{\mu}]$, we let $M^m_{\rho,\varphi}$ be the set of all functions $\sigma \in C^\infty(X \times \mathbb{R}^n)$, $X \subset \mathbb{R}^n$ is open set, such that for all multi-indices γ with $\gamma_j \in \{0,1\}, \quad j = 1,2,...,n$,

$$\xi^\gamma \partial^\gamma \sigma(x,\xi) \in S^m_{\rho,\varphi}.$$

In the case $\varphi(\xi) = \sqrt{1+|\xi|^2}$, for $\xi \in \mathbb{R}^n$, $M^m_{\rho,\varphi}$ can be found in the [10].

2. preliminaries

In this section we will express definitions and examples of these definitions and basic results for which we will need in the next section.

2.1. Definitions

Definition 2.1. *A function $\sigma \in C^{\infty}(X \times \mathbb{R}^n)$, for an open set $X \subset \mathbb{R}^n$ is called a $M-$elliptic if the following conditions are fulfilled:*

(1) $\sigma(x,\xi) \in M^m_{\rho,\varphi}$,

(2) there exists positive constants C and R for which

$$|\sigma(x,\xi)| \geq C\varphi(\xi)^m$$

for $|\xi| \geq R$.

Definition 2.2. *A function $\sigma \in C^{\infty}(X \times \mathbb{R}^n)$, is called a $M-$hypoelliptic symbol if the following conditions are fulfilled:*

(1) there exists real number t such that $\sigma(x,\xi) \in M^t_{\rho,\varphi}$,

(2) there exist real numbers m_0, m such that for an arbitrary compact set $K \subset X$ one can find positive constants c_1, c_2 and R for which

$$c_1\varphi(\xi)^m \leq |\sigma(x,\xi)| \leq c_2\varphi(\xi)^{m_0},$$

where $|\xi| \geq R$, $x \in K$,

(3) there exist number $0 \leq \delta < \rho \leq 1$ and for each compact set $K \subset X$ a constant R such that for any multi-indices α and β

$$|(\partial^{\alpha}_x \partial^{\beta}_{\xi} \sigma(x,\xi))\sigma^{-1}(x,\xi)| \leq C_{\alpha,\beta,K}[\varphi(\xi)]^{-\rho|\alpha|+\delta|\beta|}$$

for $|\xi| \geq R, x \in K$, with some constant $C_{\alpha,\beta,K}$.

We denote by $H_M S^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ the class of symbols satisfying in definition (2.2) for fixed m, m_0, ρ and δ with weight φ. If domain X is obvious then we will denote this space simply by $H_M S^{m,m_0}_{\rho,\delta}$. From definition (2.2) it obviously follows that

$$H_M S^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n) \subset S^m_{\rho,\varphi}(X \times \mathbb{R}^n).$$

We will denote by $H_M L^{m,m_0}_{\rho,\delta}(X)$ the class of properly supported pseudo differential operator A for which $\sigma_A(x,\xi) \in H_M S^{m,m_0}_{\rho,\delta}$.

Definition 2.3. *A pseudo differential operator A is called $M-$elliptic ($M-$hypoelliptic) if there exists a properly supported pseudo differential operator $A_1 \in H_M L^{m,m_0}_{\rho,\delta}(X)$ for which $\sigma_{A_1}(x,\xi)$ is $M-$elliptic ($M-$hypoelliptic) and $A = A_1 + R_1$, where $R_1 \in L^{-\infty}(X)$.*

Examples

Example 2.4. *The Laplace operator* $\Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ *in* $\mathbb{R}^n$ *and Cauchy-Riemann operator*

$$\frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x_1} + i[\frac{\partial}{\partial x_2}]$$

in $\mathbb{R}^2$ *are elliptic operators.*

Example 2.5. *We consider the partial differential operator* $\sigma(D)$ *on* $\mathbb{R}^2$ *in the paper [3] by Garello and Morando given by*

$$\sigma(D) = (\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2})(\frac{\partial}{\partial x_2} - \frac{\partial^2}{\partial x_1^2}).$$

The symbol σ *of the operator above is given by*

$$\sigma(\xi_1, \xi_2) = \xi_1^2 \xi_2^2 - \xi_1 \xi_2 + i(\xi_1^3 + \xi_2^3),$$

for $\xi_1, \xi_2 \in \mathbb{R}$. *It can be shown that symbol* σ *is* $M-$*elliptic of order* 1 *with respect to weight* φ *on* $\mathbb{R}^2$ *given by*

$$\varphi(\xi_1, \xi_2) = \sqrt{1 + \xi_1^6 + \xi_1^4 \xi_2^4 + \xi_2^6},$$

for $\xi_1, \xi_2 \in \mathbb{R}$.

Example 2.6. *The heat operator. The symbol of the heat operator*

$$\frac{\partial}{\partial x_1} - \frac{\partial^2}{\partial x_2^2},$$

on $\mathbb{R}^2$ *is given by*

$$\sigma(\xi_1, \xi_2) = i\xi_1 + \xi_2^2,$$

for $\xi_1, \xi_2 \in \mathbb{R}$. *By consider* $m_0 = -3, m = 2, \rho = \frac{1}{4}$ *and positive constants* c_1, c_2, R_1, R_2 *with appropriate weight*

$$\varphi(\xi_1, \xi_2) = 1 + \xi_1^2 + \xi_2^4,$$

then we have

$$c_1 \varphi(\xi_1, \xi_2)^{\frac{1}{2}} \leq |\sigma(\xi_1, \xi_2)| \leq c_2 \varphi(\xi_1, \xi_2)^2,$$

for $|\xi_1| \geq R_1, |\xi_2| \geq R_2$. *Therefore* σ *is an* $M-$*hypoelliptic in* $M^2_{\frac{1}{4},\varphi}$.

2.1.1. Basic results

Lemma 2.7. *[9] The following conditions are equivalent for a differential operator A: (1) A is elliptic (2)* $A \in HL^{m_0,m}_{1,0}(X)$.

