Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 89-97
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1419.1363

Interpolation of fuzzy data by using flat end fuzzy splines

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(Communicated by M. Eshaghi)

Abstract

In this paper, a new set of spline functions called "Flat End Fuzzy Spline" is defined to interpolate given fuzzy data. Some important theorems on these splines together with their existence and uniqueness properties are discussed. Then numerical examples are presented to illustrate the differences between of using our spline and other interpolations that have been studied before.

Keywords: fuzzy interpolation; extension principle; fuzzy splines.
2010 MSC: Primary 03C40; Secondary 03B52.

1. Introduction

The following problem was first posed by L. A. Zadeh, see for example [11]. Suppose that we have $n+1$ distinct real numbers $x_0, x_1, \ldots, x_n$ and for each of these numbers a fuzzy value in R, rather than a crisp value, is given. Zadeh asked the question whether it is possible to construct some kind of smooth function on R to fit with the collection of fuzzy data at these $n+1$ points.

Lagrange interpolation for fuzzy data was first investigated by Lowen [11]. Later, Kaleva [8], avoided the well-known computational troubles associated with crisp Lagrange interpolation by using linear spline and not-a-knot cubic spline approximations. If the fuzzy data are not convex, then a technical difficulty arises and in this case the Bernestein approximation can be constructed, see for example Diamond and Ramor [5]. The interpolation of fuzzy data by using spline functions of odd degree was considered in [1] with complete splines, in [2] with natural splines, and in [4] with fuzzy

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Received: June 2016 *Revised:* February 2017

splines and finally in [3] with $E(3)$ cubic splines. Constructing consistent fuzzy surfaces from fuzzy data in sense of Lagrange polynomials, linear splines and not-a-knot cubic splines were described in [10]. To see the other works on the interpolation of fuzzy data, one can refer to [14, 15, 16, 17, 18, 19, 20].

In this paper, in Section 3, we will introduce a new set of fuzzy splines interpolate the fuzzy data. Then some important theorems on these splines together with their existence and uniqueness properties will be discussed. Finally, in Section 4, to illustrate the differences between of using our spline and other interpolations that have been studied before, some numerical examples will be presented.

2. Preliminaries

In this section, we recall some fundamental results of fuzzy numbers and fuzzy interpolations.

Definition 2.1. A fuzzy number is a mapping $u : R \to I = [0, 1]$ with the following properties, see [9]:

(i) u is an upper semi-continuous function on R,
(ii) $u(x) = 0$ outside of some interval $[c, d] \subset R$,
(iii) there exist real numbers a, b, such that $c \le a \le b \le d$, and

1. $u(x)$ is a monotonic increasing function on $[c, a]$
2. $u(x)$ is a monotonic decreasing function on $[b, d]$
3. $u(x) = 1$, for all x in $[a, b]$.

The set of all fuzzy numbers is denoted by $\mathcal{F}$. A popular type of fuzzy number is the set of triangular fuzzy number $u = (c, \alpha, \beta)$ defined by

$$u(x) = \begin{cases} \dfrac{x - c + \alpha}{\alpha}, & c - \alpha \le x \le c, \\ \dfrac{c + \beta - x}{\beta}, & c \le x \le c + \beta, \\ 0, & \text{otherwise}, \end{cases}$$

where $\alpha > 0$ and $\beta > 0$. Note that the triangular fuzzy numbers are special cases of $L - L$ fuzzy numbers, see [6].

Definition 2.2. If $u \in \mathcal{F}$ then the α−level set of u is denoted by $[u]^\alpha$ and defined by $[u]^\alpha = \{x \in \underline{R|u(x) \ge \alpha}\}$, where $0 < \alpha \le 1$. Also, $[u]^0$ is called the support of u and it is given by $[u]^0 = \overline{\cup_{\alpha \in (0,1]} [u]^\alpha}$. It follows that the level sets of u are closed and bounded intervals in R.

