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Almost n-multiplicative maps between Fréchet algebras

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Abstract

For the Fréchet algebras $(A, (p_k))$ and $(B, (q_k))$ and $n \in \mathbb{N}$, $n \geq 2$, a linear map $T : A \to B$ is called *almost n-multiplicative*, with respect to (p_k) and (q_k), if there exists $\varepsilon \geq 0$ such that

$$q_k(Ta_1a_2 \cdots a_n - Ta_1Ta_2 \cdots Ta_n) \leq \varepsilon p_k(a_1)p_k(a_2) \cdots p_k(a_n),$$

for each $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A$. The linear map T is called *weakly almost n-multiplicative*, if there exists $\varepsilon \geq 0$ such that for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ with

$$q_k(Ta_1a_2 \cdots a_n - Ta_1Ta_2 \cdots Ta_n) \leq \varepsilon p_{n(k)}(a_1)p_{n(k)}(a_2) \cdots p_{n(k)}(a_n),$$

for each $k \in \mathbb{N}$ and $a_1, a_2, \ldots, a_n \in A$. The linear map T is called n-multiplicative if

$$Ta_1a_2 \cdots a_n = Ta_1Ta_2 \cdots Ta_n,$$

for every $a_1, a_2, \ldots, a_n \in A$.

In this paper, we investigate automatic continuity of (weakly) almost n-multiplicative maps between certain classes of Fréchet algebras, including Banach algebras. We show that if $(A, (p_k))$ is a Fréchet algebra and $T : A \to \mathbb{C}$ is a weakly almost n-multiplicative linear functional, then either T is n-multiplicative, or it is continuous. Moreover, if $(A, (p_k))$ and $(B, (q_k))$ are Fréchet algebras and $T : A \to B$ is a continuous linear map, then under certain conditions T is weakly almost n-multiplicative for each $n \geq 2$. In particular, every continuous linear functional on A is weakly almost n-multiplicative for each $n \geq 2$.

Keywords: multiplicative maps (homomorphisms); almost multiplicative maps; automatic continuity; Fréchet algebras; Banach algebras.
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1. Introduction

A subset V of a complex algebra A (an algebra over the complex field) is *multiplicative* or *idempotent*, if $VV \subseteq V$, and it is called *balanced* if $\lambda V \subseteq V$ for all scalars λ such that $|\lambda| \leq 1$. A topological algebra is *locally convex* if there is a base of neighbourhoods of zero consisting of convex sets. Since each locally convex algebra has a base of neighbourhoods of zero consisting of absolutely convex (convex and balanced) sets, we may always assume that a locally convex algebra has a base of neighbourhoods of zero consisting of absolutely convex sets. An algebra A is called a *Fréchet algebra* if it is a complete metrizable topological linear space and has a neighbourhood basis (V_n) of zero such that V_n is *multiplicative (idempotent)* and *absolutely convex* for all $n \in \mathbb{N}$. The topology of a Fréchet algebra A can be generated by a sequence (p_k) of separating submultiplicative seminorms, i.e.,

$$p_k(xy) \leq p_k(x)p_k(y),$$

for all $k \in \mathbb{N}$ and $x, y \in A$. Moreover, we may take the sequence (p_k) such that

$$p_k(x) \leq p_{k+1}(x),$$

for all $k \in \mathbb{N}$ and $x \in A$. The Fréchet algebra A with the above generating sequence of seminorms (p_k) is denoted by $(A, (p_k))$. Note that a sequence (x_n) in the Fréchet algebra $(A, (p_k))$ converges to $x \in A$ if and only if $p_k(x_n - x) \to 0$ for each $k \in \mathbb{N}$, as $n \to \infty$. Banach algebras are important examples of Fréchet algebras.

The (*Jacobson*) *radical* of an algebra A, denoted by $radA$, is the intersection of all maximal left (right) ideals in A. The algebra A is called *semisimple* if $radA = \{0\}$. If A is a commutative Fréchet algebra, then $radA = \bigcap_{\varphi \in M(A)} ker\varphi$, where $M(A)$ is the continuous character space of A, i.e., the space of all continuous non-zero multiplicative linear functionals on A. See, for example, [5, Proposition 8.1.2].