Remark 2.8. *First of all note, that it makes sense to say that* $\sigma(x,\xi) \in S^m_{\rho,\delta}(U)$, *where* U *is an arbitrary region in* $\mathbb{R}^n \times R^N$, *which is conic with respect to* ξ. *Indeed, we will write that* $\sigma(x,\xi) \in S^m_{\rho,\delta}(U)$, *if for any compact* $K \subset (\mathbb{R}^n \times S^{N-1}) \cap U$, *where* S^{N-1} *is the unit sphere in* $\mathbb{R}^N$ *and for arbitrary multi-indices* α, β, *there is a constant* $C_{\alpha,\beta,K} > 0$ *such that*

$$|\partial_\xi^\alpha \partial_\xi^\beta \sigma(x,\xi)| \leq C_{\alpha,\beta,K} \langle \xi \rangle^{m-\rho|\alpha|+\delta|\beta|},$$

where $(x, \frac{\xi}{|\xi|}) \in K$ *and* $|\xi| \geq 1$. *Now assume, that we are given a diffeomorphism from a conical region* $V \subset \mathbb{R}^{n_1} \times \mathbb{R}^{N_1}$ *onto the conical region* $U \subset \mathbb{R}^n \times \mathbb{R}^N$, *commuting with the natural action of the multiplicative group* $\mathbb{R}_+$, *of positive numbers, i.e. a diffeomorphism which maps a* $(y, \eta) \in V$ *to point* $(x(y, \eta), \xi(y, \eta)) \in U$, *where* $x(y, \eta)$ *and* $\xi(y, \eta)$ *are positively homogenous in* η *of degree* 0 *and* 1 *respectively. Change the variables in* $\sigma(x, \xi)$:

$$\sigma_1(y, \eta) = \sigma(x(y, \eta), \xi(y, \eta)).$$

Lemma 2.9. *[9] Let* $\sigma(x, \xi) \in S^m_{\rho,\delta}(U)$ *and assume that one of the following three assumption hold: (1)* $\rho + \delta = 1$ *(2)* $\rho + \delta \geq 1$ *and* $x = x(y)$ *(3)* $x = x(y)$ *and* $\xi = \xi(y, \eta)$. *Then*

$$\sigma_1(y, \eta) = \sigma(x(y, \eta), \xi(y, \eta)) \in S^m_{\rho,\delta}(V),$$

where $V \subset \mathbb{R}^{n_1} \times \mathbb{R}^{N_1}$ *and* $U \subset \mathbb{R}^n \times \mathbb{R}^N$ *are conical regions.*

Theorem 2.10. *[11] Let* T *and* S *be two properly supported pseudo differential operators in a domain* $X \subset \mathbb{R}^n$ *and let their symbols be* σ_T, σ_S *respectively. The composition* $Z = S.T$ *is then a properly supported pseudo differential operator whose its symbol satisfies the relation,*

$$\sigma_{ST}(x, \xi) \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha_\xi \sigma_S(x, \xi) D^\alpha_x \sigma_T(x, \xi)$$

Theorem 2.11. *[11] Suppose that* $\sigma \in M^{m_1}_{\rho,\varphi}$ *and* $\tau \in M^{m_2}_{\rho,\varphi}$ *then* $\lambda \in M^{m_1+m_2}_{\rho,\varphi}$, *where*

$$\lambda \sim \sum_{\alpha} \frac{1}{\alpha!} \partial^\alpha_\xi \sigma \partial^\alpha_x \tau \quad .$$

3. Main results

properties of $M-$***hypoelliptic symbols***

If $\sigma(x, \xi)$ belong to $S^m_{\rho,\delta}$ or $H_M S^{m,m_0}_{\rho,\delta}$ for large ξ then multiplying by a smooth cut-off function $\psi(x, \xi)$ that for $|\xi| \geq R(K) + 2$ equal 1 and for $|\xi| \leq R(K) + 1$ equal zero, where K is arbitrary compact set. Then we obtain a symbol σ_1 belong to $S^m_{\rho,\delta}(X \times \mathbb{R}^n)$ or $H_M S^{m,m_0}_{\rho,\delta}(X \times \mathbb{R}^n)$ respectively, which coincides with $\sigma(x, \xi)$ for large ξ.

Lemma 3.1. *If* $\sigma(x, \xi) \in H_M S^{m,m_0}_{\rho,\delta}$ *for large* ξ, *then* $\sigma^{-1}(x, \xi) \in H_M S^{-m_0,-m}_{\rho,\delta}$ *for large* ξ.

Proof . We shall prove that $\sigma^{-1}(x, \xi)$ satisfies in conditions definition of $M-$hypoelliptic. First we show that $\sigma^{-1} \in M^{-m}_{\rho\varphi}$. Let γ withe $\gamma_j \in \{0.1\}$, $j = 1, 2, ..., n$, and α, β be arbitrary multi-inddices then we estimate

$$|\partial^\alpha_\xi \partial^\beta_x (\xi^\gamma \partial^\gamma_\xi \sigma^{-1}(x, \xi))|.$$

To this end, by Libnitz formula,

$$\partial^\alpha_\xi (\xi^\gamma \partial^\beta_x \partial^\gamma_\xi \sigma^{-1}(x, \xi)) = \sum_{\theta \leq \alpha} \binom{\alpha}{\theta} (\partial^{\alpha-\ \theta} \xi^\gamma)(\partial^\beta_x \partial^{\theta+\gamma}_\xi \sigma^{-1}(x, \xi)).$$

we set

$$I = \partial^{\alpha-\ \theta} \xi^\gamma \qquad \textit{and} \qquad II = \partial^\beta_x \partial^{\theta+\gamma}_\xi \sigma^{-1}(x, \xi).$$

Then we obtain

$$I = \partial^{\alpha-\ \theta}\xi^{\gamma} = (\alpha-\theta)!\binom{\gamma}{\alpha-\theta}|\xi|^{|\gamma|-|\alpha-\theta|}$$

if$\alpha-\theta\leq\gamma$, otherwise $I=0$. Also

$$II = \partial_x^{\beta}\partial_{\xi}^{\theta+\gamma}\sigma^{-1}(x,\xi) = \partial_x^{\beta}(\sum_{\theta+\gamma} C_{\theta_1\ldots\theta_n,\gamma_1\ldots\gamma_n}\frac{(\partial^{\theta+\gamma(1)}\sigma)\ldots(\partial^{\theta+\gamma(n)}\sigma)}{\sigma^{n+1}})$$

$$= \sum_{\theta+\gamma} C_{\theta_1\ldots\theta_n,\gamma_1\ldots\gamma_n}\partial_x^{\beta}(\frac{(\partial^{\theta+\gamma(1)}\sigma)\ldots(\partial^{\theta+\gamma(n)}\sigma)}{\sigma^{n+1}}).$$

