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# $C$-class and $F(\psi, \varphi)$-contractions on $M$-metric spaces

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**Abstract**

Partial metric spaces were introduced by Matthews in 1994 as a part of the study of denotational semantics of data flow networks. In 2014 Asadi and *et al.* [New Extension of $p$-Metric Spaces with Some fixed point Results on $M$-metric paces, J. Ineq. Appl. 2014 (2014): 18] extend the Partial metric spaces to $M$-metric spaces. In this work, we introduce the class of $F(\psi, \varphi)$-contractions and investigate the existence and uniqueness of fixed points for the new class $\mathcal{C}$ in the setting of $M$-metric spaces. The theorems that we prove generalize many previously obtained results. We also give some examples showing that our theorems are indeed proper extensions.

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## 1. Introduction and preliminaries

The notion of metric space was introduced by Fréchet [3] in 1906. Later, many authors attempted to generalize the notion of metric space such as pseudo metric space, quasi metric space, semi metric spaces and partial metric spaces. In this paper, we consider another generalization of a metric space, so called $M$-metric space. This notion was introduced by Asadi *et al.* (see e.g. [2]) to solve some difficulties in domain theory of computer science. Geraghty in 1973 introduced an interesting class of auxiliary function to refine the Banach contraction mapping principle. Let $\mathcal{F}$ denote all functions $\beta : [0, \infty) \to [0, 1)$ which satisfies the condition:

$$\lim_{n\to\infty} \beta(t_n) = 1 \text{ implies } \lim_{n\to\infty} t_n = 0.$$

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By using the function $\beta \in \mathcal{F}$ Geraghty [4] proved the following remarkable theorem.

**Theorem 1.1.** (Geraghty [4]) Let $(X, d)$ be a complete metric space and $T : X \to X$ be an operator. Suppose that there exists $\beta : [0, \infty) \to [0, 1)$ satisfying the condition,

$$\beta(t_n) \to 1 \text{ implies } t_n \to 0$$

If $T$ satisfies the following inequality

$$d(Tx, Ty) \leq \beta(d(x, y))d(x, y), \text{ for any } x, y \in X, \tag{1.1}$$

then $T$ has a unique fixed point.

In 2014 Asadi *et al.* [2] introduced the $M$-metric space which extends partial metric space [7], by some of certain fixed point theorems obtained therein, and they have given a theorem that its proof is still open as follows.

**Theorem 1.2.** Let $(X, m)$ be a complete $M$-metric space and $T : X \to X$ be mapping satisfying:

$$\exists k \in [0, \frac{1}{2}) \quad \text{such that} \quad m(Tx, Ty) \leq k(m(x, Ty) + m(y, Tx)) \quad \forall x, y \in X.$$

Monfared *et al.* [8] presented a proof when $k \in [0, \frac{\sqrt{3}-1}{2})$, also they proved the Matkowski, Boyd and Wong's, fixed point theorems [11] which covered Ćirić-contractions and also Geraghty's fixed point theorem for $M$-metric spaces. In this work, we introduce the class of $F(\psi, \varphi)$-contractions and investigate the existence and uniqueness of fixed points for the new class $\mathcal{C}$ in the setting of $M$-metric spaces. Also we show that the Geraghty's fixed point theorem in [4] is one of corollaries of our theorems in this work. An illustrated example is also given and whenever every $p$-metric and ordinary metric are $m$-metric, all of the related results automatically hold. In this section, we review some of definitions and auxiliary results to use.

Admissible mappings have been defined recently by Samet et al [12] and employed quite often in order to generalize the results on various contractions. So we state next definitions of $\alpha$-admissible mapping and triangular $\alpha$-admissible mapping.

**Definition 1.3.** Let $\alpha : X \times X \to [0.\infty)$. A self-mapping $T : X \to X$ is called $\alpha$-admissible if the condition

$$\alpha(x, y) \geq 1 \Rightarrow \alpha(Tx, Ty) \geq 1, \tag{1.2}$$

is satisfied for all $x, y \in X$.

**Definition 1.4.** A mapping $T : X \to X$ is called triangular $\alpha$-admissible if it is $\alpha$-admissible and satisfies

$$\alpha(x, y) \geq 1, \alpha(y, z) \geq 1 \Rightarrow \alpha(x, z) \geq 1. \tag{1.3}$$

where $x, y, z \in X$ and $\alpha : X \times X \to [0.\infty)$ is a given function.

In what follows we recall the notion of (triangular) $\alpha$-orbital admissible, introduced by Popescu [10], that is inspired from [12].

**Definition 1.5.** ([10]) For a fixed mapping $\alpha : X \times X \to [0,\infty)$, we say that a self-mapping $T : X \to X$ is an $\alpha$-orbital admissible if

$$(O1)\ \ \alpha(u,Tu) \geq 1 \Rightarrow \alpha(Tu,T^2u) \geq 1.$$

Let $\mathcal{A}$ be the collection of all $\alpha$-orbital admissible $T : X \to X$.

In addition, $T$ is called triangular $\alpha$-orbital admissible if $T$ is $\alpha$-orbital admissible and

$$(O2)\ \ \alpha(u,v) \geq 1 \text{ and } \alpha(v,Tv) \geq 1 \Rightarrow \alpha(u,Tv) \geq 1$$

Let $\mathcal{O}$ be the collection of all triangular $\alpha$-orbital admissible $T : M \to M$.

**Definition 1.6.** ([2]) Let $X$ be a non empty set. A function $m : X \times X \to \mathbb{R}^+$ is called $M$-metric if the following conditions are satisfied:

(m1) $m(x,x) = m(y,y) = m(x,y) \iff x = y$,

(m2) $m_{xy} \leq m(x,y)$,

(m3) $m(x,y) = m(y,x)$,

(m4) $(m(x,y) - m_{xy}) \leq (m(x,z) - m_{xz}) + (m(z,y) - m_{zy})$.

