Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 15-20
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.273

On the quadratic support of strongly convex functions

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(Communicated by Themistocles M. Rassias)

Abstract

In this paper, we first introduce the notion of c-affine functions for $c > 0$. Then we deal with some properties of strongly convex functions in real inner product spaces by using a quadratic support function at each point which is c-affine. Moreover, a Hyers-Ulam stability result for strongly convex functions is shown.

Keywords: strongly convex function; Hahn–Banach theorem; c-affine functions; quadratic support.
2010 MSC: Primary 26A25; Secondary 39B62.

1. Introduction and preliminaries

During the 20th century mathematicians introduced and investigated many generalizations of convexity. As it is well known, the notion of the classical convexity can be expressed in terms of affine functions. It is well known that any convex function defined on a real interval admits an affine support at every interior point of a domain (the converse is also true). In this paper, we generalize this idea to strong convexity. We first introduce the notion of c-affine functions and then characterize the strongly convex functions defined on real inner product spaces.

In the following theorem, the convex version of Hahn–Banach theorem is given.

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Received: December 2013 *Revised:* November 2014

Theorem 1.1. (Roberts and Verberg [11]) Let f be convex on an open set U of a normed linear space L and let V_0 be a nontrivial subspace such that $V_0 \cap U \neq \emptyset$. If $A_0 : V_0 \longrightarrow \mathbb{R}$ is affine and $A_0(x) \leq f(x)$ on $V_0 \cap U$, then there is an affine extension $A : L \longrightarrow \mathbb{R}$ of A_0 such that $A(x) \leq f(x)$ on U.

In what follows, $(X, \|.\|)$ is a real inner product space, D stands for an open convex subset of X and c is a positive constant.

A function $f : D \longrightarrow \mathbb{R}$ is called strongly convex with modulus c if

$$f\Big(tx + (1-t)y\Big) \leq tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2$$

for all $x, y \in D$ and $t \in (0,1)$.

Strongly convex functions were introduced by Polyak [9]. They have useful properties in optimization theory. For instance, if f is strongly convex, then it is bounded from below, its level sets $x \in I : f(x) \leq \lambda$ are bounded for each λ and f has a unique minimum on every closed subinterval of I ([11], p. 268). Nikodem et al. have obtained some interesting properties of strongly convex functions (see [3, 6, 7, 8]).

Any strongly convex function defined on a real interval admits a quadratic support at every interior point of its domain.

Lemma 1.2. (Nikodem and Páles [7]) A function $f : D \longrightarrow \mathbb{R}$ is strongly convex with modulus c if and only if the function $g : D \longrightarrow \mathbb{R}$ defined by $g = f - c\|\cdot\|^2$ is convex.

Lemma 1.3. (Roberts and Verberg [11]) Call $f : (a,b) \longrightarrow \mathbb{R}$ strongly convex with modulus c if and only if for each $x_0 \in (a,b)$, there is a linear function T such that $f(x) \geq f(x_0) + T(x-x_0) + c(x-x_0)^2$.

We say a function $f : D \longrightarrow \mathbb{R}$ has support at $x_0 \in D$ if there is an affine function $A : D \longrightarrow \mathbb{R}$ such that $A(x_0) = f(x_0)$ and $A(x) \leq f(x)$ for every $x \in D$.

A convex function $f : D \longrightarrow \mathbb{R}$ is characterized by having a line of support at each point of domain.

Theorem 1.4. (Roberts and Verberg [11]) The function $f : D \longrightarrow \mathbb{R}$ is convex if and only if f has support at each point of D.

We define the class of c-affine functions as follows.

Definition 1.5. A function $q : D \longrightarrow \mathbb{R}$ is called c-affine if

$$q\Big(tx + (1-t)y\Big) = tq(x) + (1-t)q(y) - ct(1-t)\|x-y\|^2$$

for all $x, y \in D$.

Remark 1.6. A function $f : D \longrightarrow \mathbb{R}$ has a quadratic support at $x_0 \in D$ if there exists a c-affine function $q : D \longrightarrow \mathbb{R}$ with $q(x_0) = f(x_0)$ such that $q(x) \leq f(x)$ for all $x \in D$.