Now we set $III = \partial_x^{\beta}(\frac{(\partial^{\theta+\gamma(1)}\sigma)\ldots(\partial^{\theta+\gamma(n)}\sigma)}{\sigma^{n+1}})$ and we obtain it as the following

$$III = \sum_{\eta\leq\beta}\binom{\beta}{\eta}[\partial^{\beta-\eta}(\frac{1}{\sigma^{n+1}})]\ \partial_x^{\beta}[(\partial^{\theta+\gamma(1)}\sigma)\ldots(\partial^{\theta+\gamma(n)}\sigma)].$$

We take $Iv = \partial^{\beta-\eta}(\frac{1}{\sigma^{n+1}})$ and $V = \partial_x^{\beta}[(\partial^{\theta+\gamma(1)}\sigma)\ldots(\partial^{\theta+\gamma(n)}\sigma)]$. Then,

$$IV = \sum C_{\beta1\ldots\beta n,\eta1\ldots\eta n}\frac{(\partial^{\beta-\eta(1)}\sigma^{n+1})\ldots(\partial^{\beta-\eta(n)}\sigma^{n+1})}{\sigma^{(n+1)^2}}\ .$$

Now, by above conclusions we obtain that

$$|\partial_{\xi}^{\alpha}\partial_x^{\beta}(\xi^{\gamma}\partial_{\xi}^{\gamma}\sigma^{-1}(x,\xi))| \leq \sum_{\theta\leq\alpha}\binom{\alpha}{\theta}|I|\ |II| \leq$$

$$\sum_{\theta\leq\alpha}\binom{\alpha}{\theta}(\alpha-\theta)!\binom{\gamma}{\alpha-\theta}\varphi^{|\gamma|-|\alpha-\theta|}\ .\ \sum_{\theta+\gamma}|C_{\theta_1\ldots\theta_n,\gamma_1\ldots\gamma_n}|\ .\ |III| \leq$$

$$\sum_{\theta\leq\alpha}(\alpha-\theta)!\binom{\gamma}{\alpha-\theta}\varphi^{|\gamma|-|\alpha-\theta|}\ .\ \sum_{\theta+\gamma}|C_{\theta_1\ldots\theta_n,\gamma_1\ldots\gamma_n}|\ .\ \sum_{\eta\leq\beta}\binom{\eta}{\beta}\ .\ |IV|\ .\ |V| \leq$$

$$\sum_{\theta\leq\alpha}(\alpha-\theta)!\binom{\gamma}{\alpha-\theta}\varphi^{|\gamma|-|\alpha-\theta|}\ .\ \sum_{\theta+\gamma}|C_{\theta_1\ldots\theta_n,\gamma_1\ldots\gamma_n}|\ .\ \sum_{\eta\leq\beta}\binom{\beta}{\eta}\ .$$

$$\sum_{\beta,\gamma}|C_{\beta_1\ldots\beta_n,\eta_1\ldots\eta_n}|\frac{|C_{\beta-\eta(1)}\ldots C_{\beta-\eta(n)}|}{|\sigma|^{(n+1)^2}}\varphi(\xi)^{n(n+1)m}|C_{\beta,\theta+\gamma(1)}\ldots C_{\beta,\theta+\gamma(n)}|\varphi(\xi)^{nm-\rho[|\theta+\gamma|]}.$$

Since $c_1\varphi(\xi)^{m_0}\leq|\sigma(x,\xi)|\leq c_2\varphi(\xi)^m$ there exists positive constant D_n such that

$$\frac{D_n}{c_2^{(n+1)^2}}\varphi(\xi)^{-m_0(n+1)^2} \leq \frac{D_n}{|\sigma|^{(n+1)^2}} \leq \frac{D_n}{c_1^{(n+1)^2}}\varphi(\xi)^{-m(n+1)^2}.$$

Then

$$|\partial_{\xi}^{\alpha}\partial_x^{\beta}[\xi^{\gamma}\partial_{\xi}^{\gamma}\sigma^{-1}(x,\xi)]| \leq$$

$$\sum_{\theta\leq\alpha}(\alpha-\theta)!\binom{\gamma}{\alpha-\theta}\varphi^{\rho[\ |\gamma|-|\alpha-\theta|\ ]}\ .\ \sum_{\theta+\gamma}|C_{\theta_1\ldots\theta_n,\gamma_1\ldots\gamma_n}|\ .\ \sum_{\eta\leq\beta}\binom{\beta}{\eta}\ \times$$

$$\sum_{\beta,\gamma}|C_{\beta_1...\beta_n,\eta_1...\eta_n}||C_{\beta-\eta(1)}...C_{\beta-\eta(n)}||C_{\beta,\theta+\gamma(1)}...C_{\beta,\theta+\gamma(n)}|\frac{D_n}{c_2^{(n+1)^2}}\times$$

$$\varphi(\xi)^{\rho\ [\ |\gamma|-|\alpha-\theta|]}\ .\ \varphi(\xi)^{n(n+1)m}\ .\ \varphi(\xi)^{-(n+1)^2m}\ .\ \varphi(\xi)^{nm-\rho|\alpha-\theta|}$$

$$\leq C_{\alpha,\beta,n}\varphi(\xi)^{-m-\rho|\alpha|}.$$

Therefore $\sigma^{-1}\in M^{-m}_{\rho,\varphi}$. Second condition of $M-$hypoellipitcity is clearly. Finally, let α and β be arbitrary multi-indices then

$$|\ [\partial_\xi^\alpha\partial_x^\beta\sigma^{-2}(x,\xi)]\sigma(x,\xi)|\leq|\ [\partial_\xi^\alpha(\sum_\beta C_{\beta_1...\beta_n}\frac{(\partial^{\beta_1}\sigma)...(\partial^{\beta_n}\sigma)}{\sigma^{n+1}})]\sigma(x,\xi)\ |\leq$$

$$\sum_\beta|C_{\beta_1...\beta_n}|\sum_{\theta\leq\alpha}\binom{\alpha}{\theta}\ [\ |\partial^{\alpha-\theta}\frac{1}{\sigma^{n+1}}](\partial_\xi^\alpha((\partial^{\beta_1}\sigma)...(\partial^{\beta_n}\sigma)))|\ |\sigma(x,\xi)|\leq$$