Where

$$m_{xy} := \min\{m(x,x), m(y,y)\} = m(x,x) \vee m(y,y),$$

Then the pair $(X,m)$ is called a $M$-metric space.

The following notation are useful in the sequel.

$$m_{xy} := \max\{m(x,x), m(y,y)\} = m(x,x) \wedge m(y,y).$$

The next examples state that $m^s$ and $m^w$ are ordinary metric.

**Example 1.7.** ([2]) Let $m$ be a $m$-metric. Put

1. $m^w(x,y) = m(x,y) - 2m_{xy} + M_{xy}$,
2. $m^s(x,y) = m(x,y) - m_{xy}$ when $x \neq y$ and $m^s(x,y) = 0$ if $x = y$.

Then $m^w$ and $m^s$ are ordinary metrics.

In the following example we present an example of a $m$-metric which is not $p$-metric.

**Example 1.8.** ([2]) Let $X = \{1,2,3\}$. Define

$$m(1,2) = m(2,1) = m(1,1) = 8$$

$$m(1,3) = m(3,1) = m(3,2) = m(2,3) = 7 \quad m(2,2) = 9 \quad m(3,3) = 5$$

so $m$ is $m$-metric but $m$ is not $p$-metric. Since $m(2,2) \not\leq m(1,2)$ means $m$ is not $p$-metric. If $D(x,y) = m(x,y) - m_{x,y}$ then $m(1,2) = m_{1,2} = 8$ but it means $D(1,2) = 0$ while $1 \neq 2$ which means $D$ is not metric.

**Example 1.9.** ([2]) Let $(X,d)$ be a metric space. And $\phi : [0,\infty) \to [\phi(0),\infty)$ be an one to one and nondecreasing or strictly increasing mapping with $\phi(0) \geq 0$ is defined, such that

$$\phi(x+y) \leq \phi(x) + \phi(y) - \phi(0), \quad \forall x,y \geq 0.$$

Then $m(x,y) = \phi(d(x,y))$ is a $m$-metric.

**Remark 1.10.** ([2]) For every $x, y \in X$

1. $0 \leq M_{xy} + m_{xy} = m(x,x) + m(y,y)$
2. $0 \leq M_{xy} - m_{xy} = |m(x,x) - m(y,y)|$
3. $M_{xy} - m_{xy} \leq (M_{xz} - m_{xz}) + (M_{zy} - m_{zy})$

**Lemma 1.11.** ([2]) Every $p$-metric is a $m$-metric.

## 2. Topology for $M$-metric space

It is clear that each $m$-metric $p$ on X generates a $T_0$ topology $\tau_m$ on $X$. The set

$$\{B_m(x,\varepsilon) : x \in X, \varepsilon > 0\},$$

where

$$B_m(x,\varepsilon) = \{y \in X : m(x,y) < m_{x,y} + \varepsilon\},$$

for all $x \in X$ and $\varepsilon > 0$, forms the base of $\tau_m$.

**Definition 2.1.** ([2]) Let $(X,m)$ be a $M$-metric space. Then:

1. A sequence $\{x_n\}$ in a $M$-metric space $(X,m)$ converges to a point $x \in X$ if and only if
$$\lim_{n\to\infty} (m(x_n,x) - m_{x_n,x}) = 0. \tag{2.1}$$
2. A sequence $\{x_n\}$ in a $M$-metric space $(X,m)$ is called a $m$-Cauchy sequence if
$$\lim_{n,m\to\infty} (m(x_n,x_m) - m_{x_n,x_m}) \quad \& \quad \lim_{n,m\to\infty} (M_{x_n,x_m} - m_{x_n,x_m}) \tag{2.2}$$
there exist (and are finite).
3. A $M$-metric space $(X,m)$ is said to be complete if every $m$-Cauchy sequence $\{x_n\}$ in $X$ converges, with respect to $\tau_m$, to a point $x \in X$ such that
$$\left(\lim_{n\to\infty} (m(x_n,x) - m_{x_n,x}) = 0 \quad \& \quad \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0\right).$$

**Lemma 2.2.** ([2]) Let $(X,m)$ be a $M$-metric space. Then:

1. $\{x_n\}$ is a $m$-Cauchy sequence in $(X,m)$ if and only if it is a Cauchy sequence in the metric space $(X,m^w)$.
2. A $M$-metric space $(X,m)$ is complete if and only if the metric space $(X,m^w)$ is complete. Furthermore,
$$\lim_{n\to\infty} m^w(x_n,x) = 0 \iff \left(\lim_{n\to\infty} (m(x_n,x) - m_{x_n,x}) = 0 \quad \& \quad \lim_{n\to\infty} (M_{x_n,x} - m_{x_n,x}) = 0\right).$$

Likewise above definition holds also for $m^s$.

**Lemma 2.3.** ([2]) Assume that $x_n \to x$ and $y_n \to y$ as $n \to \infty$ in a $M$-metric space $(X,m)$. Then

$$\lim_{n\to\infty} (m(x_n, y_n) - m_{x_n,y_n}) = m(x,y) - m_{xy}.$$

**Lemma 2.4.** ([2]) Assume that $x_n \to x$ as $n \to \infty$ in a $M$-metric space $(X,m)$. Then

$$\lim_{n\to\infty} (m(x_n, y) - m_{x_n,y}) = m(x,y) - m_{x,y},$$

for all $y \in X$.

**Lemma 2.5.** ([2]) Assume that $x_n \to x$ and $x_n \to y$ as $n \to \infty$ in a $M$-metric space $(X,m)$. Then $m(x,y) = m_{xy}$. Further if $m(x,x) = m(y,y)$, then $x = y$.

**Definition 2.6.** ([6]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:
($i$) $\psi$ is non-decreasing and continuous,
($ii$) $\psi(t) = 0$ if and only if $t = 0$.

**Remark 2.7.** We let $\Psi$ denote the class of the altering distance functions.

**Definition 2.8.** ([1]) An ultra altering distance function is a continuous, nondecreasing mapping $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) > 0$ for $t > 0$ and $\varphi(0) \geq 0$.