It is well-known that the addition and multiplication operations of real numbers can be extended to $\mathcal{F}$. In other words, for any $0 < \alpha \le 1$, $\lambda \in R$ and $u, v \in \mathcal{F}$, we have:

$$[u + v]^\alpha = [u]^\alpha + [v]^\alpha \quad and \quad [\lambda u]^\alpha = \lambda [u]^\alpha.$$

Consider $n + 1$ distinct real numbers $x_0 \le x_1 \le x_2 \le \ldots \le x_n$. For each x_i we associate a fuzzy number $u_i \in \mathcal{F}$. To solve Zadeh's problem, we must find a continuous function F :R→ $\mathcal{F}$ such that $F(x_i) = u_i$; for $i = 0, 1, \ldots, n$.

Let $P_{y_0,y_1\dots,y_n}(x)$ be the Lagrange interpolation polynomial of degree n which interpolates the data $(x_i,\, y_i)\,;\, i = 0,1,\dots,n$. According to the extension principle [6], we can write the membership function $F(x)$ for each $x \in R$ as follows:

$$\begin{aligned}\mu_{F(x)}(t) &= \sup_{\substack{y_0,y_1,\dots,y_n \\ t=P_{y_0\cdots y_n}(x)}} \min_{i=0,1,\dots,n} \mu_{u_i}(y_i); \quad \text{if} \quad P^{-1}_{y_0\cdots y_n}(t) \neq \emptyset, \\ &= \quad 0, \qquad\qquad otherwise,\end{aligned}$$

where μ_{u_i} is the membership function of u_i.

For each $\alpha \in$(0,1] and $i = 0,1,\dots,n$, let $J_i^\alpha = [u_i]^\alpha = \mu_{u_i}^{-1}[\alpha,1]$, and $F^\alpha(x)$ be the $\alpha-$level sets of u_i and $F(x)$, respectively. Hence,

$$\begin{aligned}F^\alpha(x) &= \{t \in R | \mu_{F(x)}(t) \geq \alpha\} \\ &= \{t \in R | \exists\; y_0, y_1,\dots,y_n : \mu_{u_i}(y_i) \geq \alpha, i = 0,1,\dots,n \text{ and } P_{y_0,y_1\dots,y_n}(x) = t\} \\ &= \{t \in R\,|\exists\; \overline{y} \in \prod_{i=0}^{n} J_i^\alpha \;: P_{y_0,\,y_1\dots,\,y_n}(x) = t\}\end{aligned}$$

where $\overline{y} = (y_0,\, y_1,\dots,y_n) \in \mathrm{R}^{n+1}$. Now we have

$$\mu_{F(x)}(t) \;= \sup \left\{ \alpha \in (0,1]\,|\,\exists\; \overline{y} \in \prod_{i=0}^{n} J_i^\alpha \;: P_{y_0,y_1\dots,y_n}(x) = t \right\},$$

where, as mentioned by Lowen in [11], the supremum is attained and hence from Nguyen [12], we have

$$F^{\,\alpha}(x) = \{y \in \; R | y = P_{y_0,y_1\dots,y_n}(x),\; y_i \in J_i^\alpha\}.$$

But, from Lagrange interpolation formula, we have

$$F^{\,\alpha}(x) = \sum_{i=0}^{n} L_i(x) J_i^\alpha,$$

where $L_i(x)$ represents the Lagrange polynomials.

3. Fuzzy splines

In this section, we introduce a set of special spline functions of odd degree, called “Flat End Fuzzy Spline” for interpolation of given fuzzy data.

Definition 3.1. A spline function with zero derivatives at endpoints with knots $x_0 \leq x_1 \leq \dots \leq x_n$, is a piecewise polynomial function $s : [x_0,\, x_n] \rightarrow\; R$ of degree $l = 2m-1$ with $m = 2k,\; k \geq 1$, that possesses the following conditions:

(i) $s \in C^{l-1}[x_0,\, x_n]$,

(ii) in each subinterval $[x_{i-1}, x_i],\; s(x)$ is a polynomial of degree l,

(iii) $s^{(\nu)}(x_0) = 0$, for $\nu = 1,2,\dots,\frac{m}{2}$,

(iv) $s^{(\nu)}(x_n) = 0$, for $\nu = \frac{m}{2}, \frac{m}{2}+1,\dots,2m-3$.