$$\sum_{\theta\leq\alpha}\binom{\alpha}{\theta}[\sum_{\alpha-\theta}C_{\alpha_1...\alpha_n,\theta_1...\theta_n}\frac{(\partial^{\alpha-\theta(1)}\sigma^{n+1})...(\partial^{\alpha-\theta(n)}\sigma^{n+1})}{\sigma^{(n+1)^2}}](\partial_\xi^\alpha\ [\ (\partial^{\beta_1}\sigma)...(\partial^{\beta_n}\sigma)\ ])\ |\sigma(x,\xi)|\leq$$

$$\sum_{\theta\leq\alpha}\sum_\beta|C_{\beta_1...\beta_n}|\sum_{\alpha,\theta}|C_{\alpha_1...\alpha_n,\theta_1...\theta_n}||C_{\alpha-\theta(1)}...C_{\alpha-\theta(n)}|C_{\theta,\beta_1}...C_{\theta,\beta_n}|\frac{D_n}{c_2^{(n+1)^2}}\times$$

$$\varphi(\xi)^{n(n+1)m-\rho|\alpha\theta|}\varphi(\xi)^{-(n+1)^2m}\varphi(\xi)^m\leq$$

$$C_{\alpha,\beta,n}\ \varphi(\xi)^{-\rho|\alpha|+\rho|\theta|}.$$

Therefore $\sigma^{-1}\in H_MS^{-m,-m_0}_{\rho,\delta}$. $\square$

Lemma 3.2. *If $\sigma(x,\xi)\in H_MS^{m,m_0}_{\rho,\delta}$ and $r(x,\xi)\in S^{m_1}_{\rho,\delta}$ where $m_1<m_0$ then $\sigma+r\in H_MS^{m,m_0}_{\rho,\delta}$.*

Proof . We must show that for any multi-indices γ with $\gamma_j\in\{0,1\}$, $j=1,...,n$,

$$\xi^\gamma\partial_\xi^\gamma(\sigma+r)\in S^m_{\rho,\varphi}.$$

To this, suppose that α and β are arbitrary multi-indices then by Libnitz formula,

$$|\ \partial_\xi^\alpha\partial_x^\beta[\xi^\gamma\partial_\xi^\gamma(\sigma+r)(x,\xi)]\ |\leq|\partial_\xi^\alpha\partial_x^\beta(\xi^\gamma\partial_\xi^\gamma\sigma(x,\xi))|+|\partial_\xi^\alpha\partial_x^\beta(\xi^\gamma r(x,\xi))|.$$

We set

$$I=\partial_\xi^\alpha\partial_x^\beta(\xi^\gamma\partial_\xi^\gamma\sigma(x,\xi))$$

then

$$|I|=|\sum_{\theta\leq\alpha}\binom{\alpha}{\theta}\ [\partial^{\alpha-\theta}\xi^\gamma]\ (\partial_x^\beta\partial_\xi^{\theta+\gamma}\sigma(x,\xi))|\leq$$

$$\sum_{\theta\leq\alpha}\binom{\alpha}{\theta}(\alpha-\theta)!\binom{\gamma}{\alpha-\theta}C'_{\theta,\beta,\gamma}\times\varphi(\xi)^{\rho\ [|\gamma|-|\alpha-\theta|]}\ .\ C''_{\beta,\theta,\gamma}\varphi(\xi)^{m-\rho|\theta+\gamma|}$$

$$\sum_{\theta\leq\alpha}\binom{\alpha}{\theta}(\alpha-\theta)!\binom{\gamma}{\alpha-\theta}C_{\theta,\beta,\gamma}\varphi(\xi)^{m-\rho|\alpha|}.$$

On the other hand we estimate

$$|II| = |\partial_\xi^\alpha \partial_x^\beta (\xi^\gamma r(x,\xi))| \leq \sum_{\eta\leq\alpha} \binom{\alpha}{\eta} (\alpha-\eta)! \binom{\gamma}{\alpha-\eta} C'_{\beta,\eta,\gamma} \varphi(\xi)^{\rho[|\gamma|-|\alpha-\eta|]} \varphi(\xi)^{m_1-\rho|\eta+\gamma|}.$$

Since $m_1 < m_0$ therefore

$$|II| \leq \sum_{\eta\leq\alpha} \binom{\alpha}{\eta} (\alpha-\eta)! \binom{\gamma}{\alpha-\eta} C'_{\beta,\eta,\gamma} \varphi(\xi)^{m_0-\rho|\alpha|}.$$

Therefore

$$|\ \partial_\xi^\alpha \partial_x^\beta [\xi^\gamma \partial_\xi^\gamma (\sigma+r)(x,\xi)]\ | \leq |I| + |II| \leq C^*_{\alpha,\beta,n} \varphi(\xi)^{m-\rho|\alpha|}$$

where

$$C^*_{\alpha,\beta,n} = \sum_{\theta\leq\alpha} \binom{\alpha}{\theta} (\alpha-\theta)! \binom{\gamma}{\alpha-\theta} C'_{\theta,\beta,\gamma} C''_{\beta,\theta,\gamma} C_{\theta,\beta,\gamma}$$

$$+ \sum_{\eta\leq\alpha} \binom{\alpha}{\eta} (\alpha-\eta)! \binom{\gamma}{\alpha-\eta} C'_{\beta,\eta,\gamma}.$$

Finally, for any multi-indices α and β we estimate

$$|\ [\partial_\xi^\alpha \partial_x^\beta (\sigma+r)(x,\xi)] (\sigma+r)^{-1}(x,\xi)\ | \leq' C_{\alpha,\beta} \varphi(\xi)^{m-\rho|\alpha|}\ |(\sigma+r)^{-1}(x,\xi)| \leq$$

$$C'_{\alpha,\beta} \varphi(\xi)^{m-\rho|\alpha|} C_{0,0} \varphi(\xi)^{-m} \leq C_{\alpha,\beta} \varphi(\xi)^{-\rho|\alpha}.$$

The proof is complete. □

Theorem 3.3. *Let $\sigma(x,\xi) \in H_M S^{m,m_0}_{\rho,\delta}$ for large ξ and let $\sigma_1(y,\eta) = \sigma(x(y,\eta),\xi(y,\eta))$, where the map $(y,\eta) \mapsto x(y,\eta),\xi(y,\eta)$ is a C^∞ map from $X_1 \times (\mathbb{R}^n - \{0\})$ in to $X \times (\mathbb{R}^n - \{0\})$ and where $\xi(y,\eta)$ is positive homogenous of degree 1 in η. Assume that $1-\rho \leq \delta \leq \rho$. Then*

$$\sigma_1(y,\eta) \in H_M S^{m,m_0}_{\rho,\delta},$$

for large η.