**Remark 2.9.** We let $\Phi$ denote the class of the ultra altering distance functions.

**Definition 2.10.** ([1]) A mapping $F : [0,\infty)^2 \to \mathbb{R}$ is called $C$-*class* function if it is continuous and satisfies following axioms:

1. $F(s,t) \leq s$;
2. $F(s,t) = s$ implies that either $s = 0$ or $t = 0$; for all $s, t \in [0,\infty)$.

Note for some $F$ we have that $F(0,0) = 0$.
We denote $C$-class functions as $\mathcal{C}$.

**Example 2.11.** ([1]) The following functions $F : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathcal{C}$, for all $s, t \in [0,\infty)$:

1. $F(s,t) = s - t$.
2. $F(s,t) = ms$, $0<m<1$.
3. $F(s,t) = \frac{s}{(1+t)^r}$; $r \in (0,\infty)$.
4. $F(s,t) = \log(t + a^s)/(1+t)$, $a > 1$.
5. $F(s,t) = \ln(1 + a^s)/2$, $a > e$.
6. $F(s,t) = (s + l)^{(1/(1+t)^r)} - l$, $l > 1, r \in (0,\infty)$.
7. $F(s,t) = s \log_{t+a} a$, $a > 1$.
8. $F(s,t) = s - (\frac{1+s}{2+s})(\frac{t}{1+t})$.
9. $F(s,t) = s\beta(s)$, $\beta : [0,\infty) \to (0,1)$ and is continuous.
10. $F(s,t) = s - \frac{t}{k+t}$.

11. $F(s,t) = s - \varphi(s)$ where $\varphi : [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0$.
12. $F(s,t) = sh(s,t)$ where $h : [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0$.
13. $F(s,t) = s - (\frac{2+t}{1+t})t$.
14. $F(s,t) = \sqrt[n]{\ln(1+s^n)}$.
15. $F(s,t) = \phi(s)$ where $\phi : [0,\infty) \to [0,\infty)$ is a upper semicontinuous function such that $\phi(0) = 0$ and $\phi(t) < t$ for $t > 0$.
16. $F(s,t) = \frac{s}{(1+s)^r}$, $r \in (0,\infty)$.

## 3. The main result

Admissible mappings have been defined recently by Samet *et al.* [12] and employed quite often in order to generalize the results on various contractions. Therefore we state the following definitions and $\alpha$-admissible mapping and triangular $\alpha$-admissible mapping.

**Definition 3.1.** Let $(X,m)$ be an $M$-metric space, and let $T : X \to X$ be a given mapping. We say that $T$ is $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\psi(m(Tx,Ty)) \leq F(\psi(m(x,y)), \varphi(m(x,y))), \tag{3.1}$$

**Definition 3.2.** Let $(X,m)$ be an $M$-metric space, and let $T : X \to X$ be an $\alpha$-admissible mapping. We say that $T$ is an $\alpha$-admissible $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\alpha(x,y)\psi(m(Tx,Ty)) \leq F(\psi(m(x,y)), \varphi(m(x,y))), \tag{3.2}$$

**Definition 3.3.** Let $(X,m)$ be an $M$-metric space, and let $T : X \to X$ be an $\alpha$-admissible mapping. We say that $T$ is a generalized $\alpha$-admissible $F(\alpha,\psi)$-contractive mapping if there exist $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$ such that

$$\alpha(x,y)\psi(m(Tx,Ty)) \leq F(\psi(M(x,y)), \varphi(M(x,y))), \tag{3.3}$$

Where $M(x,y) = \max\{m(x,y), m(x,Tx), m(y,Ty)\}$

**Theorem 3.4.** $(X,m)$ *be a complete $M$-metric space, and let $T : X \to X$ be a generalized $\alpha$-admissible $F(\alpha,\psi)$-contractive mapping, and satisfies the following conditions:*
*(i) $T \in \mathcal{O}$;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) $T$ is continuous.*
*Then $T$ has a fixed point $v \in X$ and $\{T^n x_0\}$ converges to $v$.*

**Proof .** Due to the assumption of the theorem, there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Let $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$. Set-up an iterative sequence $\{x_n\}$ in $X$ by letting $x_{n+1} = Tx_n$ for all $n \geq 0$, $n \in \mathbb{N}$. If $x_n = x_{n+1}$ for some $n \in \mathbb{N} \cup \{0\}$, then $x^* = x_n$ is a fixed point of $T$. Assume further that $x_n \neq x_{n+1}$ for each $n \in \mathbb{N} \cup \{0\}$. Define

$$m_n := m(x_n, x_{n+1}).$$

Regarding that T is $\alpha$-admissible, we derive

$$\alpha(x_0,x_1)=\alpha(x_0,Tx_0)\geq 1 \Rightarrow \alpha(Tx_0,Tx_1)=\alpha(x_1,x_2)\geq 1.$$

Recursively, we obtain that

$$\alpha(x_n,x_{n+1})\geq 1 \quad \text{for all } n=0,1,\dots \tag{3.4}$$

Therefore, by (3.3) and (3.4)

$$\begin{aligned}\psi(m(x_n,x_{n+1})) &\leq \alpha(x_n,x_{n+1})\psi(m(x_n,x_{n+1}))\\ &\leq F(\psi(M(x_{n-1},x_n)),\varphi(M(x_{n-1},x_n)))\\ &\leq \psi(M(x_{n-1},x_n)),\end{aligned} \tag{3.5}$$

for each $n\geq 1$, where

$$\begin{aligned}M(x_{n-1},x_n) &= \max\{m(x_{n-1},x_n),m(x_{n-1},x_n),m(x_n,x_{n+1})\}\\ &= \max\{m(x_{n-1},x_n),m(x_n,x_{n+1})\}.\end{aligned} \tag{3.6}$$