Homomorphisms between different classes of topological algebras, including Fréchet algebras and Banach algebras, have been widely studied by many authors. For the automatic continuity of homomorphisms between Banach algebras, Fréchet algebras and topological algebras one may refer to the monographs of M. Fragoulopoulou [3], K. Jarosz [10], and A. Mallios [14].

In 1985, K. Jarosz introduced the concept of *almost multiplicative maps* between Banach algebras [10]. For the Banach algebras A and B, a linear map $T : A \to B$ is called *almost multiplicative* if there exists $\varepsilon \geq 0$ such that

$$\|Tab - TaTb\| \leq \varepsilon\|a\|\|b\|,$$

for every $a, b \in A$. K. Jarosz investigated the problem of automatic continuity for almost multiplicative linear maps between Banach algebras. Later, in 1986, B. E. Johnson obtained some results on the continuity of *approximately* (*almost*) *multiplicative functionals* [11] and then in 1988, he extended his results to *approximately (almost) multiplicative maps* between Banach algebras [12]. Since then, many authors investigated almost multiplicative maps between different classes of Banach algebras. See, for example, [13] and [15].

In 2006, Honary investigated the automatic continuity of homomorphisms between Banach algebras and Fréchet algebras [7]. Later, Honary, Omidi and Sanatpour investigated the problem of automatic continuity for (weakly) almost multiplicative linear operators between Fréchet algebras [8]. In 2005, the concept of n-homomorphims was studied for complex algebras by Hejazian, Mirzavaziri and Moslehian [6]. Then, in 2011, Honary and Shayanpour obtained some results on the automatic continuity of n-homomorphims between topological algebras [9]. In 2015, Shayanpour, Honary and

Hashemi characterized certain properties of n-characters and n-homomorphims on topological algebras [16]. Also, in 2015, Eshaghi Gordji, Jabbari and Karapinar proved that every surjective n-homomorphism from a Banach algebra A into a semisimple Banach algebra B is automatically continuous [2]. In this paper, we investigate (weakly) almost n-multiplicative maps between Fréchet algebras and we also obtain some results on the automatic continuity of such maps.

2. Automatic continuity of weakly almost n-multiplicative maps between Fréchet algebras

In this section, among other results, we show that every continuous linear functional on a Fréchet algebra is weakly almost n-multiplicative.

Definition 2.1. Let $(A,(p_k))$ and $(B,(q_k))$ be Fréchet algebras, $n \in \mathbb{N}$, $n \geq 2$ and $\varepsilon \geq 0$. A linear map (operator) $T : (A,(p_k)) \to (B,(q_k))$ is called (ε, n)-*multiplicative*, with respect to (p_k) and (q_k), if

$$q_k(Ta_1a_2\cdots a_n - Ta_1Ta_2\cdots Ta_n) \leq \varepsilon p_k(a_1)p_k(a_2)\cdots p_k(a_n),$$

for each $k \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A$. The linear map T is called weakly (ε, n)-multiplicative, with respect to (p_k) and (q_k), if for every $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$q_k(Ta_1a_2\cdots a_n - Ta_1Ta_2\cdots Ta_n) \leq \varepsilon p_{n(k)}(a_1)p_{n(k)}(a_2)\cdots p_{n(k)}(a_n),$$

for each $k \in \mathbb{N}$ and every $a_1, a_2, \ldots, a_n \in A$. A linear map $T : A \to B$ is called almost n-multiplicative or weakly almost n-multiplicative if it is (ε, n)-multiplicative or weakly (ε, n)-multiplicative, respectively, for some $\varepsilon \geq 0$. A linear map T is called n-multiplicative if

$$Ta_1a_2\cdots a_n = Ta_1Ta_2\cdots Ta_n,$$

for every $a_1, a_2, \ldots, a_n \in A$. In particular, when $n = 2$ the corresponding linear map $T : A \to B$ is called (weakly) (almost) multiplicative.

An example of a weakly almost multiplicative continuous map, which is not multiplicative, has been given in [8].

Remark 2.2. It is clear that every (ε, n)-multiplicative map is weakly (ε, n)-multiplicative, and since (p_k) is a separating sequence of seminorms, it is clear that (ε, n)-multiplicative and weakly (ε, n)-multiplicative maps turn out to be n-multiplicative, whenever $\varepsilon = 0$. Moreover, any n-multiplicative map is (ε, n)-multiplicative for every $\varepsilon \geq 0$.

