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On a Hilbert Gołąb-Schinzel type functional equation

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Abstract

Let X be a vector space over a field K of real or complex numbers. We will prove the superstability of the following Gołąb-Schinzel type equation

$$f(x + g(x)y) = f(x)f(y), x, y \in X,$$

where $f, g : X \to K$ are unknown functions (satisfying some assumptions). Then we generalize the superstability result for this equation with values in the field of complex numbers to the case of an arbitrary Hilbert space with the Hadamard product. Our result refers to papers by Chudziak and Tabor [J. Math. Anal. Appl. 302 (2005) 196-200], Jabłońska [Bull. Aust. Math. Soc. 87 (2013), 10-17] and Rezaei [Math. Ineq. Appl., 17 (2014), 249-258].

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1. Introduction

Let X be a vector space over a field K of real or complex numbers. The partially pexiderized Gołąb-Schinzel equation, i.e. the equation

$$f(x + g(x)y) = f(x)f(y) \text{ for } x, y \in X, \tag{1.1}$$

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in the class of unknown functions $f, g : X \to K$ generalizes the exponential equation

$$f(x + y) = f(x)f(y) \quad \text{for} \quad x, y \in X,$$

which is very well-known, as well as the Gołąb-Schinzel equation

$$f(x + f(x)y) = f(x)f(y) \quad \text{for}\, x, y \in X, \tag{1.2}$$

which appeared in 1959 in [20] and has been studied under various regularity assumptions, e.g., in [1], [4]-[9] and [20]. For more details concerning (1.1), its applications and further generalizations we refer to a survey paper [10] and concerning the equation (1.2) we refer to [17], [18] and [22], [23]. The stability problem for (1.2) and its generalization

$$f(x + f(x)^k y) = tf(x)f(y) \text{ for } x, y \in X, \tag{1.3}$$

where $k \in \mathbb{N},\ t \in K\backslash\{0\}$ are fixed, has been considered in [13]-[16]. It has been proved in [14] that for every $k \in \mathbb{N}$ the equation (1.3) is superstable in the class of functions $f : X \to K$ continuous at 0 on rays, i.e. every such function satisfying the inequality

$$\left| f(x + f(x)^k y) - tf(x)f(y) \right| \le \varepsilon \text{ for } x, y \in X,$$

where ε is a fixed positive real number, either is bounded or satisfies (1.3). The first results of that kind have been established in [2] for the exponential equation, in [3] for the cosine equation on an abelian group and in [28], [31]-[37] for trigonmetric type functional equations on a group that need not be abelian. For more informations concerning stability of functional equation we refer to [11]-[16], [19], [21], [24]-[37]. Let H be a Hilbert space with a countable orthonormal basis $\{e_n,\, n \in \mathbb{N}\}$. For two vectors $x, y \in H$, we have the Hadamard product, also known as the entrywise product on Hilbert space H as the following:

$$x * y = \sum_{n=0}^{+\infty} \langle x, e_n\rangle \langle y, e_n\rangle e_n, \quad x, y \in H. \tag{1.4}$$

The Cauchy-Schwartz inequality together with the Parseval identity insure that Hadamard multiplication is well defined. In fact

$$\|x * y\| \le \left(\sum_{n=0}^{+\infty} \langle x, e_n\rangle^2 \right)^{\frac{1}{2}} \left(\sum_{n=0}^{+\infty} \langle y, e_n\rangle^2 \right)^{\frac{1}{2}} = \|x\| \, \|y\| \,. \tag{1.5}$$

Superstability results for the approximately exponential and cosine Hilbert-valued functional equation by Hadamard product, have been started in [26] and [27].

In the present paper, we will prove the superstability of the Gołąb-Schinzel type equation (1.1). We find forms of solutions of the pexiderized Gołąb-Schinzel Hilbert-valued functional equation by Hadamard product, i.e. the equation

$$f(x + g(x)y) = f(x) * f(y) \quad \text{for} \quad x, y \in X, \tag{1.6}$$

in the class of unknown functions $f : X \to H$ and $g : X \to K$. Then we state, with the assumption that g is a solution of (1.2), a superstability result for the equation (1.6). As consequences, if

$L : X \to H$ is a linear functional, we investigate the superstability problem of the following functional equation

$$f(x + y + L(x)y) = f(x) * f(y) \text{ for } x, y \in X,$$

a result which we have not been able to find in the literature, and we prove that if a surjective function $f : H \to H$ satisfies the inequality

$$\|f(x + g(x)y) - f(x) * f(y)\| \leq \varepsilon,$$

for some $\varepsilon \geq 0$ and for all $x, y \in H$, then the pair (f, g) must be a solution, with this product, of the equation (1.6). These results are derived in Section 3.