Proof . First, we prove that $\sigma_1(y,\eta) \in M^m_{\rho,\varphi}$. to this, suppose that α,β and γ with $\gamma_j \in \{0,1\}$, $j = 1,...,n$, are arbitrary multi-indices then by Libnitz formula we estimate

$$|\partial_\eta^\alpha \partial_y^\beta\ [\eta^\gamma \partial_\eta^\gamma \sigma_1(y,\eta)]\ | \leq \sum_{\theta\leq\alpha} \binom{\alpha}{\theta} |\ (\partial^{\alpha-\theta} \eta^\gamma)\ (\partial_y^\beta \partial_\eta^{\theta+\gamma} \sigma_1(y,\eta))\ | \leq$$

$$\sum_{\theta\leq\alpha} \binom{\alpha}{\theta} (\alpha-\theta)! \binom{\gamma}{\alpha-\theta}\ (1+|\eta|)^{\rho\ [|\gamma|-|\alpha-\theta|]} C'_{\beta,\theta,\gamma} \varphi(\eta)^{m-\rho|\theta+\gamma|} \leq$$

$$\sum_{\theta\leq\alpha} \binom{\alpha}{\theta} (\alpha-\theta)! \binom{\gamma}{\alpha-\theta}\ (1+|\eta|)^{\rho\ [|\gamma|-|\alpha-\theta|]} C'_{\beta,\theta,\gamma} \varphi(\eta)^{m-\rho\ [|\alpha|+|\gamma|]}.$$

Therefore $\sigma_1 \in M^m_{\rho,\varphi}$.

Now we prove that there exist real numbers m, m_0 such that for any compact set $K \subset X$ one can find positive constants R, c_1 and c_2 for which

$$c_1 \varphi(\eta)^{m_0} \leq |\sigma_1(y,\eta)| \leq c\varphi(\eta)^m.$$

By assumptions we have $\sigma_1(y,\eta)=\sigma(x(y,\eta),\xi(y,\eta))$, so by differentiating of $\sigma_1(y,\eta)$ we obtain

$$\frac{\partial\sigma_1}{\partial\eta_l}=\sum_j\frac{\partial\sigma}{\partial\xi_j}(x(y,\eta),\xi(y,\eta))\frac{\partial\xi_j}{\partial\eta_l}+\sum_k\frac{\partial\sigma}{\partial x_k}(x(y,\eta),\xi(y,\eta))\frac{\partial x_k}{\partial\eta_l}\qquad(3.1.3)$$

and

$$\frac{\partial\sigma_1}{\partial y_r}=\sum_j\frac{\partial\sigma}{\partial\xi_j}(x(y,\eta),\xi(y,\eta))\frac{\partial\xi_j}{\partial y_r}+\sum_k\frac{\partial\sigma}{\partial x_k}(x(y,\eta),\xi(y,\eta))\frac{\partial x_k}{\partial y_r}\qquad(3.1.4).$$

The functions $\frac{\partial\xi_j}{\partial\eta_l},\frac{\partial x_k}{\partial\eta_l},\frac{\partial\xi_j}{\partial y_r}$ and $\frac{\partial x_k}{\partial y_r}$ are positively homogenous in η of degree $0,-1,1,$ and 0 respectively. We obtain for $|\eta|\geq R$, where R is positive constant and $c_1,c_2>0$

$$c_1(\varphi(\eta)^{m_0-\rho}+\varphi(\eta)^{m_0+\delta-1})\leq|\frac{\partial\sigma_1}{\partial\eta_l}|\leq c_2(\varphi(\eta)^{m-\rho}+\varphi(\eta)^{m+\delta-1})$$

and there exist positive constants d_1,d_2 such that

$$d_1(\varphi(\eta)^{m_0-\rho+1}+\varphi(\eta)^{m_0+\delta})\leq|\frac{\partial\sigma_1}{\partial y_r}|\leq d_2(\varphi(\eta)^{m-\rho+1}+\varphi(\eta)^{m+\delta}),$$

for $|\eta|\geq R, y\in K$, where K is some compact set. If $m+\delta-1\leq m-\rho$ and $m_0+\delta-1\leq m_0-\rho$ then $\rho+\delta\leq1$, therefore from (3.1.3) we have that

$$c_1\varphi(\eta)^{m_0-\rho}\leq|\frac{\partial\sigma_1}{\partial\eta_l}|\leq c_2\varphi(\eta)^{m-\rho}\qquad(3.1.5).$$

IF $x=x(y)$ then $\frac{\partial x_k}{\partial\eta_l}=0$, so we obtain the estimate (3.1.5) without $\rho+\delta\leq1$. Similarly, if $m-\rho+1\leq m+\delta$ and $m_0-\rho+1\leq m_0+\delta$ then $\rho+\delta\geq1$ therefore it follows from (3.14) that]]

$$d_1\varphi(\eta)^{m_0+\delta}\leq|\frac{\partial\sigma_1}{\partial y_r}|\leq d_2\varphi(\eta)^{m+\delta}\qquad(3.1.6),$$

and the same estimate is obtained by assumption $1-\rho\leq\delta<\rho$. The necessary estimates of the from in definition (2.2) are thus verified when m,m_0 and positive constants c_1,c_2,d_1,d_2 and R are chosen nicely. Also when $|\alpha+\beta|\leq1$ for an arbitrary function $\sigma(x,\xi)\in H_MS^{m,m_0}_{\rho,\delta}$, the assertion of theorem is true. Now, inductively, suppose that the estimates hold for $|\alpha+\beta|\leq k$ and arbitrary $\sigma(x,\xi)\in H_MS^{m,m_0}_{\rho,\delta}$. We obtain that for the derivatives of order equal or less than k of