Remark 2.3. Note that if $(A,(p_k))$ is a Fréchet algebra and $T : (A,(p_k)) \to \mathbb{C}$ is a weakly (ε, n)-multiplicative linear functional, then there exists smallest $m \in \mathbb{N}$ such that

$$|Ta_1a_2\cdots a_n - Ta_1Ta_2\cdots Ta_n| \leq \varepsilon p_m(a_1)p_m(a_2)\cdots p_m(a_n),$$

for every $a_1, a_2, \ldots, a_n \in A$. In the sequel we use this fixed m for a weakly (ε, n)-multiplicative linear functional.

Definition 2.4. A Fréchet algebra $(A,(p_k))$ with unit $e_{_A}$, is called *unital* if for every $k \in \mathbb{N}$, $p_k(e_{_A}) = 1$.

Proposition 2.5. *Let $(A,(p_k))$ and $(B,(q_k))$ be unital Fréchet algebras and $T : A \to B$ be a unital $(Te_{_A} = e_{_B})$ weakly almost n-multiplicative map. Then T is weakly almost multiplicative.*

Proof . Let $\varepsilon \geq 0$ and T be a weakly (ε, n)-multiplicative map. Then, for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that for every $x, y \in A$,

$$\begin{aligned} q_k(Txy - TxTy) &= q_k(Txye_{_A}e_{_A}\cdots e_{_A} - TxTyTe_{_A}Te_{_A}\cdots Te_{_A}) \\ &\leq \varepsilon p_{n(k)}(x)p_{n(k)}(y)p_{n(k)}(e_{_A})^{n-2} \\ &= \varepsilon p_{n(k)}(x)p_{n(k)}(y). \end{aligned}$$

Therefore, T is a weakly almost multiplicative map. □

Definition 2.6. A Fréchet algebra $(A,(p_k))$ is a uniform Fréchet algebra if

$$p_k(a^2) = (p_k(a))^2,$$

for all $k \in \mathbb{N}$ and $a \in A$.

The following result has appeared in the form of remarks in [4, Page 8] and [5, Page 73], without proof. Here we present the proof for easy reference.

Proposition 2.7. *Let $(A,(p_k))$ and $(B,(q_k))$ be Fréchet algebras and $T : A \to B$ be a linear map. Then, T is continuous if and only if for each $k \in \mathbb{N}$ there exist $n(k) \in \mathbb{N}$ and $c_k > 0$ such that*

$$q_k(T(a)) \leq c_k p_{n(k)}(a), \tag{2.1}$$

for each $a \in A$. Moreover, if $(B,(q_k))$ is a uniform Fréchet algebra and T is a continuous homomorphism, then we may choose $c_k = 1$ for all $k \in \mathbb{N}$.

Proof . Let $T : A \to B$ be continuous. Then, for each neighbourhood W of zero in B, there exists a neighbourhood V of zero in A such that $T(V) \subseteq W$. On the other hand, by [5, page 63], the sequences of neighbourhoods

$$V_k = \{x \in A : \ p_k(x) < \frac{1}{k}\},$$
$$W_k = \{y \in B : \ q_k(y) < \frac{1}{k}\},$$

are local bases in A and B, respectively. Therefore, for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that $T(V_{n(k)}) \subseteq W_k$. Note that for each $a \in A$ and $k \in \mathbb{N}$, if $p_{n(k)}(a) \neq 0$ then $\frac{a}{(n(k)+1)p_{n(k)}(a)} \in V_{n(k)}$. Consequently, $\frac{T(a)}{(n(k)+1)p_{n(k)}(a)} \in W_k$ and hence

$$q_k(T(a)) \leq \frac{1}{k}(n(k)+1)p_{n(k)}(a).$$

Therefore, by taking $c_k = \frac{1}{k}(n(k)+1)$, the inequality (2.1) holds for each $a \in A$ and $k \in \mathbb{N}$ with $p_{n(k)}(a) \neq 0$. If $p_{n(k)}(a) = 0$ for some $a \in A$ and $k \in \mathbb{N}$, then for each $r > 0$ we have $p_{n(k)}(ra) = 0$. Consequently, $ra \in V_{n(k)}$ and since $T(V_{n(k)}) \subseteq W_k$ we get $q_k(T(a)) = \frac{1}{r}q_k(T(ra)) < \frac{1}{rk}$. Since $r > 0$ is arbitrary, we conclude that $q_k(T(a)) = 0$. Hence, the inequality (2.1) also holds in this case.

