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Simulation and perturbation analysis of escape oscillator

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Abstract

The dynamical behaviour of the forced escape oscillator, which depends on the parameter values we considered, have been studied numerically using the techniques of phase portraits and Poincaré sections. Also, we employed perturbation methods such as Lindstedt's method to obtain the frequency-amplitude relation of escape oscillator.

Keywords: Escape oscillator; Perturbation analysis; Lindstedt's method.
2010 MSC: Primary 34C15; Secondary 34D10.

1. Introduction

Oscillatory phenomena are ubiquitous in physical sciences and engineering world. Pulsating stars are studied in astrophysics, while in astronomy the motions of the planets in their orbits have an oscillatory nature. Physics is full of oscillatory phenomena, such as, electromagnetic fields whose intensity changes periodically with time, known as electromagnetic waves, atomic vibrations in solid state physics and modes of oscillation of the atom nucleus.

Electrical and mechanical oscillators, as well as vibrations in structures are everyday elements in the world of engineering. Oscillatory behaviours are often found in the life sciences too, these include the circadian rhythms, the beats of the hearts, and the oscillations of the membrane potential in the axons of the neurons, among many others. The study of the dynamical behaviour of oscillators is therefore a central issue in sciences and engineering. Oscillatory phenomena can easily modeled with the help of differential equations.

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Consider the particle moves in a force field which is generated by a potential $V(x)$, then the general equation of motion of the particle is given by

$$\frac{d^2x}{dt^2} + \frac{dV(x)}{dx} = 0 \tag{1.1}$$

Therefore, the general form of damped, driven oscillator is written in the following form:

$$\frac{d^2x}{dt^2} + f\left(x, \frac{dx}{dt}\right) + \frac{dV(x)}{dx} = F cos\,(\omega t) \tag{1.2}$$

Different oscillators may be obtained depending on the potential $V(x)$ acting on the particle. If one take the potential $V(x) = -cos(x)$ the equation (1.1) reduces to the well known pendulum equation

$$\frac{d^2x}{dt^2} + sin(x) = 0$$

If we take $V(x)$, to be a polynomial function of fourth order in x, then we get the Duffing oscillator, and if we take $V(x)$, to be a polynomial function of third order in x then we get the escape oscillator.

Taking $V(x) = \alpha\frac{x^2}{2} + \beta\frac{x^3}{3}$, in (1.2), we obtain

$$\frac{d^2x}{dt^2} + \mu\frac{dx}{dt} + \alpha x + \beta x^2 = F cos\,(\omega t) \tag{1.3}$$

which is the differential equation of damped, forced escape oscillator.

The dynamics of this simple nonlinear oscillator (1.3), known as escape oscillator (Helmholtz oscillator) include, the capsizing of a ship [10], nonlinear dynamics of a drop in a time-periodic flow [6] or in a time-periodic electric field [5]. It appears in relation to the randomization of solitary-like waves in boundary-layer flows [1] and in the three-wave interaction, also referred to as resonant triads [3]. It also, gives a model equation for nonlinear soil-mass oscillator which is useful for the study of landmine oscillator [7]. In [9], the numerical solution of the heat equation is obtained using finite-difference schemes. Cai et al. [2], used the multiple scale method to obtain the asymptotic solution of quadratic and cubic nonlinear oscillator. Cvetićanin [4], used the Lambert W-function to analyze the oscillations of a system with strong quadratic damping.

The aim of this paper, is to study the various cases of eq. (1.3) both numerically and analytically. We will also use Lindstedt's perturbation method to derive frequency-amplitude relation of escape oscillator.

2. Equilibrium Points and Dynamics of the unforced, undamped escape oscillator

The differential equation of unforced, undamped escape oscillator is given by

$$x^{''} + \alpha x + \beta x^2 = 0 \tag{2.1}$$

From (2.1), we obtain the autonomous dynamical system

$$\begin{cases} x^{'} = y; \\ y^{'} = -\alpha x - \beta x^2 \end{cases} \tag{2.2}$$

with $x^{'} = 0$, we observe that $y = 0$. The condition $y^{'} = 0$ implies that $x = 0$ or $x = -\frac{\alpha}{\beta}$.