$$\frac{\partial\sigma}{\partial\xi_j}(x(y),\xi(y,\eta))\quad\text{and}\quad\frac{\partial\sigma}{\partial\xi_k}(x(y),\xi(y,\eta))$$

the estimates of class $H_MS^{m-\rho,m_0-\rho}$ and $H_MS^{m+\delta,m_0+\delta}$ respectively hold. From (3.1.3) and (3.1.4) by similarly reasoning we obtain that these estimates hold for arbitrary $\sigma\in H_MS^{m,m_0}_{\rho,\delta}$. Finally, for arbitrary multi-indices α β we obtain

$$|\ [\partial^\alpha_\eta\partial^\beta_y\sigma_1(y,\eta)]\sigma_1^{-1}(y,\eta)\ |\leq C'_{\alpha,\beta}\varphi(\eta)^{m-\rho\ [|\alpha+|\gamma|]}C_{0,0}\varphi(\eta)^{-m}$$
$$\leq C_{\alpha,\beta}\varphi(\eta)^{-\rho|\alpha|+\rho|\gamma|}.$$

□

Proposition 3.4. *If* $\sigma \in H_M S^{m,m_0}_{\rho,\delta}$ *and* $\tau \in H_M S^{m',m'_0}_{\rho,\delta}$ *then* $\sigma.\tau \in H_M S^{m+m',m_0+m'_0}_{\rho,\delta}$.

proof. First, we show that for any multi-indices α, β and γ with γ_j, $j = 1, 2, ..., n$

$$|\partial^\alpha_\xi \partial^\beta_x [\xi^\gamma \partial^\gamma_\xi (\sigma.\tau)(x,\xi)]| \leq \sum_{\theta \leq \alpha} \binom{\alpha}{\theta} |[\partial^{\alpha-\theta} \xi^\gamma] \; [\partial^\beta_x \partial^{\theta+\gamma}_\xi (\sigma.\tau)(x,\xi)]|$$

$$\sum_{\theta \leq \alpha} (\alpha - \theta)! \binom{\gamma}{\alpha - \theta} \; |\xi|^{\rho[\; |\gamma| - |\alpha - \theta|]} \; |\partial^\beta_x \partial^{\theta+\gamma}_\xi (\sigma.\tau)(x,\xi)|.$$

We set $I = \partial^\beta_x \partial^{\theta+\gamma}_\xi (\sigma.\tau)(x,\xi)$ then by Libnitz rule

$$|I| \leq \sum_{\eta \leq \theta+\gamma} \binom{\beta}{\eta} \; [\; |\partial^{\beta-\eta}_x \partial^{\theta+\gamma-\eta}_\xi \sigma(x,\xi)] \; | \;\; |[\partial^{\eta+\theta+\gamma}_\xi \tau(x,\xi)] \; |$$

$$\sum_{\lambda \leq \beta} \binom{\beta}{\lambda} \sum_{\eta \leq \theta+\gamma} \binom{\theta+\gamma}{\eta} C'_{\beta,\theta,\gamma,\eta,\lambda} \varphi(\xi)^{m-\rho[\; |\theta+\gamma-\eta| \;]} C''_{\beta,\theta,\gamma,\eta,\lambda} \varphi(\xi)^{m'-\rho[\; |\eta+\theta+\gamma| \;]}$$

$$\leq \sum_{\lambda \leq \beta} \sum_{\eta \leq \theta+\gamma} \binom{\beta}{\lambda} \binom{\theta+\gamma}{\eta} C'_{\beta,\theta,\gamma,\eta,\lambda} \; C''_{\beta,\theta,\gamma,\eta,\lambda} \varphi(\xi)^{m+m'-\rho \; [\; |\theta+\gamma| \;]}.$$

Therefore

$$|\partial^\alpha_\xi \partial^\beta_x [\xi^\gamma \partial^\gamma_\xi (\sigma.\tau)(x,\xi)]| \leq \sum_{\theta \leq \alpha} \binom{\alpha}{\theta} (\alpha-\theta)! \binom{\gamma}{\alpha-\theta} \varphi(\xi)^{\rho \; [|\gamma| - |\alpha-\theta| \;]} \; |I|$$

$$\leq C_{\alpha,\beta} \varphi(\xi)^{m+m'\rho|\alpha|}.$$

Hence $\sigma.\tau \in M^{m+m'}_{\rho,\varphi}$. By assumptions $\sigma.\tau$ satisfies in conditions of definition of $M-$hypoelliptic symbol therefore $\sigma.\tau \in H_M S^{m+m',m_0+m'_0}_{\rho,\delta}$.

properties of $M-$hypoelliptic operators

Lemma 3.5. *If* $T \in H_M L^{m,m_0}_{\rho,\delta}(X)$, *and* $R \in L^{m_1}_{\rho,\delta}(X)$ *with* $m_1 < m_0$ *where* R *is properly supported, then* $T + R \in H_M L^{m,m_0}_{\rho,\delta}(X)$.

Proof . The statement follows immediately from lemma (3.1). □

Remark 3.6. *Given a diffeomorphism* $\chi : X \to X_1$ *from one open set* $X \subset \mathbb{R}^n$ *onto another open set* $X_1 \subset \mathbb{R}^n$, *the induced transformation* $\chi^* : C^\infty(X_1) \to C^\infty(X)$, *taking a function* u *to the function* $uo\chi$, *is an isomorphism and transforms* $C^\infty_0(X_1)$ *into* $C^\infty_0(X)$. *Let* T *be a pseudo differential operator on* X *and define* $T_1 : C^\infty_0(X_1) \to C^\infty(X_1)$ *with the help of the commutative diagram:*

$$\begin{array}{ccc} C^\infty_0(X_1) & \xrightarrow{T_1} & C^\infty(X_1) \\ \chi^* \downarrow & & \downarrow \chi^* \\ C^\infty_0(X) & \xrightarrow{T} & C^\infty(X) \end{array} .$$

We obtain

$$T_1u(x) = \int \exp^{(\chi_1(x)-\chi_1(z)).\xi} a(\chi_1(x), \chi_1(z), \xi)|det\acute{\chi}_1(z)|u(z)dzd\xi,$$

where $\chi_1(z) = y$ *and* $\acute{\chi}_1$ *is the Jacobi matrix of the transformationa* χ_1.