Conversely, if the inequality (2.1) holds for each $a \in A$ and $k \in \mathbb{N}$, then clearly $T : A \to B$ is continuous at zero, which is equivalent to the continuity of $T : A \to B$ at all points of A.

In the case that $(B,(q_k))$ is a uniform Fréchet algebra and $T : A \to B$ is a continuous homomorphism, we apply the inequality (2.1) for each $m \in \mathbb{N}$, and obtain

$$(q_k(T(a)))^{2^m} = q_k(T(a^{2^m})) \leq c_k p_{n(k)}(a^{2^m}) \leq c_k \left(p_{n(k)}(a)\right)^{2^m}.$$

Therefore, $q_k(T(a)) \leq (c_k)^{\frac{1}{2^m}} p_{n(k)}(a)$, and hence by letting $m \to \infty$, we get $q_k(T(a)) \leq p_{n(k)}(a)$, which is the desired result. □

Definition 2.8. Let $(A,(p_k))$ and $(B,(q_k))$ be Fréchet algebras. A linear map $T : A \to B$ is called *quasi uniformly continuous*, if there exists a constant $c > 0$ such that for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ satisfying the inequality

$$q_k(Ta) \leq c\, p_{n(k)}(a),$$

for every $a \in A$.

Remark 2.9. It is clear that, if $(A,(p_k))$ is a Fréchet algebra, $(B, \|\cdot\|)$ is a (complete) normed algebra and $T : A \to B$ is a continuous linear map, then T is quasi uniformly continuous, by Proposition 2.7.

Theorem 2.10. *Let $(A,(p_k))$ and $(B,(q_k))$ be Fréchet algebras and $T : A \to B$ be a quasi uniformly continuous map. Then, T is weakly almost n-multiplicative for each $n \in \mathbb{N}$, $n \geq 2$.*

Proof . By the hypothesis, there exists $c > 0$ and $n(k) \in \mathbb{N}$ for each $k \in \mathbb{N}$, such that

$$q_k(Ta) \leq c\, p_{n(k)}(a),$$

for every $a \in A$. Then, for each $a_1, a_2, \ldots, a_n \in A$ and for all $k \in \mathbb{N}$, we have

$$\begin{aligned} & q_k(Ta_1a_2\cdots a_n - Ta_1Ta_2\cdots Ta_n) \\ & \quad \leq q_k(Ta_1a_2\cdots a_n) + q_k(Ta_1)q_k(Ta_2)\cdots q_k(Ta_n) \\ & \quad \leq c\, p_{n(k)}(a_1a_2\cdots a_n) + c\, p_{n(k)}(a_1) c\, p_{n(k)}(a_2)\cdots c\, p_{n(k)}(a_n) \\ & \quad \leq (c + c^n)\, p_{n(k)}(a_1)p_{n(k)}(a_2)\cdots p_{n(k)}(a_n). \end{aligned}$$

Therefore, T is weakly almost n-multiplicative. □

Corollary 2.11. Let $(A,(p_k))$ be a Fréchet algebra, B be a (complete) normed algebra and $T : A \to B$ be a continuous linear map. Then, T is weakly almost n-multiplicative for each $n \in \mathbb{N}$, $n \geq 2$. In particular, every continuous linear functional on A is weakly almost n-multiplicative.

Theorem 2.12. *Let $(A,(p_k))$ be a unital Fréchet algebra and $n \in \mathbb{N}$, $n \geq 2$. If $T : A \to \mathbb{C}$ is a weakly almost n-multiplicative map and $\varphi : A \to \mathbb{C}$ is defined by $\varphi(a) = T(e_A)^{n-2}T(a)$ for every $a \in A$, then φ is a weakly almost multiplicative map.*

Proof . Let $\varepsilon \geq 0$ and T be a weakly (ε, n)-multiplicative map. By Remark 2.3 there exists $m \in \mathbb{N}$, such that for every $a, b \in A$

$$\begin{aligned} |\varphi(ab) - \varphi(a)\varphi(b)| &= |T(e_A)^{n-2}T(ab) - T(e_A)^{n-2}T(a)T(e_A)^{n-2}T(b)| \\ &\leq |T(e_A)^{n-2}||T(a(e_A)^{n-2}b) - T(a)T(e_A)^{n-2}T(b)| \\ &\leq \varepsilon|T(e_A)^{n-2}|p_m(a)p_m(e_A)^{n-2}p_m(b) \\ &= \varepsilon|T(e_A)^{n-2}|p_m(e_A)^{n-2}p_m(a)p_m(b) \\ &= \delta p_m(a)p_m(b), \end{aligned}$$

where $\delta = \varepsilon|T(e_A)^{n-2}|p_m(e_A)^{n-2}$. Hence, φ is weakly almost multiplicative. □

3. Automatic continuity of weakly almost n-multiplicative functionals on Fréchet algebras

We first bring two lemmas and then present the main result of this paper.