In what follows, $\mathbb{N}$, $\mathbb{R}$ stand for the sets of all positive integers and real numbers respectively and ε, a nonnegative real number. X is a vector space over a field K of real or complex numbers and H is a Hilbert space with a countable orthonormal basis $\{e_n,\ n \in \mathbb{N}\}$ with the Hadamard product defined as in (1.4).

Definition 1.1. We say that a function $f : X \to H$ is continuous on rays if for every $x \in X$ the function $f_x : K \to H$ defined by $f_x(t) = f(tx)$ for $t \in K$ is continuous.

2. Solution of the equation (1.6)

In [17] and [18], among others, J. Chudziak determined all real solutions of equation (1.1) in the class of pairs of functions (f, g) such that f, g satisfy some regularity assumptions. In [22], E. Jabłońska gives all solutions $f, g : X \to \mathbb{R}$ of (1.1) in the case where X is a real linear space under the assumption that the function f is continuous on rays. It turned out that the following theorem holds (which is essential in our considerations):

Theorem 2.1. [22] Let X be a real linear space. Functions $f, g : X \to \mathbb{R}$ satisfy (1.1) and f is continuous on rays if and only if one of the following conditions holds:

(i) $f = 0$ or $f = 1$;

(ii) $g = 1$ and there exist a linear functional $L : X \to \mathbb{R}$ such that $f = \exp L$;

(iii) There are a non trivial linear functional $L : X \to \mathbb{R}$ and some nonnegative real number r such that f, g have one of the following form:

$$\begin{cases} g(x) = L(x) + 1 \text{ for } x \in X, \\ f(x) = \mid L(x) + 1 \mid^r sgn(L(x) + 1) \text{ for } x \in X, \end{cases} \tag{2.1}$$

$$\begin{cases} g(x) = L(x) + 1 \text{ for } x \in X, \\ f(x) = \mid L(x) + 1 \mid^r \text{ for } x \in X, \end{cases} \tag{2.2}$$

$$\begin{cases} g(x) = \max(L(x) + 1, 0) \text{ for } x \in X, \\ f(x) = \max(L(x) + 1, 0)^r \text{ for } x \in X. \end{cases} \tag{2.3}$$

Now we shall extend this result to the pexiderized Gołąb-Schinzel Hilbert-valued functional equation by Hadamard product.

Theorem 2.2. *Let H be a separable real Hilbert space, X be a real linear space. Functions $f : X \to H$ and $g : X \to \mathbb{R}$ wher f is continuous on rays satisfy the functional equation* (1.6), *if and only if one of the following statements holds:*

(i) There exist a positive integer N such that $f = \sum_{k=1}^{N} \alpha_k e_k$ where $\alpha_k = 1$ or 0;

(ii) $g = 1$ and there exist linear functionals $L_k : X \to \mathbb{R}$ and a positive integer N such that $f(x) = \sum_{k=1}^{N} \exp L_k(x) e_k$;

(iii) There are a non trivial linear functional $L : X \to \mathbb{R}$, some positive real numbers r_k and a positive integer N such that f, g have one of the following forms:

$$\begin{cases} g(x) = \max(L(x)+1, 0), & x \in X, \\ f(x) = \sum_{k=1}^{N} \max(L(x)+1, 0)^{r_k} e_k, & x \in X, \end{cases} \tag{2.4}$$

$$\begin{cases} g(x) = L(x)+1, & x \in X, \\ f(x) = \sum_{k=1}^{N} |L(x)+1|^{r_k} \alpha_k e_k, & x \in X, \end{cases} \tag{2.5}$$

with $\alpha_k = 1$, $\alpha_k = 0$ or $\alpha_k(x) = sgn(L(x)+1)$.