We denote the family of these splines with $S_l(x_0, x_n)$. If the base splines $s_i \in S_l(x_0, x_n)$ are such that $s_i(x_j) = 1$ for $i = j$ and $s_i(x_j) = 0$ for $i \neq j$, then similar to Lagrange interpolation polynomial, the fuzzy spline

$$S_{y_0,y_1\ldots,y_n}(x) = \sum_{i=0}^{n} s_i(x)\, y_i$$

interpolates $(x_i,\, y_i);\ i = 0, 1, \ldots, n$. Hence from Section 2, we have

$$F^{\,\alpha}(x) = \{t \in R | \exists \overline{y} \in \prod_{i=0}^{n} J_i^{\alpha}\ : S_{y_0,y_1\ldots,y_n}(x) = t\} = \sum_{i=1}^{n} s_i(x)\, J_i^{\alpha},$$

and

$$F(x) = \sum_{i=0}^{n} s_i(x) u_i.$$

Hence if all u_i are $L - L$ fuzzy numbers, then $F(x)$ is an $L - L$ fuzzy number for all $x \in [x_0, x_n]$.

The rest of Section 3 is devoted to prove some properties of the spline function defined at Definition 3.1.

Definition 3.2. A point $\alpha \in [x_\nu, x_{\nu+1}) \subset [x_0, x_n],\ 0 \leq \nu \leq n - 1$ is called an essential zero of a spline $s \in S_l(x_0, x_n)$, provided that $s(\alpha) = 0$, but $s(x) \neq 0$ on $[x_\nu, x_{\nu+1})$, see [7].

Definition 3.3. The number of all essential zeros of s in $[x_0, x_n]$ is denoted by $Z(s)$, where each zero is counted according to its multiplicity.

Theorem 3.4. ***Theorem 1.*** *If $s \in S_l(x_0,\, x_n)$ then $Z(s) \leq n + l - 1$ and if $[x_\nu,\, x_{\nu+\eta}]$ is the maximal subinterval that s vanishes, then $x_{\nu+\eta}$ is an $l-$fold essential zero of s.*

Proof . For proof, see [7]. □

Theorem 3.5. *If $[x_0, x_\sigma]$ is the maximal interval where $s_i \in S_l(x_0,\, x_n)$ vanishes identically and s_i doesn't vanish identically on any subinterval $[x_\nu,\ x_{\nu+1}]$ for $\nu > \sigma$ then $s_i^{(2m-2)}$ has at least $n-\sigma+\frac{3}{2}m-2$ essential zeros on $[x_\sigma,\ x_n]$.*

Proof . Since $s_i(x_i) = 1$, we have $\sigma < i$ and hence $s_i(x) = 0$ for $x = x_j;\ j = \sigma, \ldots, i-1, i+1, \ldots, n$. But we know that x_σ is an $l-$fold zero of s_i and by using Rolle's Theorem , ${s_i}'$ has at least $n - \sigma$ essential zeros on $[x_\sigma,\, x_n]$. By repeating this argument, we can see that the functions $s_i^{(2)},\, s_i^{(3)},\, \ldots,\, s_i^{(\frac{m}{2}-1)}$ have at least $n-\sigma$ essential zeros on $[x_\sigma, x_n]$. But by Definition 3, $s_i^{(\nu)}(x_n) = 0$, for $\nu = \frac{m}{2}, \frac{m}{2} + 1, \ldots, 2m - 3$. Hence, by virtue of Rolle's Theorem, $s_i^{(\frac{m}{2})}$ has at least $n - \sigma + 1$ essential zeros and consequently $s_i^{(2m-3)}$ and $s_i^{(2m-2)}$ have at least $n - \sigma + \frac{3}{2}m - 2$ essential zeros on $[x_\sigma,\, x_n]$. □

Theorem 3.6. *Suppose that $l \geq 3$. For all $s_i \in S_l(x_0,\, x_n);\, i = 1, \ldots, n$:*

(i) s_i is not identically zero on any subinterval $[x_j, x_{j+1}]$,

(ii) the sign of s_i does not change on $[x_j, x_{j+1}]$,

(iii) the sign of s_i, changes at x_j for all $j \neq i$.