Lemma 3.1. *Let $(A,(p_k))$ be a Fréchet algebra and $T: A \to \mathbb{C}$ be a weakly (ε, n)-multiplicative linear functional. Then, for every $x_1, x_2, \ldots, x_n, z \in A$, we have*

$$\begin{aligned}&|Tz|^{n-1} \cdot |Tx_1x_2\cdots x_n - Tx_1Tx_2\cdots Tx_n|\\ &\quad \leq \varepsilon\ (2p_m(x_1)p_m(x_2)\cdots p_m(x_{n-1}) + |Tx_1Tx_2\cdots Tx_{n-1}|)p_m(x_n)(p_m(z))^{n-1}.\end{aligned} \tag{3.1}$$

Proof . Let $x_1, x_2, \ldots, x_n, z \in A$. Then,

$$\begin{aligned}&|Tz|^{n-1} \cdot |Tx_1x_2\cdots x_n - Tx_1Tx_2\cdots Tx_n|\\ &\quad = |Tx_1x_2\cdots x_n(Tz)^{n-1} - Tx_1Tx_2\cdots Tx_n(Tz)^{n-1}|\\ &\quad \leq |Tx_1x_2\cdots x_n(Tz)^{n-1} - Tx_1x_2\cdots x_nz^{n-1}|\\ &\qquad + |Tx_1x_2\cdots x_nz^{n-1} - Tx_1Tx_2\cdots Tx_{n-1}Tx_nz^{n-1}|\\ &\qquad + |Tx_1Tx_2\cdots Tx_{n-1}Tx_nz^{n-1} - Tx_1Tx_2\cdots Tx_n(Tz)^{n-1}|\\ &\quad \leq \varepsilon\ p_m(x_1x_2\cdots x_n)p_m(z)^{n-1} + \varepsilon\ p_m(x_1)\cdots p_m(x_{n-1})p_m(x_nz^{n-1})\\ &\qquad + \varepsilon\ |Tx_1\cdots Tx_{n-1}|p_m(x_n)p_m(z)^{n-1}\\ &\quad \leq \varepsilon\ (p_m(x_1)\cdots p_m(x_{n-1}) + p_m(x_1)\cdots p_m(x_{n-1})\\ &\qquad + |Tx_1\cdots Tx_{n-1}|)p_m(x_n)p_m(z)^{n-1}\\ &\quad \leq \varepsilon\ (2p_m(x_1)\cdots p_m(x_{n-1}) + |Tx_1\cdots Tx_{n-1}|)p_m(x_n)p_m(z)^{n-1}.\end{aligned}$$

□

Lemma 3.2. *Let $(A,(p_k))$ be a Fréchet algebra and $T: A \to \mathbb{C}$ be a weakly almost n-multiplicative functional. Then, at least one of the following holds:*

(i) T is n-multiplicative,

(ii) for every $a \in A$, if $p_m(a) = 0$ then $Ta = 0$.

Proof . Let T be a weakly (ε, n)-multiplicative for some $\varepsilon \geq 0$ and suppose that (ii) does not hold, that is, $p_m(a) = 0$ and $Ta \neq 0$ for some $a \in A$. Then, by letting $z = a$ in Lemma 3.1, we have $Tx_1x_2\cdots x_n = Tx_1Tx_2\cdots Tx_n$ for all $x_1, x_2, \ldots, x_n \in A$. Hence, T is n-multiplicative. □

Theorem 3.3. *Let $(A,(p_k))$ be a Fréchet algebra and $T: A \to \mathbb{C}$ be a weakly almost n-multiplicative linear functional. Then, at least one of the following holds:*

(i) T is n-multiplicative,

(ii) T is continuous.