Proof . Let $\{e_k,\ k \in \mathbb{N}\}$ be an orthonormal basis for H. For every integer $k \geq 0$, consider the function $f_k : X \to K$ defined by

$$f_k(x) = \langle f(x), e_k \rangle \text{ for } x \in X.$$

Since the pair (f, g) satisfies (1.6), we have for all $x, y \in H$

$$\sum_{k=0}^{+\infty} \langle f(x+g(x)y), e_k \rangle e_k = \sum_{k=0}^{+\infty} \langle f(x) * f(y), e_k \rangle e_k, \tag{2.6}$$

$$= \sum_{k=0}^{+\infty} \langle f(x), e_k \rangle \langle f(y), e_k \rangle e_k, \tag{2.7}$$

this yields that

$$f_k(x+g(x)y) = f_k(x) f_k(y),$$

for all $k \in \mathbb{N}$ and $x, y \in X$. In view of Theorem 2.1, one of the following statements holds:

a) $f_k = 0$.

b) $f_k = 1$.

c) $g = 1$ and there exists a linear functional $L_k : X \to \mathbb{R}$ such that $f_k(x) = \exp L_k(x)$ for $x \in X$.

d) There exist a linear functional L and some nonnegative real number r_k such that $g(x) = L(x)+1$ and $f_k(x) = (L(x)+1)^{r_k}$ or $f_k(x) = (L(x)+1)^{r_k} sgn(L(x)+1)$ for $x \in X$.

e) There exist a linear functional L and some nonnegative real number r_k such that $g(x) = \max(L(x)+1, 0)$ and $f_k(x) = (\max(L(x)+1, 0))^{r_k}$ for all $x \in X$.

We have

$$f(x) = \sum_{k=0}^{+\infty} \langle f(x), e_k \rangle e_k \tag{2.8}$$

$$= \sum_{k=0}^{+\infty} f_k(x) e_k. \tag{2.9}$$

The continuation of the proof depends on the dimension of H. In fact if H is infinite dimensional, since

$$f_k(x) \to 0,$$

for every $x \in X$ as $n \to \infty$. The statements (b), (c),(d) and (e) are not possibles for infinitely many possible integer n, hence there exists some positive integer N such that for every $k > N$, $f_k = 0$. Thus f can be represented as

$$f(x) = \sum_{k=0}^{N} f_k(x) e_k. \tag{2.10}$$

In the case that, H is of finite dimensional type the proof is clear. Then each pair of functions f, g satisfying (1.6) falls into one of the categories (i)-(iii). □

Corollary 2.3. *Let a be a real number, $i \in \{1, 2, ..., n\}$, H be a separable real Hilbert space and $f : \mathbb{R}^n \to H$ be a continuous on rays mapping satisfying the functional equation*

$$f(x + y + ax_i y) = f(x) * f(y) \text{ for all } x, y \in \mathbb{R}^n$$

where $x = (x_1, x_2, ..., x_n)$. Then one of the following statements holds:

i) There exist a positive integer N such that $f = \sum\limits_{k=1}^{N} \alpha_k e_k$ where $\alpha_k = 1$ or 0 ;

ii) There are some nonnegative real numbers r_k and a positive integer N such that f has the following forms:

$$f(x) = \sum_{k=1}^{N} \alpha_k(x) \left| ax_i + 1 \right|^{r_k} e_k \quad \text{for } x \in \mathbb{R}^n,$$

where $\alpha_k = 1, 0$ or $\alpha_k(x) = sgn(ax_i + 1)$.

Proof . The proof follows on putting $g(x) = 1 + L(x) = 1 + ax_i$ in Theorem 2.2. □

3. Superstability of the equation (1.6)

We will begin this section by stating a superstability result for the the equation (1.1) in the class of unknown functions $f, g : X \to K$. For the proof of our result we need some lemmas.

Lemma 3.1. *Let $f, g : X \to K$ be two functions satisfying*

$$\mid f(x + g(x)y) - f(x)f(y) \mid \leq \varepsilon \text{ for all } x, y \in X, \tag{3.1}$$

such that f is bounded. Then

$$\mid f(x) \mid \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2} \text{ for all } x \in X. \tag{3.2}$$

Proof . Let $f, g : X \to K$ be two functions satisfying (3.1) such that f is bounded, let

$$M = \sup_{x\in X} \mid f(x) \mid .$$

We have

$$\mid f(x)f(y) \mid - \mid f(x+g(x)y) \mid \leq \varepsilon \text{ for } x, y \in X,$$

thus

$$M^2 - M \leq \varepsilon,$$

which implies that

$$\mid f(x) \mid \leq \frac{1+\sqrt{1+4\varepsilon}}{2} \text{ for all } x \in X.$$