The equilibrium points of the unforced, undamped escape oscillator are thus of the form (0,0) and (x,0). The Jacobian of system of equations (2.2) is given by

$$A = [\begin{pmatrix} 0 & 1 \\ -\alpha - 2\beta x & 0 \end{pmatrix}.] \tag{2.3}$$

It's eigenvalues satisfy

$$\lambda^2 = -\alpha - 2\beta \tag{2.4}$$

where x, denotes the x-coordinate of an equilibrium point.

When $\alpha = 1, \beta = 1$ there is a spiral equilibrium point at (0,0) with eigenvalues $\lambda = \pm\iota$. A phase portrait for this situation is shown in Figure 1.

When $\alpha = -1, \beta = 1$ there is a spiral equilibrium point at (1,0) with eigenvalues $\lambda = \pm\iota$. A phase portrait for this situation is shown in Figure 2.

When $\alpha = 1, \beta = -1$ there is a spiral equilibrium point at (0,0) with eigenvalues $\lambda = \pm\iota$. A phase portrait for this situation is shown in Figure 3.

When $\alpha = -1, \beta = -1$ there is a spiral equilibrium point at (-1,0) with eigenvalues $\lambda = \pm\iota$. A phase portrait for this situation is shown in Figure 4.

When $\alpha = 1, \beta = -0.1$ there is a spiral equilibrium point at (0,0) with eigenvalues $\lambda = \pm\iota$. A phase portrait for this situation is shown in Figure 5.

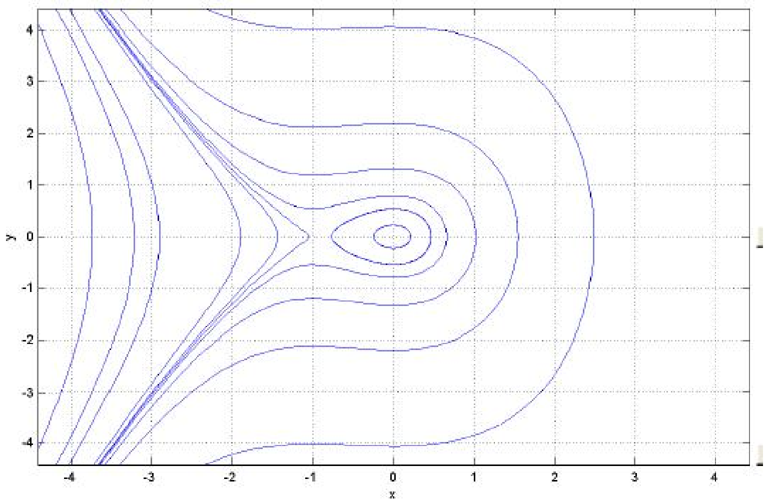

Fig. 1: Phase plane of eq. 2.1 for $\alpha = 1$ and $\beta = 1$.

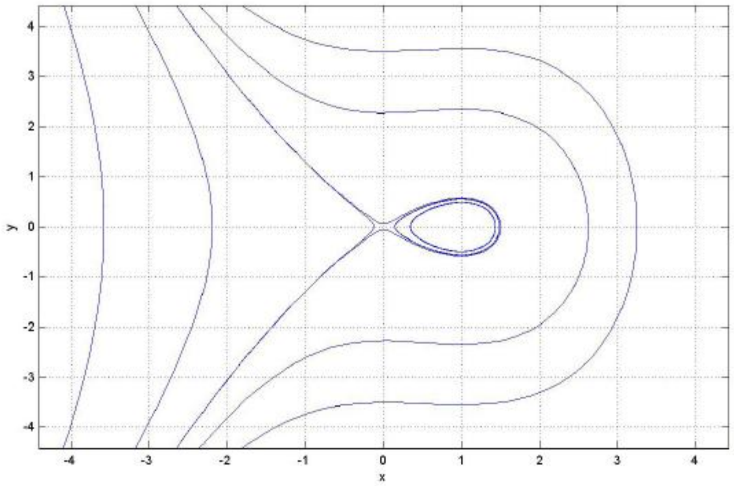

Fig. 2: Phase plane of eq. 2.1 for $\alpha = -1$ and $\beta = 1$.