Proof . Let T be weakly (ε, n)-multiplicative for some $\varepsilon \geq 0$. Then, we have

$$|Ta_1a_2\cdots a_n - Ta_1Ta_2\cdots Ta_n| \leq \varepsilon\ p_m(a_1)p_m(a_2)\cdots p_m(a_n),$$

for every $a_1, a_2, \dots, a_n \in A$. We can choose $c > 1$ such that $c^n - c \geq \varepsilon$. If

$$|Ta| \leq c\ p_m(a), \tag{3.2}$$

for all $a \in A$, then T is continuous. If (3.2) does not hold for some $a_0 \in A$, we have

$$|Ta_0| > c\ p_m(a_0). \tag{3.3}$$

Therefore, $Ta_0 \neq 0$. If $p_m(a_0) = 0$, then by Lemma 3.2, T is n-multiplicative. Let $p_m(a_0) \neq 0$ in (3.3). By considering $\frac{a_0}{p_m(a_0)}$ instead of a_0 in (3.3), we may assume that $p_m(a_0) = 1$ and $|Ta_0| > c$. Therefore, we can write $|Ta_0| = c + r$, for some $r > 0$.

Now, by induction we prove that $|Ta_0^{n^k}| > c + kr$, for each $k \in \mathbb{N}$. For $k = 1$ we have

$$\begin{aligned}
|Ta_0^n| &\geq |(Ta_0)^n| - |Ta_0^n - (Ta_0)^n| \\
&\geq |Ta_0|^n - \varepsilon p_m(a_0)^n = (c+r)^n - \varepsilon \\
&> c^n + \binom{n}{1} c^{n-1} r - \varepsilon \\
&\geq c + nc^{n-1} r > c + r.
\end{aligned}$$

Let $|Ta_0^{n^k}| > c + kr$, for $k \in \mathbb{N}$. Then, for $k+1$ we have

$$\begin{aligned}
|Ta_0^{n^{k+1}}| &= |Ta_0^{n^k.n}| = |Ta_0^{n^k + n^k + \cdots + n^k}| = |Ta_0^{n^k} a_0^{n^k} \dots a_0^{n^k} a_0^{n^k}| \\
&= |Ta_0^{n^k} a_0^{n^k} \dots a_0^{n^k} a_0^{n^k} - (Ta_0^{n^k})^n + (Ta_0^{n^k})^n| \\
&\geq |Ta_0^{n^k}|^n - |Ta_0^{n^k} a_0^{n^k} \dots a_0^{n^k} a_0^{n^k} - (Ta_0^{n^k})^n| \\
&\geq (c+kr)^n - \varepsilon p_m(a_0^{n^k})^n = (c+kr)^n - \varepsilon \\
&> c^n + \binom{n}{1} c^{n-1} kr - \varepsilon \geq c + nc^{n-1} kr \\
&> c + nkr > c + (k+1)r.
\end{aligned}$$

Therefore, we have $|Ta_0^{n^k}| > c + kr$ for all $k \in \mathbb{N}$. For every $x_1, x_2, \dots, x_n, z \in A$, by letting $z = a_0^{n^k}$ in the equation (3.1) of Lemma 3.1, we get

$$\begin{aligned}
&|Tx_1 x_2 \cdots x_n - Tx_1 Tx_2 \cdots Tx_n| \\
&\quad \leq \frac{\varepsilon\ (2p_m(x_1) \cdot p_m(x_{n-1}) + |Tx_1 \cdot Tx_{n-1}|) p_m(x_n) p_m^{n-1}(a_0^{n^k})}{|Ta_0^{n^k}|^{n-1}} \\
&\quad \leq \frac{\varepsilon(2p_m(x_1) \dots p_m(x_{n-1}) + |Tx_1 \dots Tx_{n-1}|) p_m(x_n)}{(c+kr)^{n-1}}.
\end{aligned} \tag{3.4}$$

Hence, by letting $k \to \infty$, we get $Tx_1 x_2 \cdots x_n = Tx_1 Tx_2 \cdots Tx_n$, which completes the proof. □

Definition 3.4. A topological algebra A is called *n-functionally continuous*, if each n-multiplicative linear functional on A is continuous.

Theorem 3.5. [16, Corollary 2.2]. Let A be a topological algebra. Then A is n-functionally continuous if and only if it is functionally continuous.