Proposition 3.7. *If* $1-\rho \leq \delta < \rho$, *then* $H_ML^{m,m_0}_{\rho,\delta}(X)$ *is invariant with respect to change of variables i.e. if we are given a diffeomorphism* $\chi : X \to X_1$ *and an operator* T_1 *is defined as remark* (2.8) *then* $T_1 \in H_ML^{m,m_0}_{\rho,\delta}(X)$

Proof . By theorem (3.3) we have

$$\sigma_T(x^{-1}(y),^t \acute{\chi}_1(y)^{-1}\eta) \in H_MS^{m,m_0}_{\rho,\delta}(X).$$

But by theorem (4.2) in Subin [9] and lemma (3.1) we have

$$\sigma_{T_1}(y,\eta) = \sigma_T(x^{-1}(y),^t \acute{\chi}_1(y)^{-1}\eta)(1 + r(y,\eta)),$$

where $r(y,\eta) \in S^{-2(\rho-\frac{1}{2})}_{\rho,\delta}$ for large η. Now, by theorem (2.11) and lemma (3.1) the assertion is proved. □

The parametrix and the regularity theorem

Theorem 3.8. *Let* $T \in H_ML^{m,m_0}_{\rho,\delta}(X)$ *with either* $1-\rho \leq \delta < \rho$ *or* $\delta < \rho$ *and* X *a domain in* $\mathbb{R}^n$. *Then there exists an operator* $S \in H_ML^{-m,-m_0}_{\rho,\delta}(X)$, *such that*

$$ST = I + R_1 \quad and \quad TS = I + R_2 \quad (3.3.1)$$

where $R_j \in L^{-\infty}(X)$ *for* $j = 1, 2$ *and* I *is the identity operator. If,* $\acute{S}$ *is another pseudo differential operator for which*

$$T\acute{S} = I + \acute{R}_2 \quad and \quad \acute{S}T = I + \acute{R}_1$$

where $\acute{R}_j \in L^{-\infty}(X)$ *for* $j = 1, 2$, *then* $\acute{S} - S \in L^{-\infty}(X)$.

Proof . Consider a function $b_0(x,\xi) \in H_MS^{-m,-m_0}_{\rho,\delta}(X\times\mathbb{R}^n)$ for which $\sigma_T^{-1}(x,\xi) = b_0(x,\xi)$ for large ξ. Now, corresponding to this symbol we can chose a properly supported operator $S_0 \in H_ML^{-m,-m_0}_{\rho,\delta}(X)$ such that

$$\sigma_{S_0} - b_0 \in M^{-\infty}_{\rho,\varphi}(X \times \mathbb{R}^n).$$

Let us show that

$$S_0T = I + R_0 \quad (3.3.2) \quad ,$$

where $R_0 \in L^{-(\rho-\delta)}_{\rho,\delta}(X)$. By theorems (2.10) and (2.11) we have for large ξ

$$\sigma_{S_0T}(x,\xi) \sim 1 + \sum_{|\alpha|\geq 1} \frac{1}{\alpha!}\partial^{\alpha}_{\xi}\sigma_T^{-1} D^{\alpha}_x \sigma_T =$$

$$1 + \sum_{|\alpha|\geq 1} \frac{1}{\alpha!} \frac{\partial^{\alpha}_{\xi}\sigma_T^{-1}}{\sigma_T^{-1}} \frac{D^{\alpha}_x \sigma_T}{\sigma_T}.$$

Now, by lemma (3.1) we have

$$\frac{\partial^{\alpha}_{\xi}\sigma_T^{-1}}{\sigma_T^{-1}} \in S^{-\rho|\alpha|}_{\rho,\delta} \quad \text{and} \quad \frac{D^{\alpha}_x \sigma_T}{\sigma_T} \in S^{\delta|\alpha|}_{\rho,\delta}.$$

Then $\frac{\partial^{\alpha}_{\xi}\sigma_T^{-1}}{\sigma_T^{-1}} \frac{D^{\alpha}_x \sigma_T}{\sigma_T} \in S^{-|\alpha|(\rho-\delta)}_{\rho,\delta}$. Therefore $R_0 \in L^{-(\rho-\delta)}_{\rho,\delta}(X)$. Now, let C_0 be a properly supported pseudo differential operator for which

$$C_0 \sim \sum_{j=0}^{-\infty} (-1)^j R_0^j \qquad (3.3.3)$$

or

$$\sigma_{C_0} \sim \sum_{j=0}^{-\infty} (-1)^j \sigma_{R_0^j} \qquad (3.3.4).$$

From (3.3.3) we have

$$C_0(I + R_0) - I \in L^{-\infty},$$

so that putting $S_1 = C_0 S_0$ therefore $S_1 T = I + R_1$ (3.3.5)where $R_1 \in L^{-\infty}(X)$. From construction it implies that $S_1 \in H_M L^{-m,-m_0}_{\rho,\delta}(X)$. We can similarly construct an operator $S_2 \in H_M L^{-m,-m_0}_{\rho,\delta}(X)$ for which $S_2 T = I + R_2$ (3.3.6) where $R_2 \in L^{-\infty}(X)$. Let us now verify that if S_1 and $S_2$are two arbitrary pseudo differential operator such that (3.3.5) and (3.3.6) hold then $S_1 - S_2 \in L^{-\infty}(X)$. This will demonstrate the existence of the required S and its uniqueness. Note that S_1 and S_2 can be taken to be properly supported. From (3.35) we have

$$S_1 T = I + R_1 \Rightarrow S_1 T S_2 = S_2 R_1 S_2 = S_1(I + R_2) = S_2(I + R_1)$$

therefore $S_1 - S_2 = R_1 S_2 - S_1 R_2$. Since $S_1 R_2$ and $R_1 S_2$ both belong to $L^{-\infty}(X)$ therefore $S_1 - S_2 \in L^{-\infty}(X)$. □

Corollary 3.9. *If T is a $M-$hypoelliptic pseudo differential operator on X then there exists a properly supported pseudo differential operator ,S, for which* (3.3.1) *holds.*

Corollary 3.10. *Any $M-$hypoelliptic operator $T \in L^m_{\rho,\delta}(X)$ has a parametrix S in $H_M L^{-m,-m0}_{\rho,\delta}(X)$.*

$M-$hypoelliptic pseudo-differential operators

In this section we want to study some properties of $M-$hypoelliptic pseudo-differential operators on $L^p(\mathbb{R}^n)$, $1 < p < \infty$. We prove that if $T \in H_M L^{m,m_0}_{\rho,\delta}(X)$ and $Dom(T) = S(\mathbb{R}^n)$, then T is closable, where $S(\mathbb{R}^n)$ is the Schwartz space. This result is proposition (3.1) in Wong [11] if $\delta = 0, \rho = 1$ and $m = m_0$.