2. The quadratic support of strongly convex function

In order to prove the fundamental theorem on the quadratic support of strongly convex functions, we need a version of the Hahn–Banach theorem which we now prove.

Theorem 2.1. *Let Y be a subspace of a finite dimensional vector space X and f be a strongly convex function with modulus c on Y. If $g : Y \longrightarrow [0,\infty)$ is c-affine and $g(x) \le f(x)$ on Y, then there is a c-affine extension $h : X \longrightarrow [0,\infty)$ of g such that $h(x) \le f(x)$ on X.*

Proof . Define $f(x)$ to be $+\infty$ if $x \in X \setminus D$ and note that f now is strongly convex on all of X. Choose a fixed $w \in X \setminus V_0$. Then for $x, y \in V_0$, $r > 0$ and $s > 0$

$$\begin{aligned}
\frac{r}{r+s}q_0(x) + \frac{s}{r+s}q_0(y) - \frac{crs\|rx-sy\|^2}{(r+s)^4} &= q_0\left(\frac{r}{r+s}x + \frac{s}{r+s}y\right) \\
&\le f\left(\frac{r}{r+s}x + \frac{s}{r+s}y\right) = f\left(\frac{r}{r+s}(x-sw) + \frac{s}{r+s}(y+rw)\right) \\
&\le \frac{r}{r+s}f(x-sw) + \frac{s}{r+s}f(y+rw) - \frac{crs\|rx-sy\|^2}{(r+s)^4}.
\end{aligned}$$

Multiplying by $r+s$ gives

$$rq_0(x) + sq_0(y) \le rf(x-sw) + sf(y+rw)$$

or

$$g(x,s) = \frac{q_0(x) - f(x-sw)}{s} \le \frac{f(y+rw) - q_0(y)}{r} = h(y,r).$$

It follows that $\sup g \le \inf h$ on $V_0 \times P$ where P denotes the positive real line. Moreover, if $x_0 \in V_0 \cap D$ and s_0 is small enough so that both $x_0 - s_0 w$ and $x_0 + s_0 w$ are in D, then both $g(x_0, s_0)$ and $h(x_0, s_0)$ are finite. Hence, $\sup g$ and $\inf h$ are finite. In particular

$$\frac{q_0(x) - f(x-sw)}{s} \le \alpha \le \frac{f(x+rw) - q_0(x)}{r}$$

for $x \in V_0$, $r > 0$ and $s > 0$. Substituting $t = -s$ when $t < 0$ and $t = r$ for $t > 0$ leads immediately to

$$q_0(x) + \alpha t \le f(x+tw)$$

for all $x \in V_0$ and $t \in \mathbb{R}$.

Now, let $V_1 = \{x + tw : x \in V_0, t \in \mathbb{R}\}$. V_1 is a subspace which properly contains V_0. Define $q_1 : V_1 \longrightarrow \mathbb{R}$ by $q_1(x+tw) = q_0(x) + t\alpha$. Then q_1 is c-affine on V_1, $q_1 = q_0$ on V_0 and we have just established that $q_1(x+tw) \le f(x+tw)$. If $V_1 = X$, our theorem is proved. If not, we may proceed by mathematical induction to extend q_0 to all of X since $\dim V_n = \dim V_{n-1} + 1$. $\square$

As an application of Hahn–Banach theorem, we show that a c-affine function defined on a subspace of real inner product space X can be extended to a c-affine function on X with preserving of its original norm.