Corollary 3.6. *Let $(A,(p_k))$ be a functionally continuous Fréchet algebra. Then, every weakly almost n-multiplicative linear functional T on A is automatically continuous.*

Proof . The result follows from Theorems 3.3 and 3.5. □

The next preliminary result is on the composition of two linear operators.

Theorem 3.7. *Let $(A,(p_k))$ and $(B,(q_k))$ be Fréchet algebras and $(D,(r_n))$ be a uniform Fréchet algebra. If $T : A \to B$ is a weakly almost n-multiplicative map and $\varphi : B \to D$ is a continuous homomorphism, then $\varphi \circ T$ is a weakly almost n-multiplicative map.*

Proof . Let T be a weakly (ε, n)-multiplicative map for some $\varepsilon \geq 0$. Since φ is continuous, by Proposition 2.7 it follows that for each $k \in \mathbb{N}$ there exists $n(k) \in \mathbb{N}$ such that

$$\begin{aligned} &r_k((\varphi \circ T)(a_1 a_2 \cdots a_n) - (\varphi \circ T)(a_1)(\varphi \circ T)(a_2) \cdots (\varphi \circ T)(a_n)) \\ &= r_k(\varphi(Ta_1 a_2 \cdots a_n - Ta_1 Ta_2 \cdots Ta_n)) \\ &\leq q_{n(k)}(Ta_1 a_2 \cdots a_n - Ta_1 Ta_2 \cdots Ta_n), \end{aligned}$$

for every $a_1,\ a_2, \ldots,\ a_n \in A$. Since T is weakly (ε, n)-multiplicative, there exists $m_{n(k)} \in \mathbb{N}$ such that

$$q_{n(k)}(Ta_1 a_2 \cdots a_n - Ta_1 Ta_2 \cdots Ta_n) \leq \varepsilon\, p_{m_{n(k)}}(a_1) p_{m_{n(k)}}(a_2) \cdots p_{m_{n(k)}}(a_n).$$

Therefore,

$$\begin{aligned} &r_k((\varphi \circ T)(a_1 a_2 \cdots a_n) - (\varphi \circ T)(a_1)(\varphi \circ T)(a_2) \cdots (\varphi \circ T)(a_n)) \\ &\leq \varepsilon\, p_{m_{n(k)}}(a_1) p_{m_{n(k)}}(a_2) \cdots p_{m_{n(k)}}(a_n), \end{aligned} \tag{3.5}$$

for every $a_1,\ a_2, \ldots,\ a_n \in A$. Consequently, $\varphi \circ T$ is a weakly n-multiplicative map. □

Corollary 3.8. Let $(A,(p_k))$ and $(B,(q_k))$ be Fréchet algebras. If $T : A \to B$ is weakly almost n-multiplicative and $\varphi \in M(B)$, then $\varphi \circ T$ is a weakly almost n-multiplicative functional.

The following theorem is due to E. Ansari-Piri and N. Eghbali, concerning the automatic continuity of almost n-multiplicative maps between Banach algebras.

Theorem 3.9. [1, Theorem 4.2] Let A and B be Banach algebras such that B is semisimple and commutative. If $T : A \to B$ is an almost n-multiplicative map, then it is automatically continuous.

The following theorem is a generalization of the theorem above.

Theorem 3.10. *Let A be a Banach algebra and $(B,(q_k))$ be a semisimple commutative Fréchet algebra. If $T : A \to B$ is a weakly almost n-multiplicative map, then it is automatically continuous.*

Proof . Let $\varphi \in M(B)$, then by Corollary 3.8, $\varphi \circ T : A \to \mathbb{C}$ is a weakly almost n-multiplicative map and hence it is automatically continuous by Theorem 3.9. Now, suppose that $a_n \to a$ in A and $Ta_n \to b$ in B. By the continuity of $\varphi \circ T$, we have

$$(\varphi \circ T)(a_n) \to (\varphi \circ T)(a),$$

as $n \to \infty$. On the other hand,

$$\lim_{n\to\infty} (\varphi \circ T)(a_n) = \varphi(\lim_{n\to\infty} (Ta_n)) = \varphi(b).$$

Hence, $\varphi(Ta - b) = 0$, for each $\varphi \in M(B)$, that is

$$(Ta - b) \in \bigcap_{\varphi \in M(B)} ker\varphi = radB = \{0\}.$$

Therefore, $Ta = b$ and hence, by the Closed Graph Theorem, T is continuous. □

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