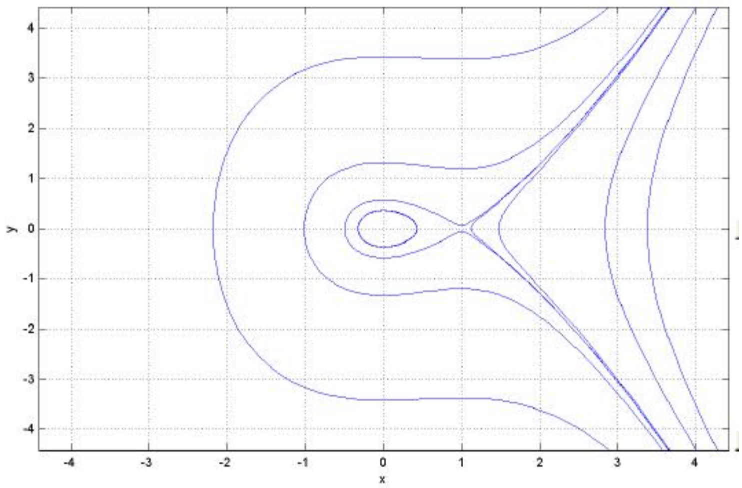

Fig. 3: Phase plane of eq. 2.1 for $\alpha = 1$ and $\beta = -1$.

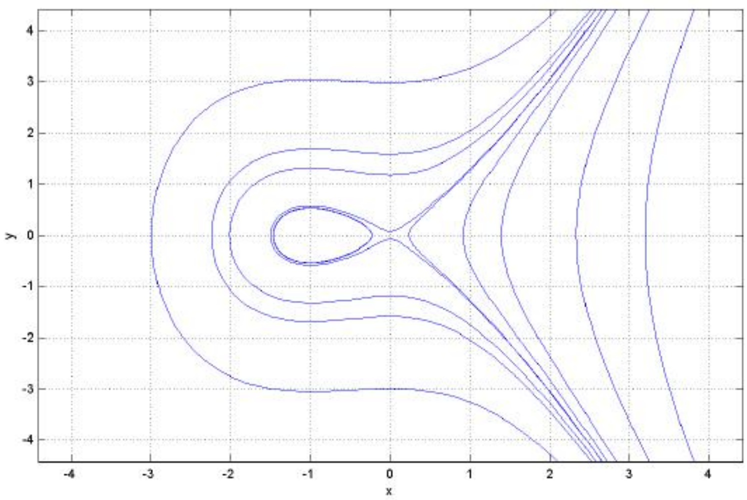

Fig. 4: Phase plane of eq. 2.1 for $\alpha = -1$ and $\beta = -1$.

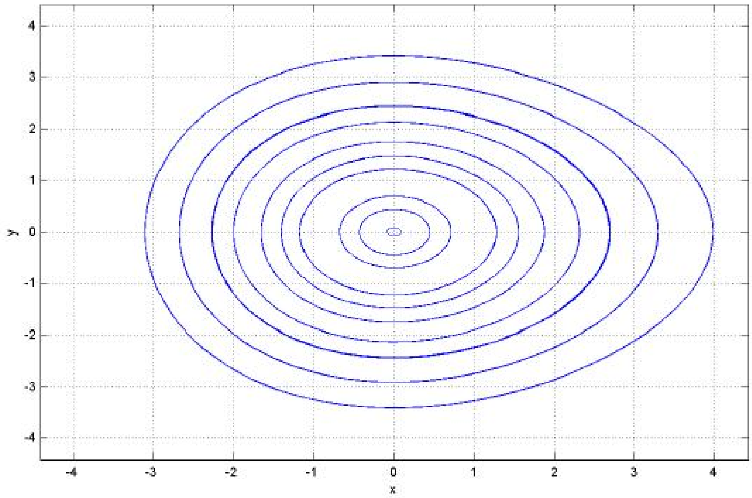

Fig. 5: Phase plane of eq. 2.1 for $\alpha = 1$ and $\beta = -0.1$.