Corollary 2.2. *Let Y be a subspace of a finite dimensional real inner product space X and $g : Y \longrightarrow [0, \infty)$ be a $\|g\|$-affine function ($\|g\| = \sup\{|g(x)| : x \in Y\}$). Then g can be extended to a $\|g\|$-affine function $h : X \longrightarrow \mathbb{R}$ such that $\|h\| = \|g\|$.*

Proof . Define $f : X \longrightarrow [0, \infty)$ by $f(x) = \|g\|\Big(1 + \|x\| + \|x\|^2\Big)$ for each $x \in X$. Note that $g(x) \le f(x)$ and

$$\begin{aligned} f\Big(tx + (1-t)y\Big) &= \|g\|\Big(1 + \|tx + (1-t)y\| + \|tx + (1-t)y\|^2\Big) \\ &\le \|g\|\Big(t + (1-t) + t\|x\| + (1-t)\|y\| + t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x-y\|^2\Big) \\ &= t\|g\|\Big(1 + \|x\| + \|x\|^2\Big) + (1-t)\|g\|\Big(1 + \|y\| + \|y\|^2\Big) - \|g\|t(1-t)\|x-y\|^2 \\ &= tf(x) + (1-t)f(y) - \|g\|t(1-t)\|x-y\|^2 \end{aligned}$$

which shows that f is strongly convex with modulus $\|g\|$ on X. By Theorem 2.1 there exists a $\|g\|$-affine extension h of g to all of X such that $h(x) \le f(x)$ holds for all $x \in X$. This implies that $\|h\| \le \|g\|$.

On the other hand,

$$\|g\| = \sup\Big\{|g(x)| : x \in Y\Big\} \le \sup\Big\{|h(x)| : x \in X\Big\} = \|h\|$$

also holds. So, $\|g\| = \|h\|$ and h is the desired extension of g to all of X. □

We are now ready to prove the fundamental theorem on the quadratic support of a strongly convex function.

Theorem 2.3. *The function f is strongly convex with modulus $c > 0$ on an open convex set D of X if and only if f has quadratic support at each point of D.*

Proof . Suppose f has support at each point of D. Let $x_1, x_2 \in D$ and $x_0 = tx_1 + (1-t)x_2$ where $t \in [0, 1]$ and let $q : D \longrightarrow \mathbb{R}$ be a c-affine function such that supports f at x_0. Then

$$\begin{aligned} f(x_0) = q(x_0) &= tq(x_1) + (1-t)q(x_2) - ct(1-t)\|x_1 - x_2\|^2 \\ &\le tf(x_1) + 1 - t)f(x_2) - ct(1-t)\|x_1 - x_2\|^2 \end{aligned}$$

which establishes the strong convexity of f.

Now suppose f is strongly convex on D and choose $x_0 \in D$. Take u to be a unit vector with the same direction as x_0 (or an arbitrary unit vector if $x_0 = 0$), and define the vector subspace $V_0 = \{x_0 + tu : t \in \mathbb{R}\}$. The open interval $I = \{t : (x_0 + tu) \in D\}$ contains 0 and we may define a convex function $g(t) = f(x_0 + tu)$ in the now customary way. According to Lemma 1.3 there is a c-affine function $q : \mathbb{R} \longrightarrow \mathbb{R}$ such that $G(0) = g(0)$ and $G(t) \le g(t)$ on I. We use this G to define $q_0 : V_0 \longrightarrow \mathbb{R}$ by $q_0(x_0 + tu) = G(t)$. Now q_0 is $c-$quadratic affine, $q_0(x_0) = f(x_0)$ and

$$q_0(x_0 + tu) = G(t) \le g(t) = f(x_0 + tu)$$

for $(x_0 + tu) \in V_0 \cap D$. It follows from Theorem 2.1 that q_0 can be extended to a function $f : X \longrightarrow \mathbb{R}$ where $q(x) \le f(x)$ on D. That is, q is a quadratic support function for f at x_0. □

Remark 2.4. The theorem above can be proved by using Lemmas 1.2 and 1.4. By Lemma 1.2 a function $f : D \longrightarrow \mathbb{R}$ is strongly convex with modulus $c > 0$ if and only if the function $g = f - c\|.\|^2$ is convex. By Lemma 1.4, g is convex if and only if g has support at each point of D. Let $A : D \longrightarrow \mathbb{R}$ be an affine function that supports g at x_0. So $A(x_0) = g(x_0)$ and $A(x) \le g(x) = f(x) - c\|x\|^2$ for every $x \in D$. Then, function $q : D \longrightarrow \mathbb{R}$ defined by $q(x) = A(x) + c\|x\|^2$ is a quadratic support for f at x_0.