If $\max\{m(x_{n-1},x_n),m(x_n,x_{n+1})\}=m(x_n,x_{n+1})$, then by (3.5), we get

$$\psi(m(x_n,x_{n+1}))\leq F(\psi(m(x_n,x_{n+1})),\varphi(m(x_n,x_{n+1})))\leq \psi(m(x_n,x_{n+1})),$$

so, $\psi(m(x_n,x_{n+1}))=0$ or $\varphi(m(x_n,x_{n+1}))=0$ therefore $m(x_n,x_{n+1})=0$, which is a contradiction. Hence $\max\{m(x_{n-1},x_n),m(x_n,x_{n+1})\}=m(x_{n-1},x_n)$ therefore (3.5) gives

$$\psi(m(x_n,x_{n+1}))\leq F(\psi(m(x_{n-1},x_n)),\varphi(m(x_{n-1},x_n)))\leq \psi(m(x_{n-1},x_n)), \tag{3.7}$$

for all $n\in\mathbb{N}$. This yield that, for each $n\in\mathbb{N}$

$$m(x_n,x_{n+1})\leq m(x_{n-1},x_n) \tag{3.8}$$

Thus, we conclude that the sequence $\{m(x_n,x_{n+1})\}$ is nonnegative and nonincreasing. As a result, there exists $t\geq 0$ such that $\lim_{n\to\infty} m(x_n,x_{n+1})=t$. We claim that $t=0$. Suppose, on the contrary, that $t>0$. Then, on account of (3.5), we get that

$$\psi(t)\leq F(\psi(t),\varphi(t)),$$

which yields that $\psi(t)=0$, or $,\varphi(t)=0$. We derive

$$\lim_{n\to\infty} m(x_n,x_{n+1})=0. \tag{3.9}$$

We shall prove that $\{x_n\}$ is $M$-Cauchy. We have

1. $\lim_{n\to\infty} m(x_n,x_{n+1})=0$.
2. $0\leq m_{x_n,x_{n+1}}\leq m(x_n,x_{n+1})\Rightarrow \lim_{n\to\infty} m_{x_n,x_{n+1}}=0$.
3. $m_{x_n,x_{n+1}}=\min\{m(x_n,x_n),m(x_{n+1},x_{n+1})\}\Rightarrow \lim_{n\to\infty} m(x_n,x_n)=0$.

On the other hand

$$m_{x_n,x_m} = \min\{m(x_n,x_n), m(x_m,x_m)\} \quad \text{so,} \quad \lim_{n,m\to\infty} m_{x_n,x_m} = 0,$$

therefore,

$$\lim_{n,m\to\infty}(M_{x_n,x_m} - m_{x_n,x_m}) = 0.$$

We show

$$\lim_{n,m\to\infty}(m(x_n,x_m) - m_{x_n,x_m}) = 0.$$

Define

$$M^*(x,y) := m(x,y) - m_{x,y}, \quad \forall x,y \in X.$$

If $\lim_{n,m\to\infty} M^*(x_n,x_m) \neq 0$, so there exist $\varepsilon > 0$ and $\{l_k\} \subset \mathbb{N}$ such that

$$M^*(x_{l_k}, x_{n_k}) \geq \varepsilon.$$

suppose that $k$ is the smallest number which satisfies above equation such that

$$M^*(x_{l_k-1}, x_{n_k}) < \varepsilon.$$

Now by (m4) we have

$$\varepsilon \leq M^*(x_{l_k}, x_{n_k}) \leq M^*(x_{l_k}, x_{l_k-1}) + M^*(x_{l_k-1}, x_{n_k}) < m_{l_k-1} + \varepsilon,$$

Thus

$$\lim_{k\to\infty} M^*(x_{l_k}, x_{n_k}) = \varepsilon,$$

which means

$$\lim_{k\to\infty} m(x_{l_k}, x_{n_k}) - m_{x_{l_k},x_{n_k}} = \varepsilon.$$

On the other hand

$$\lim_{k\to\infty} m_{x_{l_k},x_{n_k}} = 0,$$

so we have

$$\lim_{k\to\infty} m(x_{l_k}, x_{n_k}) = \varepsilon.$$

Now

$$\begin{aligned}
\varepsilon &\leq M^*(x_{l_k}, x_{n_k}) \leq M^*(x_{l_k}, x_{l_{k+1}}) + M^*(x_{l_{k+1}}, x_{n_{k+1}}) + M^*(x_{n_{k+1}}, x_{n_k}) \\
&= m_{l_k} - m_{x_{l_k},x_{l_k+1}} + m(x_{l_k+1}, x_{n_k+1}) - m_{x_{l_k+1},x_{n_k+1}} + m_{n_k} - m_{x_{n_k},x_{n_k+1}}.
\end{aligned}$$

Moreover by (i) since T is triangular $\alpha$-orbital admissible we have

$$\alpha(x_{l_k}, x_{n_k}) \geq 1. \tag{3.10}$$

So by (3.3) and (3.10)

$$\begin{aligned}
\psi(\varepsilon) \leq \limsup_{k\to\infty} \psi(M^*(x_{l_k}, x_{n_k})) &\leq \limsup_{k\to\infty} \psi(m(x_{l_k+1}, x_{n_k+1})) \\
&\leq \limsup_{k\to\infty} \alpha(x_{l_k}, x_{n_k})\psi(m(x_{l_k+1}, x_{n_k+1})) \\
&\leq \limsup_{k\to\infty} F(\psi(M(x_{l_k}, x_{n_k})), \varphi((M(x_{l_k}, x_{n_k}))).
\end{aligned}$$

which

$$M(x_{l_k},x_{n_k}) \;=\; \max\big\{m(x_{l_k},x_{n_k}),m(x_{l_k},x_{l_k+1}),m(x_{n_k},x_{n_k+1})\big\}.$$