Proof . Suppose $s_i(x) = 0$ for each $x \in [x_j, x_{j+1}]$. To prove theorem, we assume that $j+1 < i$, (a similar argument holds for $j > i$). Let

$$s(x) = \begin{cases} 0, & x_0 \le x \le x_{j+1}, \\ s_i(x), & x_{j+1} < x \le x_n. \end{cases}$$

Obviously $s \in S_l(x_0, x_n)$ and by the uniqueness of spline, $s_i(x) = 0$ for all $x \in [x_0, x_{j+1}]$. Let $[x_0, x_\sigma]$ be the maximal interval where $s_i(x)$ vanishes. Since $s_i(x) = 1$, we have $\sigma < i$.

Similarly, let for $\tau > i$, $[x_\tau, x_n]$ be the maximal interval such that $s_i(x) = 0$, for all $x \in [x_\tau, x_n]$. We apply Theorem 1 to s_i restricted to $[x_\tau, x_n]$. We now consider two cases: $\tau < n$ or $\tau = n$.

If $\tau < n$, then x_σ and x_τ are $l-$ fold zeros of s_i and since x_i is not a zero, then by Theorem 1 we have,

$$2l + (\tau - \sigma - 1) - 1 \le Z(s_i) \le (\tau - \sigma) + l - 1,$$

which implies $l \le 1$ and this contradicts $l \ge 3$.

If $\tau = n$, then by Theorem 3.5, $s_i^{(2m-2)}$ has at least $n - \sigma + \frac{3}{2}m - 2$ essential zeros on $[x_\sigma, x_n]$. Now, by this fact that $s_i^{(2m-2)} \in \mathrm{S}_1(x_\sigma, x_n)$ and hence by Theorem 1,

$$n - \sigma + \frac{3}{2}m - 2 \le Z(s_i^{(2m-2)}) \le n - \sigma,$$

which implies another contradiction $m \le 1$. Hence (i) is provided.

Let r be the number of essential zeros of s_i on $[x_0, x_n]$, $r \ge n$. By Rolle's Theorem, $s_i^{(\kappa)}$, $\kappa = 1, 2, \ldots, \frac{m}{2} - 1$, has at least r essential zeros and by Definition 3.1, $s_i^{(\frac{m}{2})}$ has at least $r+1$ essential zeros. Hence $s_i^{(2m-3)}$ and $s_i^{(2m-2)}$ have $r+1$ and r essential zeros, respectively. Now $s_i^{(2m-2)}$ has at least r essential zeros. But $s_i^{(2m-2)} \in S_1(x_0, x_n)$ and hence by Theorem 3.4,

$$n \le r \le Z(s_i^{(2m-2)}) \le n.$$

It follows that the essential zeros of s_i are x_j for $j \ne i$, with multiplicity one. This proves parts (ii) and (iii). $\square$

Theorem 3.7. *If $F(x) = \sum_{i=0}^{n} s_i(x) u_i$ be the interpolating fuzzy spline, then for all $x \in (x_i, x_{i+1})$ and for all $\alpha \in (0, 1]$,*

$$lenF^\alpha(x) \ge min\{lenF^\alpha(x_i), lenF^\alpha(x_{i+1})\},$$

where len denotes the length of an interval.

Proof . Since the addition does not decreases the length of an interval we have

$$lenF^\alpha(x) \ge len(\sum_{j=i}^{i+1} s_j(x) J_j^\alpha) \ge min\{lenJ_i^\alpha, lenJ_{j+1}^\alpha\} \sum_{j=i}^{i+1} |s_j(x)|.$$