As an application, we obtain the following Hyers–Ulam type stability result related to strongly convex and c-affine functions. Let $\varepsilon > 0$, $c > 0$, $(X, \|\cdot\|)$ be a real inner product space and D be a convex subset of X. We say that a function $f : D \longrightarrow \mathbb{R}$ is ε-strongly convex with modulus c if

$$f\Big(tx + (1-t)y\Big) \le tf(x) + (1-t)f(y) - ct(1-t)\|x-y\|^2 + \varepsilon$$

for all $x, y \in D$ and $t \in (0, 1)$. The following lemma is useful in our investigations.

Lemma 2.5. (Baron et al. [2]) Real functions f and g, defined on a real interval I, satisfy

$$f_1\Big(tx + (1-t)y\Big) \le tg_1(x) + (1-t)g_1(y)$$

for all $x, y \in D$ and $t \in [0, 1]$ if and only if there exists a convex function $h : I \longrightarrow \mathbb{R}$ such that $f \le h \le g$.

Lemma 2.6. *Let $f, g : D \longrightarrow \mathbb{R}$ be functions with $f \le g$ on D. Then g is c-affine if and only if there exists a strongly convex function $h : D \longrightarrow \mathbb{R}$ such that $f \le h \le g$ on D.*

Proof . The only if part is obvious. To prove the if part, assume that g is c-affine and consider the functions $f_1, g_1 : D \longrightarrow \mathbb{R}$ defined by

$$f_1(x) = f(x) - c\|x\|^2, \quad g_1(x) = g(x) - c\|x\|^2.$$

From the c-affineness of g, we get

$$\begin{aligned}
f_1\Big(tx + (1-t)y\Big) &= f\Big(tx + (1-t)y\Big) - c\|tx + (1-t)y\|^2 \\
&\le g\Big(tx + (1-t)y\Big) - c\|tx + (1-t)y\|^2 \\
&= tg(x) + (1-t)g(y) - ct(1-t)\|x-y\|^2 - c\|tx + (1-t)y\|^2 \\
&= tg_1(x) + (1-t)g_1(y)
\end{aligned}$$

for all $x, y \in D$ and $t \in (0, 1)$. Hence, by Lemma 2.5, there exists a convex function $h_1 : D \longrightarrow \mathbb{R}$ such that $f_1 \le h_1 \le g_1$ on D. Define $h(x) = h_1(x) + c\|x\|^2$, $x \in D$. Then, by Lemma 1.2, h is strongly convex with modulus c and $f \le h \le g$ on D. □

Theorem 2.7. *Let $f : D \longrightarrow \mathbb{R}$ be ε-strongly convex with modulus c. Then there exists a strongly convex function $h : D \longrightarrow \mathbb{R}$ with modulus c such that*

$$\|f(x) - h(x)\| \le \varepsilon$$

for all $x \in D$.

Proof . Let $x_1, x_2 \in D$. Define $g : D \longrightarrow \mathbb{R}$ by $g(x) = a + \langle y, x\rangle + c\|x\|^2$ for $a \in \mathbb{R}$ and $y \in D$ with $g(x_1) = f(x_1)+\varepsilon$ and $g(x_2) = f(x_2)+\varepsilon$. One can easily show that g is c-affine. For each $z \in (x_1, x_2)$, $z = tx_1 + (1-t)x_2$ and we have

$$\begin{aligned} f(z) &= f(tx_1 + (1-t)x_2) \\ &\leq tf(x_1) + (1-t)f(x_2) - ct(1-t)\|x_1 - x_2\|^2 + \varepsilon \\ &= tg(x_1) + (1-t)g(x_2) - ct(1-t)\|x_1 - x_2\|^2 \\ &= g(z). \end{aligned}$$

Hence, according to Lemma 2.6 there exists a function $h : D \longrightarrow \mathbb{R}$ which is strongly convex with modulus c and such that $f \leq h \leq g$ on D. Since $g(x) - f(x) = \varepsilon$, then

$$\|f(x) - h(x)\| \leq \varepsilon$$

as desired. □

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