□

Lemma 3.2. *Let $f, g : X \to K$ be two functions satisfying (3.1) and f is unbounded. Then there exists a sequence $(x_n) \subset X$ such that,*

$$\lim_{n\to\infty} \mid f(x_n) \mid = +\infty, \ f(x_n) \neq 0 \textit{ and } g(x_n) \neq 0 \ \textit{ for all } \ n \in \mathbb{N}.$$

Proof . Let $f, g : X \to K$ be two functions satisfying (3.1) such that f is unbounded. Let (x_n) be a sequence such that $\lim \mid f(x_n) \mid = +\infty$ and $f(x_n) \neq 0$ for every integer n. Assume that there exists $p \in \mathbb{N}$ such that $g(x_p) = 0$. Since the pair (f, g) satisfy (3.1), then

$$\mid f(x_p + g(x_p)y) - f(x_p)f(y) \mid \leq \varepsilon \text{ for all } y \in X,$$

hence

$$\mid f(x_p)(1 - f(y)) \mid \leq \varepsilon \text{ for all } y \in X.$$

Thus f is bounded which contradict the assumption. □

Lemma 3.3. *Let $f, g : X \to K$ be functions satisfying (3.1). Then*

$$\mid f(y) - f(x)f(\frac{y-x}{g(x)}) \mid \leq \varepsilon \textit{ for all } \ x \in X \setminus g^{-1}(0), \ y \in X. \tag{3.3}$$

Proof . It is enough to replace in (3.1) y by $\frac{y-x}{g(x)}$. □

Lemma 3.4. *Let $f, g : X \to K$ be functions satisfying (3.1) and*

$$A_f = \{(x_n, n \in \mathbb{N}) \textit{ such that } (x_n) \in X \setminus f^{-1}(0) \textit{ and } \lim_{n\to+\infty} \mid f(x_n) \mid = +\infty\},$$

then for every $(x_n) \in A_f$, we have:

$$f(x) = \lim_{n\to\infty} \frac{f(x_n + g(x_n)x)}{f(x_n)} \ \textit{ for all } x \in X. \tag{3.4}$$

$$(x_n + g(x_n)x) \in A_f \textit{ for all } n \in \mathbb{N} \textit{ and } x \in X \setminus f^{-1}(0). \tag{3.5}$$

$$f(y) = \lim_{n\to\infty} \frac{f(x_n + g(x_n)x + g(x_n + g(x_n)x)y)}{f(x_n + g(x_n)x)} \ \textit{ for all } x \in X \setminus f^{-1}(0), y \in X. \tag{3.6}$$

$$(x_n + g(x_n)x + g(x_n + g(x_n)x)y, n \in \mathbb{N}) \in A_f \textit{ for all } x, y \in X \setminus f^{-1}(0). \tag{3.7}$$

Proof . (3.4) becomes easily from (3.1). (3.5) is a consequence of (3.4). Replacing y by x and x_n by $x_n + g(x_n)x$ in (3.4), we obtain (3.6). The equalities (3.5) and (3.6) implies (3.7). □

Now we prove the superstability of the equation (1.1) with the assumption that g is a solution of (1.2) and without any conditions on f.

Theorem 3.5. *Let $f, g : X \to K$ be two functions satisfying* (3.1) *such that g is a solution of* (1.2). *Then either* (3.2) *occur or*

$$f(x + g(x)y) = f(x)f(y) \text{ for all } x, y \in X.$$

Proof . From Lemma 3.1 it is enough to consider the case where f is unbounded.
Since f is unbounded then $A_f \neq \emptyset$. Fix a sequence $(x_n, n \in \mathbb{N}) \in A_f$ and let

$$c_n(x, y) = x_n + g(x_n)x + g(x_n + g(x_n)x)y,\ x, y \in X,\ n \in \mathbb{N}.$$

$$d_n(x, y) = x_n + g(x_n)x + g(x_n)g(x)y,\ \ x, y \in X,\ n \in \mathbb{N}.$$

Then from (3.4), we obtain

$$f(x + g(x)y) = \lim_{n\to\infty} \frac{f(x_n + g(x_n)(x + g(x)y))}{f(x_n)} \tag{3.8}$$

$$= \lim_{n\to\infty} \frac{f(d_n(x, y))}{f(x_n)} \text{ for } x, y \in X. \tag{3.9}$$