3. Dynamics of the forced, escape oscillator with single sinusoidal forcing term

The differential equation of damped, driven escape oscillator with single sinusoidal force is given by

$$x'' + \mu x' + \alpha x + \beta x^2 = F cos\left(\omega t\right) \tag{3.1}$$

Setting $\mu = 0$, i.e. in the absence of damping, (3.1) is written as the following system of non-autonomous first order differential equations

$$\begin{cases} x' = y; \\ y' = -\alpha x - \beta x^2 + F cos\left(\omega t\right) \end{cases} \tag{3.2}$$

The Phase portrait of (3.2), simulated in Figure 6 is obtained when $\alpha = -1, \beta = 1, F = 1.5, \omega = 4.2$.

The Poincaré map of (3.2), depicted in Figure 7 is obtained for the parameter values $\alpha = -1, \beta = 1, F = 1.5, \omega = 4.2$. The Poincaré map reveals the several resonances.

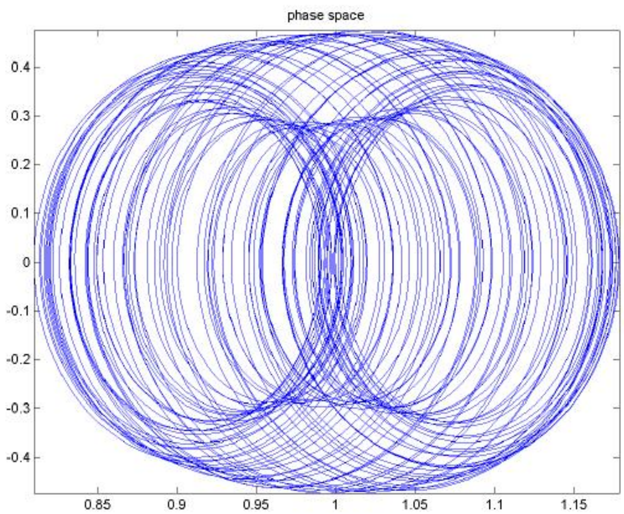

Fig. 6: Phase plane of eq. 3.2 for $\alpha = -1$ and $\beta = 1$.

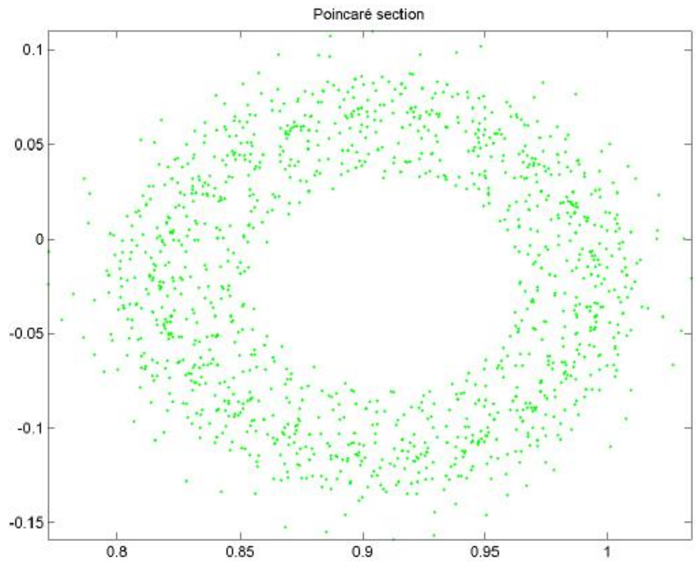

Fig. 7: Poincaré Section of eq. 3.2 for $\alpha = -1$ and $\beta = 1$.

4. Dynamics of the forced, escape oscillator with multiple-frequency sinusoidal forcing term

The differential equation of damped, driven escape oscillator with single sinusoidal force is given by

$$x'' + \mu x' + \alpha x + \beta x^2 = F_1 cos\left(\omega_1 t\right) + F_2 cos\left(\omega_2 t\right) \tag{4.1}$$

For $\mu = 0$, i.e. in the absence of damping, (4.1) is written as the following system of non-autonomous first order differential equations

$$\begin{cases} x' = y; \\ y' = -\alpha x - \beta x^2 + F_1 cos\left(\omega_1 t\right) + F_2 cos\left(\omega_2 t\right) \end{cases} \tag{4.2}$$

The Phase portrait of (4.2), simulated in Figure 8 is obtained when $\alpha = -1, \beta = 1, F_1 = 1.5 = F_2, \omega_1 = 4.2 = \omega_2$.