On the other hand, by (m4) we have

$$\begin{aligned} m(x_{l_k},x_{n_k+1})-m_{x_{l_k},x_{n_k+1}} &\le m(x_{l_k},x_{l_k+1})-m_{x_{l_k},x_{l_k+1}} \\ &+ m(x_{l_k+1},x_{n_k+1})-m_{x_{l_k+1},x_{n_k+1}} \\ &\le m(x_{l_k},x_{l_k+1})-m_{x_{l_k},x_{l_k+1}} \\ &+ m(x_{l_k+1},x_{n_k})-m_{x_{l_k+1},x_{n_k}} \\ &+ m(x_{n_k},x_{n_k+1})-m_{x_{n_k},x_{n_k+1}}. \end{aligned}$$

Thus

$$\limsup_{k\to\infty} m(x_{l_k},x_{n_k+1}) \le \limsup_{k\to\infty} m(x_{l_k+1},x_{n_k}),$$

similarly

$$\limsup_{k\to\infty} m(x_{l_k+1},x_{n_k}) \le \limsup_{k\to\infty} m(x_{l_k},x_{n_k+1}).$$

Therefore

$$\limsup_{k\to\infty} m(x_{l_k+1},x_{n_k}) = \limsup_{k\to\infty} m(x_{l_k},x_{n_k+1}).$$

On the other hand

$$\begin{aligned} \limsup_{k\to\infty} m(x_{l_k+1},x_{n_k}) &\le \limsup_{k\to\infty}(m(x_{l_k+1},x_{l_k})-m_{x_{l_k+1},x_{l_k}} \\ &+ m(x_{l_k},x_{n_k})-m_{x_{l_k},x_{n_k}}+m_{x_{l_k+1},x_{n_k}}) \\ &= 0+\lim_{k\to\infty} m(x_{l_k},x_{n_k})-0+0=\varepsilon. \end{aligned}$$

So we see

$$\lim_{k\to\infty} M(x_{l_k},x_{n_k})=\varepsilon.$$

Hence

$$\psi(\varepsilon)\le F(\psi(\varepsilon),\varphi(\varepsilon))\le\psi(\varepsilon).$$

So we get

$$F(\psi(\varepsilon),\varphi(\varepsilon))=\psi(\varepsilon),$$

and so $\psi(\varepsilon)=0$, or $\varphi(\varepsilon)=0$, hence we have $\varepsilon=0$. which is contradiction and then $\{x_n\}$ is $M$-Cauchy. So by completeness of X, $x_n\to x$ for some $v\in X$ in $\tau_m$ topology which means

$$\lim_{n\to\infty}(m(x_n,v)-m_{x_n,v})=0.$$

and

$$\lim_{n\to\infty}(M_{x_n,v}-m_{x_n,v})=0.$$

We have

$$m_{x_n,v}=\min\{m(x_n,x_n),m(v,v)\}.$$

So

$$\lim_{n\to\infty} m_{x_n,v}=0,$$

hence $\lim_{n\to\infty} m(x_n, v) = 0$ and by Remark 1.10 $m(v, v) = 0$. Now we want to show that $v$ is the fixed point of $T$. $T$ is continuous so

$$\lim_{n\to\infty}(m(Tx_n, Tv) - m_{Tx_n,Tv}) = 0,$$

that means

$$\lim_{n\to\infty}(m(x_{n+1}, Tv) - m_{x_{n+1},Tu}) = 0,$$

and similar to the above, we have $\lim_{n\to\infty} m_{x_{n+1},Tv} = 0$, hence $\lim_{n\to\infty} m(x_{n+1}, Tv) = 0$ and by Remark 1.10, $m(Tv, Tv) = 0$. On the other hand, $x_n \to v$ as $n \to \infty$ so by Lemma 2.3, we get

$$(m(x_n, Tv) - m_{x_n,Tv}) \to (m(v, Tv) - m_{v,Tv}) = m(v, Tv) \quad \text{as} \quad n \to \infty,$$

but we have

$$(m(x_n, Tv) - m_{x_n,Tv}) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus

$$m(v, Tv) = 0,$$

therefore $m(v, Tv) = m(Tv, Tv) = m(v, v) = 0$ and by (m1) we get

$$Tv = v.$$

□

**Theorem 3.5.** *Let $(X, m)$ be a complete $M$-metric space, and let $T : X \to X$ be a generalized $\alpha$-admissible $F(\alpha, \psi)$-contractive mapping. and satisfies the following conditions:*
*(i) $T \in \mathcal{O}$;*
*(ii) there exists $x_0 \in X$ such that $\alpha(x_0, Tx_0) \geq 1$;*
*(iii) if $\{x_n\}$ is a sequence in $X$ such that $\alpha(x_n, x_{n+1}) \geq 1$ for all $n$ and $x_n \to x \in X$ as $n \to \infty$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, x) \geq 1$ for all $k$.*
*Then $T$ has a fixed point $v \in X$ and $\{T^n x_0\}$ converges to $v$.*

**Proof .** Following the proof of Theorem 3.4, we know that the sequence $\{x_n\}$ defined by $x_{n+1} = Tx_n$ for all $n \geq 0$, converges for some $v \in X$. From (3.10) and (iii) , there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\alpha(x_{n_k}, v) \geq 1$ for all $k$. Applying (3.3), for all $k$, we get that

$$\begin{aligned} \psi(m(x_{n_k+1}, Tv)) &\leq \alpha(x_{n_k}, v)\psi(m(Tx_{n_k}, Tv)) \leq F(\psi(M(x_{n_k}, v)), \varphi(M(x_{n_k}, v))) \\ &\leq \psi(M(x_{n_k}, v)) \to \psi(m(v, Tv)) \text{ as } n \to \infty. \end{aligned} \tag{3.11}$$

On the other hand by Lemma 2.4

$$\lim_{k\to\infty} m(x_{n_k+1}, Tv) = m(v, Tv). \tag{3.12}$$

Now by (3.11) and (3.12) we have

$$F(\psi(m(v, Tv)), \varphi(m(v, Tv))) = \lim_{k\to\infty} F(\psi(M(x_{n_k}, v)), \varphi(M(x_{n_k}, v))) = \psi(m(v, Tv)).$$

So by the property of $F$, we get $\psi(m(v, Tv)) = 0$, thus $m(v, Tv) = 0$. Therefore by (3.12) we get

$$m(x_{n_k+1}, Tv) \to 0 \text{ as } k \to \infty.$$

Therefore as in proof of Theorem 3.4 we have $Tv = v$. □

For the uniqueness of a fixed point of a generalized $\alpha$-admissible $F(\alpha,\psi)$-contractive mapping, we shall suggest the following hypothesis.