Simiraly, by (3.4) and (3.6), we have

$$f(x)f(y) = \lim_{n\to\infty} \frac{f(x_n + g(x_n)x)}{f(x_n)} \frac{f(x_n + g(x_n)x + g(x_n + g(x_n)x)y)}{f(x_n + g(x_n)x)} \tag{3.10}$$

$$= \lim_{n\to\infty} \frac{f(c_n(x, y))}{f(x_n)} \text{ for } y \in X,\ x \in X \setminus f^{-1}(0). \tag{3.11}$$

As g is a solution of (1.2), we obtain that

$$c_n(x, y) = d_n(x, y) \text{ for all } x, y \in X.$$

From previous discussions we get

$$f(x + g(x)y) = f(x)f(y) \text{ for } x \in X \setminus f^{-1}(0),\ \ y \in X.$$

In the case where $x \in f^{-1}(0)$, we have $x \in g^{-1}(0)$, in fact if $g(x) \neq 0$, using Lemma 3.3 we obtain, $| f(y) | \leq \varepsilon$ for all $x,\ y \in X$. So f is bounded, this is a contradiction because f is unbounded. Finaly (1.1) holds, which completes the proof. □

Corollary 3.6. *Let a be a real number and $f : \mathbb{R} \to \mathbb{R}$ be a mapping satisfying the inequality*

$$|f(x + y + axy) - f(x)f(y)| \leq \varepsilon,\ x, y \in \mathbb{R}.$$

Then either f is bounded or $f(x + y + axy) = f(x)f(y)$ for all $x, y \in \mathbb{R}$.

Proof . The proof follows on putting $g(x) = 1 + L(x) = 1 + ax$ in Theorem 3.5. $\square$

Corollary 3.7. *Let X be a vector space, $f : X \to K$ be a function and $L : X \to K$ be a linear function satisfying:*

$$\|f(x + y + L(x)y) - f(x)f(y)\| \le \varepsilon, \ x, y \in X. \tag{3.12}$$

Then either f is bounded or

$$f(x + (L(x) + 1)y) = f(x)f(y) \text{ for all } x, y \in X. \tag{3.13}$$

Proof . The proof follows on putting $g(x) = 1 + L(x)$ in Theorem 3.5. $\square$

Corollary 3.8. *Let X be a vector space, $f : X \to K$ a function and $L : X \to K$ is a linear function satisfying:*

$$\|f(x + \max(L(x) + 1, 0)y) - f(x)f(y)\| \le \varepsilon, \ x, y \in X. \tag{3.14}$$

Then either f is bounded or

$$f(x + \max((L(x) + 1), 0)y) = f(x)f(y) \text{ for all } x, y \in X. \tag{3.15}$$

Proof . The proof follows on putting $g(x) = \max(L(x) + 1, 0)$ in Theorem 3.5. $\square$

Now we state a superstability result, in the sense of Rezaei-Sharifzadeh for the approximately pexiderized Gołąb-Schinzel Hilbert-valued functional equation by Hadamard product.

Theorem 3.9. *Let $f : X \to H$ and $g : X \to K$ be two functions satisfying*

$$\|f(x + g(x)y) - f(x) * f(y)\|_H \le \varepsilon, \tag{3.16}$$

for all $x, y \in X$, such that g is a solution of (1.2). Then, either there exists $k \ge 1$ such that

$$| \langle f(x), e_k \rangle | \le \frac{1 + \sqrt{1 + 4\varepsilon}}{2} \tag{3.17}$$

for all $x \in X$ or the pair (f, g) is a solution of (1.6).