The Poincaré map of (4.2), depicted in Figure 9 is obtained for the parameter values $\alpha = -1, \beta = 1, F_1 = 1.5 = F_2, \omega_1 = 4.2 = \omega_2$. The Poincaré map reveals the several different resonances.

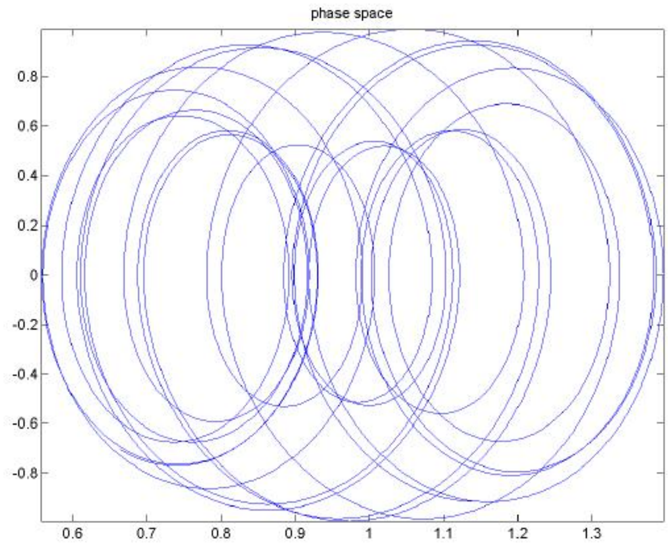

Fig. 8: Phase space of eq. 4.2 for $\alpha = -1$ and $\beta = 1$.

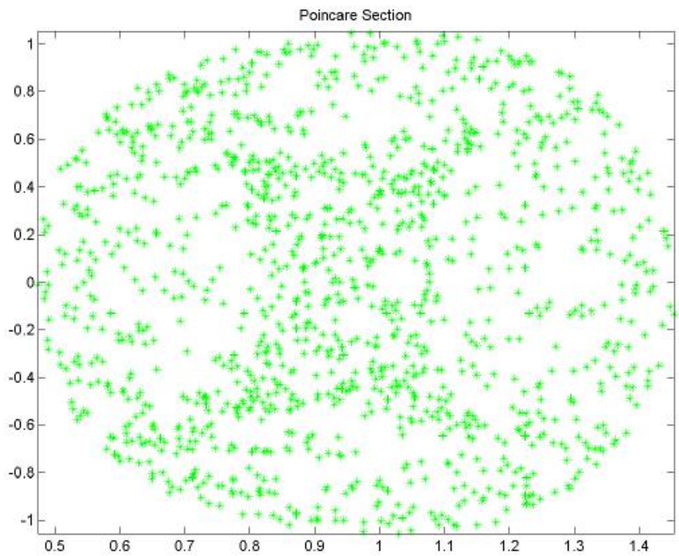

Fig. 9: Poincaré Section of eq. 4.2 for $\alpha = -1$ and $\beta = 1$.

5. Perturbation Analysis

Lindstedt's method [8], is a simple singular perturbation scheme which is used to derive the relationship between period and amplitude.

Consider the unforced, perturbed escape oscillator of the following form:

$$\frac{d^2x}{dt^2} + x + \epsilon x^2 = 0 \tag{5.1}$$

Using,

$$\tau = \omega t, \omega = 1 + \epsilon\omega_1 + \epsilon^2\omega_2 + \dots \tag{5.2}$$

in eq.(5.1), we get

$$\omega^2 \frac{d^2x}{dt^2} + x + \epsilon x^2 = 0 \tag{5.3}$$

Now expanding x in a power series in ϵ:

$$x(\tau) = x_0(\tau) + \epsilon x_1(\tau) + \epsilon^2 x_2(\tau) + \ldots \tag{5.4}$$