$(*)$ For all $x, y \in \mathrm{Fix}(T)$, we have $\alpha(x,y) \geq 1$.

Here, $\mathrm{Fix}(T)$ denotes the set of fixed points of $T$.

**Theorem 3.6.** *Adding condition $(*)$ to the hypotheses of Theorem 3.4 (resp. Theorem 3.5), we obtain that $v$ is the unique fixed point of $T$.*

**Proof .** Suppose that $u, v \in X$ are two fixed points of $T$.

$$\begin{aligned}\psi(m(u,u)) = \psi(m(Tu,Tu)) &\leq \alpha(u,u)\psi(m(Tu,Tu)) \\ &\leq F(\psi(M(u,u)), \varphi(M(u,u))) \\ &= F(\psi(m(u,u)), \varphi(m(u,u))) \leq \psi(m(u,u),\end{aligned}$$

so we have

$$F(\psi(m(u,u)), \varphi(m(u,u))) = \psi(m(u,u),$$

and by the property of functions $F$ and $\psi$ we get $m(u,u) = 0$. By similar way we have $m(u,v) = m(v,v) = 0$. hence by $(m_1)$

$$u = v.$$

□

If we let $\alpha(x,y) = 1$ for all $x, y \in X$, then we get the following Corollaries.

**Corollary 3.7.** Let $(X,m)$ be a complete $M$-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

$$\psi(m(Tx,Ty)) \leq F(\psi(M(x,y)), \varphi(M(x,y))), \tag{3.13}$$

Where $M(x,y) = \max\{m(x,y), m(x,Tx), m(y,Ty)\}$, $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then $T$ has unique fixed point $x^* \in X$ and $\{T^n x_0\}$ converges to $x^*$.

**Corollary 3.8.** Let $(X,m)$ be a complete $M$-metric space and $T : X \to X$ be a self-mapping satisfying

$$\psi(m(Tx,Ty)) \leq F(\psi(m(x,y)), \varphi(m(x,y))), \tag{3.14}$$

Where $\psi \in \Psi$, $\varphi \in \Phi$ and $F \in \mathcal{C}$. Then $T$ has unique fixed point $v \in X$ and $\{T^n x_0\}$ converges to $v$.

If we let $F(s,t) = ks$ in Corollary 3.8, we get the following result.

**Corollary 3.9.** Let $(X,m)$ be a complete $M$-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

$$\psi(m(Tx,Ty)) \leq k\psi(m(x,y))$$

Then, $T$ has unique fixed point $v \in X$, and $\{T^n x_0\}$ converges to $v$.

If we let $\psi(t) = t$ in Corollary 3.8, we get the following result.

**Corollary 3.10.** Let $(X, m)$ be a complete $M$-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

$$m(Tx, Ty) \leq F(m(x, y), \varphi(m(x, y))),$$

Then, $T$ has unique fixed point $v \in X$, and $\{T^n x_0\}$ converges to $v$.

If we take $\psi(t) = t$ in Corollary 3.7, we get the following.

**Corollary 3.11.** Let $(X, m)$ be a complete $M$-metric space, and let $T : X \to X$ be a map. Suppose that the following conditions are satisfied:

$$m(Tx, Ty) \leq F(M(x, y), \varphi(M(x, y))),$$

Where $M(x, y) = \max\{m(x, y), m(x, Tx), m(y, Ty)\}$ and $\varphi \in \Phi, F \in \mathcal{C}$. Then, $T$ has a unique fixed point $v \in X$, and $\{T^n x_0\}$ converges to $v$.

## 4. Consequences

By Corollaries 3.7 and 3.8, we obtain the following corollaries as an extension of several known results in the literature.

**Corollary 4.1.** Let $(X, m)$ be a complete $M$-metric space and $T : X \to X$ be a continuous map. Assume that there exists two function $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that

$$\psi(m(Tx, Ty)) \leq \beta(\psi(m(x, y)))\psi(m(x, y)), \tag{4.1}$$

for all $x, y \in X$. Then $T$ has unique fixed point.

**Corollary 4.2.** Let $(X, m)$ be a complete $M$-metric space and $T : X \to X$ be a continuous map. Assume that there exists two function $\beta \in \mathcal{F}$ and $\psi \in \Psi$ such that for all $x, y \in X$

$$\psi(m(Tx, Ty)) \leq \beta(\psi(M(x, y)))\psi(M(x, y)), \tag{4.2}$$

where $M(x, y) = \max\{m(x, y), m(x, Tx), m(y, Ty)\}$.Then $T$ has unique fixed point.

If we let $\psi(t) = t$, we get the following corollary that proved in [8, 9]:

**Corollary 4.3.** ([8]) Let $(X, m)$ be a complete $M$-metric space and $T : X \to X$ be a continuous map. Assume that there exists a function $\beta \in \mathcal{F}$ such that

$$m(Tx, Ty) \leq \beta(m(x, y))m(x, y), \tag{4.3}$$

for all $x, y \in X$. Then $T$ has unique fixed point.

**Corollary 4.4.** *Let $(X, m)$ be a complete $M$-metric space and $T : X \to X$ be a continuous map. Assume that there exists two function $\beta \in \mathcal{F}$ such that for all $x, y \in X$*

$$m(Tx, Ty) \leq \beta(M(x, y))M(x, y), \tag{4.4}$$

*where $M(x, y) = \max\{m(x, y), m(x, Tx), m(y, Ty)\}$. Then $T$ has unique fixed point.*

## 5. Fixed point theorems on $M$-metric spaces endowed with a partial order

In this section, we state some consequences of our main results in the context of a partially ordered $M$-metric space.