Proof . Let $f : X \to H$, $g : X \to K$ be two functions satisfying (3.16) and g is a solution of (1.2). By applying the Parseval identity and defintion of Hadamard product with the inequality (3.16), we find that each scalar valued function f_k satisfy

$$|f_k(x + g(x)y) - f_k(x)f_k(y)| \le \varepsilon \quad \text{for all } \ x, y \in X.$$

According to Theorem 3.5, since g is a solution of (1.2) we have for all $k \in \mathbb{N}$, either

$$| \ f_k(x) \ | \le \frac{1 + \sqrt{1 + 4\varepsilon}}{2} \ \text{ for all } \ x \in X,$$

or

$$f_k(x + g(x)y) = f_k(x)f_k(y) \ \text{ for all } \ x, y \in X.$$

Thus either there exists $k \ge 1$

$$| \ f_k(x) \ | \le \frac{1 + \sqrt{1 + 4\varepsilon}}{2} \ \text{ for all } \ x \in X,$$

or the pair (f, g) is a solution of (1.6). $\square$

Corollary 3.10. *Let $g : H \to K$ be a solution of* (1.2) *and $f : H \to H$ be a surjective function satisfying*

$$\|f(x + g(x)y) - f(x) * f(y)\|_H \leq \varepsilon,$$

for all $x,\ y \in H$. Then

$$f(x + g(x)y) = f(x) * f(y) \text{ for all } x, y \in H.$$

Proof . Let $g : H \to K$ be a function satisfying (1.2) and $f : H \to H$ be a surjective function satisfying (3.16). Then every component function f_k is unbounded (See [26, page7]): In fact for every integer k, there exist $x_k \in H$ such that $f(x_k) = ke_k$, so $f_k(x_k) = k$. In view of Theorem 3.9, we conclude that the pair (f, g) satisfy the equation

$$f(x + g(x)y) = f(x) * f(y) \text{ for all } x, y \in H.$$

□

Corollary 3.11. [26] *Let X be a vector space and $f : X \to H$ be a function satisfying*

$$\|f(x + y) - f(x) * f(y)\|_H \leq \varepsilon,$$

for all $x,\ y \in X$. Then either there exist an integer k such that

$$| \langle f(x), e_k\rangle | \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2}$$

or

$$f(x + y) = f(x) * f(y) \text{ for all } x, y \in X.$$

Proof . The proof follows on putting $g(x) = 1$ in Theorem 3.9. □

Corollary 3.12. *Let X be a vector space and $f : X \to H$ be a function satisfying*

$$\|f(x + (L(x) + 1)y) - f(x) * f(y)\|_H \leq \varepsilon$$

such that $L : X \to K$ be a linear function. Then either there exist an integer k such that

$$| \langle f(x), e_k\rangle | \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2},$$

or

$$f(x + (L(x) + 1)y) = f(x) * f(y) \text{ for all } x, y \in X. \tag{3.18}$$

Proof . The proof follows on putting $g(x) = 1 + L(x)$ in Theorem 3.9. □

Corollary 3.13. Let X be a vector space, $f : X \to H$ be a function and $L : X \to K$ be a linear function satisfying

$$\|f(x + \max((L(x) + 1), 0)y) - f(x) * f(y)\|_H \leq \varepsilon \ x, y \in X.$$

Then either there exist an integer k such that

$$| \langle f(x), e_k\rangle | \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2},$$

or

$$f(x + \max(L(x) + 1, 0)y) = f(x) * f(y) \text{ for all } x, y \in X. \tag{3.19}$$

Corollary 3.14. *Let $a_1, a_2, ..., a_n$ be real numbers and $f : \mathbb{R}^n \to H$ be a continuous on rays function satisfying*

$$\|f(x + y + \sum_{i=1}^{n} a_i x_i y) - f(x) * f(y)\|_H \leq \varepsilon,$$

for all $x = (x_1, x_2, ..., x_n), y = (y_1, y_2, ..., y_n) \in \mathbb{R}^n$. Then, either there exists $k \geq 1$ such that

$$| \langle f(x), e_k \rangle | \leq \frac{1 + \sqrt{1 + 4\varepsilon}}{2},$$

for all $x \in X$ or there are some nonnegative real numbers r_k and a positive integer N such that f has the form:

$$f(x) = \sum_{k=1}^{N} \left| 1 + \sum_{i=1}^{n} a_i x_i \right|^{r_k} \alpha_k(x) e_k \quad \text{for } x \in \mathbb{R}^n,$$

with $\alpha_k = 1,\ \alpha_k = 0$ or $\alpha_k(x) = sgn(\sum\limits_{i=1}^{n} a_i x_i + 1)$.

Proof . The proof follows on putting $g(x) = 1 + L(x) = 1 + \sum\limits_{i=1}^{n} a_i x_i$ in Theorem 3.9 combined with Theorem 2.2. □

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