Substitution of second of (5.2) and (5.4) into the differential eq. (5.3) and comparing the terms of same order in ϵ yield the system of equations:

$$\frac{d^2x_0}{d\tau^2} + x_0 = 0 \tag{5.5}$$

$$\frac{d^2x_1}{d\tau^2} + x_1 = -2\omega_1 \frac{d^2x_0}{d\tau^2} - x_0^2 \tag{5.6}$$

$$\frac{d^2x_2}{d\tau^2} + x_2 = -2\omega_1 \frac{d^2x_1}{d\tau^2} - \left(\omega_1^2 + 2\omega_2\right) \frac{d^2x_0}{d\tau^2} - 2x_0x_1 \tag{5.7}$$

Equation (5.5) has the solution:

$$x_0(\tau) = Acos\tau \tag{5.8}$$

where A is the amplitude of the motion. Substitution of (5.8) into (5.6), gives

$$\frac{d^2x_1}{d\tau^2} + x_1 = -2A\omega_1 cos\tau - A^2 cos^2\tau \tag{5.9}$$

or

$$\frac{d^2x_1}{d\tau^2} + x_1 = -2A\omega_1 cos\tau - \frac{A^2}{2} - \frac{A^2}{2} cos^2\tau \tag{5.10}$$

For a periodic solution, removing the resonance terms, i.e. setting coefficients of $cos\tau$ equal to zero, we get

$$2A\omega_1 = 0 \tag{5.11}$$

or

$$\omega_1 = 0 \tag{5.12}$$

Using, $\omega_1 = 0$ in (5.10) and solving it, we obtain

$$x_1(\tau) = C_1 cos\left(\tau + k_1\right) - \frac{A^2}{2} + \frac{A^2}{6} cos2\tau \tag{5.13}$$

Taking, initial conditions $x_1(0) = 0 = \frac{dx_1}{d\tau}(0)$, the above eq. becomes

$$x_1(\tau) = -\frac{A^2}{2} + \frac{A^2}{3} cos\tau + \frac{A^2}{6} cos2\tau \tag{5.14}$$

Substituting, (5.8), (5.12) and (5.14) into (5.7), we obtain

$$\frac{d^2x_2}{d\tau^2} + x_2 = 2A\omega_2 cos\tau - 2Acos\tau \left(-\frac{A^2}{2} + \frac{A^2}{3} cos\tau + \frac{A^2}{6} cos2\tau\right) \tag{5.15}$$

which becomes

$$\frac{d^2x_2}{d\tau^2}+x_2=-\frac{A^3}{3}+\left(2A\omega_2+\frac{5A^3}{6}\right)cos\tau-\frac{A^3}{3}cos2\tau-\frac{A^3}{6}cos3\tau \tag{5.16}$$

Again removing the resonance terms, i.e. setting coefficients of $cos\tau$ equal to zero, we get

$$2A\omega_2+\frac{5A^3}{6}=0 \tag{5.17}$$

or

$$\omega_2=-\frac{5A^2}{12} \tag{5.18}$$

Substituting this result into second of the ansatz (5.2), we obtain the approximate frequency-amplitude relation:

$$\omega=1-\frac{5A^2}{12}\epsilon^2+o(\epsilon^3) \tag{5.19}$$

The period, $T=\frac{2\pi}{\omega}$, may then be written as:

$$T=\frac{2\pi}{\omega}=\frac{2\pi}{1-\frac{5A^2}{12}\epsilon^2+o(\epsilon^3)} \tag{5.20}$$

or

$$T=2\pi\left(1+\frac{5A^2}{12}\epsilon^2+o(\epsilon^3)\right) \tag{5.21}$$

6. Conclusion

In this paper, we investigated forced escape oscillators using both numerical and analytical techniques. We considered the unforced, undamped escape oscillator, the forced escape oscillator with a single sinusoidal forcing term, and the forced escape oscillator with two sinusoidal forcing terms of multiple frequencies.

The dynamical behaviour of the forced escape oscillator, which depends on the parameter values we considered, have been studied numerically using phase portraits and Poincaré sections. Phase portraits and Poincaré sections also allow us to compare the effects of single-frequency and dual-frequency sinusoidal forcing with unforced situations. Finally, we employed perturbation methods such as Lindstedt's method to obtain the frequency-amplitude relation of escape oscillator.

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