**Definition 5.1.** Let $(X,\preceq)$ be a partially ordered set and $T:X\to X$ be a given mapping. We say that $T$ is nondecreasing with respect to $\preceq$ if

$$x\preceq y \Longrightarrow Tx\preceq Ty,\quad \forall x,y\in X.$$

**Definition 5.2.** Let $(X,\preceq)$ be a partially ordered set. A sequence $\{x_n\}\subset X$ is said to be nondecreasing with respect to $\preceq$ if $x_n\preceq x_{n+1}$ for all $n$.

**Definition 5.3.** Let $(X,\preceq)$ be a partially ordered set and $m$ be an $M$-metric on $X$. We say that $(X,\preceq,m)$ is regular if for every nondecreasing sequence $\{x_n\}\subset X$ such that $x_n\to x\in X$ as $n\to\infty$, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}\preceq x$ for all $k$.

We have the following results.

**Corollary 5.4.** *Let $(X,\preceq)$ be a partially ordered set and $m$ be an $M$-metric on $X$ such that $(X,m)$ is complete. Let $T:X\to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists two functions $\psi\in\Psi$ and $\beta\in\mathcal{F}$such that*

$$\psi(m(Tx,Ty))\le\beta(\psi(M(x,y)))\psi(M(x,y)),$$

*for all $x,y\in X$ with $x\succeq y$. Suppose also that the following conditions hold:*

*(i) there exists $x_0\in X$ such that $x_0\preceq Tx_0$;*

*(ii) $T$ is continuous or $(X,\preceq,m)$ is regular.*

*Then $T$ has a fixed point. Moreover, if for all $x,y\in X$ there exists $z\in X$ such that $x\preceq z$ and $y\preceq z$, we have uniqueness of the fixed point.*

**Proof .** Let us discuss the case that $(X,\preceq,m)$ is regular. Let $\{x_n\}$ be a sequence in $X$ such that $x_n\to x\in X$ as $n\to\infty$. From the regularity hypothesis, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that $x_{n(k)}\preceq x$ for all $k$. In this case, the existence of a fixed point follows from Corollary 3.7. The uniqueness follows from Corollary 3.7. □

The following results are immediate consequences of Corollary 5.4.

**Corollary 5.5.** Let $(X,\preceq)$ be a partially ordered set and $m$ be an $M$-metric on $X$ such that $(X,m)$ is complete. Let $T:X\to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists two functions $\psi\in\Psi$ and $\beta\in\mathcal{F}$such that

$$\psi(m(Tx,Ty))\le\beta(\psi(m(x,y)))\psi(m(x,y)),$$

for all $x,y\in X$ with $x\succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0\in X$ such that $x_0\preceq Tx_0$;

(ii) $T$ is continuous or $(X, \preceq, m)$ is regular.

Then $T$ has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

The following two corollaries can be concluded from the above results by taking $\psi(t) = t$.

**Corollary 5.6.** Let $(X, \preceq)$ be a partially ordered set and $m$ be an $M$-metric on $X$ such that $(X, m)$ is complete. Let $T : X \to X$ be a nondecreasing mapping with respect to $\preceq$. Suppose that there exists a function $\beta \in \mathcal{F}$ such that

$$m(Tx, Ty) \leq \beta(m(x, y))m(x, y),$$

for all $x, y \in X$ with $x \succeq y$. Suppose also that the following conditions hold:

(i) there exists $x_0 \in X$ such that $x_0 \preceq Tx_0$;

(ii) $T$ is continuous or $(X, \preceq, m)$ is regular.

Then $T$ has a fixed point. Moreover, if for all $x, y \in X$ there exists $z \in X$ such that $x \preceq z$ and $y \preceq z$, we have uniqueness of the fixed point.

In [5] Haghi *et al.* proved the following lemma.

**Lemma 5.7.** Let $X$ be a nonempty set and $f : X \to X$ a function. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one to one.

**Theorem 5.8.** *Let $(X, m)$ be an $M$-metric space and let $T, f : X \to X$ be two self maps such that $T(X) \subseteq f(X)$ and $f(X)$ is a complete subset of $X$. If there exists functions $F \in \mathcal{C}$, $\psi \in \Psi$ and $\varphi \in \Phi$ such that*

$$\psi(m(Tx, Ty)) \leq F(\psi(m(fx, fy)), \varphi(m(fx, fy))) \tag{5.1}$$

*for all $x, y \in X$. Then $T$ and $f$ have unique point of coincidence in $X$. Moreover, if $T$ and $f$ are weakly compatible, then $T$ and $f$ have a unique common fixed point.*

**Proof .** By Lemma 5.7, there exists $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one to one. We define a map $g : f(E) \to f(E)$ by $g(fx) = Tx$. Clearly, $g$ is well defined, since $f$ is one to one. Now, using (5.1), we have

$$\begin{aligned} \psi(m(g(fx), g(fy))) &= \psi(m(Tx, Ty)) \\ &\leq F(\psi(m(fx, fy)), \varphi(m(fx, fy))) \end{aligned}$$

for all $fx, fy \in f(E)$. Since $f(E) = f(X)$ is complete, therefore by Corollary 3.7, there exists $z \in X$ such that $g(fz) = fz$ which imply $Tz = fz$. Hence, $T$ and $f$ have a coincidence point. Again, if $w$ is another coincidence point of $T$ and $f$ such that $z \neq w$, then by (5.1)

$$\begin{aligned} \psi(m(Tw, Tz)) &\leq F(\psi(m(fw, fz)), \varphi(m(fw, fz))) \\ &\leq \psi(m(Tw, Tz)), \end{aligned}$$

then

$$F(\psi(m(fw, fz)), \varphi(m(fw, fz))) = \psi(m(Tw, Tz)) = \psi(m(fw, fz)),$$

and so we get $\psi(m(fw, fz)) = 0$ or $\varphi(m(fw, fz)) = 0$, hence $m(fw, fz) = 0$. Similarly we have $m(fw, fw) = 0$, and $m(fz, fz) = 0$. Therefore by (m4) $fw = fz$, which is a contradiction. Hence $z$ is unique coincidence point of $T$and $f$.It is clear that $T$ and $f$ have a unique common fixed point whenever $T$and $f$ are weakly compatible. □