Proposition 3.11. *If* $T \in H_M L^{m,m_0}_{\rho,\delta}(X)$ *and* $S \in H_M L^{m',m'_0}_{\rho,\delta}(X)$ *then*

$$ToS \in H_M L^{m+m',m_0+m'_0}_{\rho,\delta}(X).$$

proof. By theorem 2.10

$$\sigma_{TS} \sim \sigma_T(x,\xi)\sigma_S(x,\xi)\ [1 + \sum_{|\alpha|\geq 1} \frac{\partial^\alpha_\xi \sigma_T}{\sigma_T}\ .\ \frac{\partial^\alpha_x \sigma_S}{\sigma_S}\ ]$$

and from Lemma 3.1 and 3.2 we see that the series is square bracket is an asympotic sum belong to $HS^{0,0}_{\rho,\delta}$. It remaind to use proposition 3.4.

Theorem 3.12. *If* $T \in H_M L^{m,m_0}_{\rho,\delta}(X)$ *then* $T^* \in H_M L^{m,m_0}_{\rho,\delta}(X)$ *where* T^**is the adjoint of* T.

proof. It follows from proposition 3.11.

Proposition 3.13. *If* $T \in H_M L^{m,m_0}_{\rho,\delta}(X)$ *and the* $Dom(T)$ *of* T *is* $S(\mathbb{R}^n)$, *then* T *is closable.*

proof. Let $\{\varphi_k\}$ be a sequence of functions in $S(\mathbb{R}^n)$ such that $\varphi_k \to 0$, $T(\varphi_k) \to f$ in $L^p(\mathbb{R}^n)$ as $k \to \infty$. Then for any function ψ in $S(\mathbb{R}^n)$, we have

$$\langle T(\varphi_k), \psi\rangle = \langle \varphi_k, T^*(\psi)\rangle.$$

Let $k \to \infty$. Then we have $\langle f, \psi\rangle = 0$ for all function ψ in $S(\mathbb{R}^n)$. Since $S(\mathbb{R}^n)$ is dense in $L^p(\mathbb{R}^n)$ it follows that $f = 0$.

Remark 3.14. *A consequence of the above proposition is that* $T : S(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ *has a closed extension in* $L^p(\mathbb{R}^n)$. *We denote the smallest such by* T_0 *and call it the minimal operator of* T. *It can be shown easily that domain* $Dom(T_0)$ *of* T_0 *consist of all functions* $u \in L^p(\mathbb{R}^n)$ *such that a sequence* $\{\varphi_k\}$ *in* $S(\mathbb{R}^n)$ *cab be found for which* $\varphi_k \to u$ *in* $L^p(\mathbb{R}^n)$ *and* $T(\varphi_k) \to f$ *for some* f *in* $L^p(\mathbb{R}^n)$. *Moreover,* $T_0 u = f$ *see [3] by Wong.*

Theorem 3.15. *Let* $T \in H_M L^{m,m_0}_{\rho,\delta}(X)$ *for which* $1 - \rho \leq \delta < \rho$ *or* $\rho < \delta$ *and* X *is open set in* $\mathbb{R}^n$. *If there exist two constants* C, D *such that*

$$\|\varphi\|_{m,p} \leq C\ [\ \|T\varphi\|_{0,p} + \|\varphi\|_{0,p}] \leq D\|\varphi\|_{m,p},$$

for $\varphi \in S(\mathbb{R}^n)$. *Then* $Dom(T_0) = H^{m,p}$.

proof. If $u \in H^{m,p}$, then we can take a sequence $\{\varphi_k\}$ of functions in $S(\mathbb{R}^n)$ such that $\varphi_k \to u$ in $H^{m,p}$. Therefore by assumptions, $\{T\varphi_k\}$ and $\{\varphi_k\}$ are Cauchy sequences in $L^p(\mathbb{R}^n)$. Thus $\varphi_k \to u$ and $T\varphi_k \to f$ for some u and f in $L^p(\mathbb{R}^n)$. Hence $u \in Dom(T_0)$ and $T_0 u = f$. ON the other hand, if $u \in Dom(T_0)$, then we can find a sequence $\{\varphi_k\}$ in $S(\mathbb{R}^n)$ for which $\varphi_k \to u$ in $L^p(\mathbb{R}^n)$ and $T\varphi_k \to f$ for some f in $L^p(\mathbb{R}^n)$. Therefore $\{\varphi_k\}$ and $\{T\varphi_k\}$ are Cauchy sequence in $L^p(\mathbb{R}^n)$, so by assumption of theorem $\{\varphi_k\}$ is a Cauchy in $H^{m,p}$. Since $H^{m,p}$ is complete, $\varphi_k \to v$ for some $v \in H^{m,p}$. Suppose $m \geq 0$. Then the inclusion map $H^{m,p} \hookrightarrow L^p(\mathbb{R}^n)$ is continuous. Therefore $\varphi_k \to v$ in $L^p(\mathbb{R}^n)$. So $u = v$, and consequently lies in $H^{m,p}$. If $m < 0$, then the inclusion map $L^p(\mathbb{R}^n) \hookrightarrow H^{m,p}$ is continuous. Hence $\varphi_k \to u$ in $H^{m,p}$. Therefore $u = v$ and consequently lies in $H^{m,p}$.

Remark 3.16. *We can define another closed extension T_1 of T on $S(\mathbb{R}^n)$ as follows. We say that $u \in Dom(T_1)$ and $T_1 u = f$ if u and f are in $L^p(\mathbb{R}^n)$ and $\langle u, T^*\psi\rangle = \langle f, \psi\rangle$ for all $\psi \in S(\mathbb{R}^n)$. It is called the maximal or weak extension of T. It is the maximal in the sense that it is the largest closed extension having $S(\mathbb{R}^n)$ in the domain of its adjoint. Since T_1 is the maximal closed extension of T, $H^{m,p}$ is contained in the domain of T_1.*

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