**Theorem 5.9.** *Let $(X,m)$ be an $M$-metric space and let $T,f:X\to X$ be two self maps such that $T(X)\subseteq f(X)$ and $f(X)$ is a complete subset of $X$. If there exists two function $F\in\mathcal{C}$ , $\psi\in\Psi$ and $\varphi\in\Phi$ such that for all $x,y\in X$*

$$\psi(m(Tx,Ty))\leq F(\psi(M(fx,fy)),\varphi(M(fx,fy))) \tag{5.2}$$

*where, $M(x,y)=\max\{m(fx,fy),m(fx,Tx),m(fy,Ty)$. Then $T$ and $f$ have a unique point of coincidence in $X$. Moreover, if $T$ and $f$ are weakly compatible, then $T$ and $f$ have a unique common fixed point.*

**Proof .** Here, we utilize Corollary 4.4 instead of Corollary 4.3 in the proof of Theorem 5.8 and the proof completely follows the lines of the proof of Theorem 5.8 and hence it is omitted. □

Now, we present an example in support of the proved results.

**Example 5.10.** Let $X=\{0,\frac{1}{4},\frac{1}{2}\}$, $m:X\times X\to\mathbb{R}^+$ be defined by $m(x,y)=\frac{x+y}{2}$ for all $x,y\in X$ and $\psi:\mathbb{R}^+\to\mathbb{R}^+$ by $\psi(t)=\frac{t}{4}$, then $\psi\in\Psi$ and $(X,m)$ is a complete $M$-metric space. Define $T:X\to X$ by

$$Tx=\begin{cases} x^2, & x=0,\frac{1}{2};\\ 0, & x=\frac{1}{4}.\end{cases}$$

We want to show that the condition of Corollary 4.1 holds. Without loss of generality, we assume that $x\leq y$.
To proof we consider some cases.
Case 1: If $x=y=0$ the claim is obvious and hence (4.3) trivially holds.
Case 2: If $x=y=\frac{1}{4}$, we have

$$\begin{aligned}\beta(\psi(m(x,y)))\psi(m(x,y))-\psi(m(Tx,Ty)) &= \beta(\psi(m(\frac{1}{4},\frac{1}{4})))\psi(m(\frac{1}{4},\frac{1}{4}))-\psi(m(0,0))\\ &= \beta(\psi(\frac{1}{4})\psi(\frac{1}{4})-\psi(0)\\ &= \beta(\frac{1}{16})\frac{1}{16}-0>0.\end{aligned}$$

Case 3: If $x=y=\frac{1}{2}$, we have

$$\begin{aligned}\beta(\psi(m(x,y)))\psi(m(x,y))-\psi(m(Tx,Ty)) &= \beta(\psi(m(\frac{1}{2},\frac{1}{2})))\psi(m(\frac{1}{2},\frac{1}{2}))-\psi(m(\frac{1}{4},\frac{1}{4}))\\ &= \beta(\psi(\frac{1}{2}))\psi(\frac{1}{2})-\psi(m(\frac{1}{4},\frac{1}{4}))\\ &= \beta(\frac{1}{8})\frac{1}{8}-\psi(\frac{1}{4})\\ &= \frac{1}{1+\frac{1}{8}}\frac{1}{8}-\frac{1}{8}=\frac{1}{9}-\frac{1}{16}>0.\end{aligned}$$

Case 4: If $x=0,\quad y=\frac{1}{4}$, we get

$$\begin{aligned}\beta(\psi(m(x,y)))\psi(m(x,y))-\psi(m(Tx,Ty)) &= \beta(\psi(m(0,\frac{1}{4})))\psi(m(0,\frac{1}{4}))-\psi(m(0,0))\\ &= \beta(\psi(\frac{1}{8})\psi(\frac{1}{8})-\psi(0)\\ &= \beta(\frac{1}{32})\frac{1}{32}-0>0.\end{aligned}$$

Case 5: If $x=0, \quad y=\frac{1}{2}$, we have

$$\begin{aligned}
\beta(\psi(m(x,y)))\psi(m(x,y))-\psi(m(Tx,Ty)) &= \beta(\psi(m(0,\frac{1}{2})))\psi(m(0,\frac{1}{2}))-\psi(m(0,\frac{1}{4}))\\
&= \beta(\psi(\frac{1}{4})\psi(\frac{1}{4})-\psi(\frac{1}{8})\\
&= \beta(\frac{1}{16})\frac{1}{16}-\frac{1}{32}\\
&= \frac{1}{1+\frac{1}{16}}\frac{1}{16}-\frac{1}{32}=\frac{1}{17}-\frac{1}{32}>0.
\end{aligned}$$

Case 6: If $x=\frac{1}{4}$ and $y=\frac{1}{2}$, we get

$$\begin{aligned}
\beta(\psi(m(x,y)))\psi(m(x,y))-\psi(m(Tx,Ty)) &= \beta(\psi(m(\frac{1}{4},\frac{1}{2})))\psi(m(\frac{1}{4},\frac{1}{2}))-\psi(m(0,\frac{1}{4}))\\
&= \beta(\psi(\frac{3}{8})\psi(\frac{3}{8})-\psi(\frac{1}{8})\\
&= \beta(\frac{3}{32})\frac{3}{32}-\frac{1}{32}\\
&= \frac{1}{1+\frac{3}{32}}\frac{3}{32}-\frac{1}{32}=\frac{3}{35}-\frac{1}{32}=\frac{61}{1120}>0.
\end{aligned}$$

Hence the inequality (4.4) is satisfied. So, $T$ is contraction type map. Hence all the condition of Corollary 4.1 are satisfied and consequently $T$ has a unique fixed point. Here 0 is such point.

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