Now, to complete our proof, we will show that $s(x) = s_i(x) + s_{i+1}(x) \ge 1$ for all $x \in (x_i, x_{i+1})$. The polynomial $s(x) = s_i(x) + s_{i+1}(x) \in S_l(x_0, x_n)$ interpolates the data (x_j, f_j), where $f_j = 1$ for $j = i$ and $i+1$ and zero otherwise. Suppose that $0 < i < n-1$ and $s(x) < 1$ for some $x \in (x_i, x_{i+1})$. Then $s'(x)$ has at least three zero in (x_{i-1}, x_{i+2}) and by using Rolle's Theorem $s'(x)$ has at least $n+1$ zeros on $[x_0, x_n]$. Hence $s^{(\kappa)}$, $\kappa = 1, 2, \ldots, (\frac{m}{2} - 1)$, has at least $n+1$ zeros and since $s^{(\frac{m}{2})}(x_0) = s^{(\frac{m}{2})}(x_n) = 0$, then $s^{(\frac{m}{2})}(x)$ has at least $n+2$ zeros on $[x_0, x_n]$. By repeating this argument $s^{(2m-3)}(x)$ has at least $n+2$ zeros on $[x_0, x_n]$ and hence $s^{(2m-2)}$ has at least $n+1$ zeros on $[x_0, x_n]$, which is a contradiction, since $\mathrm{s}^{(2m-2)}(x) \in S_1(x_0, x_n)$. $\square$

Theorem 3.8. *For any given function $y = f(x)$ defined at x_i; $i = 0, 1, \ldots, n$, there exists a unique spline function $s(x) \in S_l(x_0,\, x_n)$ which interpolates the function values $y_i = f(x_i)$.*

Proof . The proof is similar to the proof of Theorem 5 in [4]. □

4. Numerical examples

Let $J_i^\alpha = [a_i^\alpha,\, b_i^\alpha]$. Then the upper end point of $F^\alpha(x)$ is the solution of the following problem:

$$Maximize \;\; S_{y_0 y_1 \ldots y_n}$$
$$subject\ to \;\; a_i^\alpha \le y_i \le b_i^\alpha;\; i = 0, 1, \ldots, n,$$

where the optimal solution is

$$y_i = \begin{cases} b_i^\alpha, & if \;\; s_i(x) \ge 0, \\ a_i^\alpha, & if \;\; s_i(x) < 0. \end{cases}$$

Similarly the lower end point of $F^\alpha(x)$ can be obtained. Hence if $u_i = (m_i,\, l_i,\, r_i)$ and $F(x) = (m(x),\, l(x),\, r(x))$, then we will have,

$$m(x) = \textstyle\sum_{i=0}^{n} s_i(x) m_i,$$

$$l(x) = \sum_{s_i(x)\ge 0} s_i(x) l_i - \sum_{s_i(x)<0} s_i(x) r_i;$$

$$r(x) = \sum_{s_i(x)\ge 0} s_i(x) r_i - \sum_{s_i(x)<0} s_i(x) l_i,$$

which are the same results in Kaleva [8].

Example 4.1. Suppose we have the data (x_i, u_i)

x_i	1	1.1	1.2	3	3.5	4
m_i	0	5	1	4	0	1
l_i	2	1	0	4	3	1
r_i	1	2	3	3	2	1

and $l = 3$, *i.e.* using cubic spline, similar to Example 2.1 of Kaleva [8]. For example,

$$F(2.2) = (-10.8504, 14.8322, 19.5005), F(3.1) = (3.6820, 4.6174, 1.6812).$$

Figure 1 shows the zero, 0.5 and one level sets.

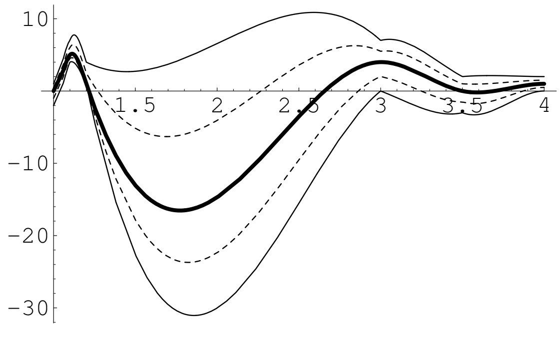

Fig. 1. The solid line represent the support and the dashed line represent 0.5-level set and the thick line represent 1-level set of $F(x)$.

Example 2. Here we have $u_i = y_i + A$; $i = 0, 1, \ldots, n$ and $A = (0, 1, 1)$ and $l = 3$, where

x_i	1	2	3	4	5	6
y_i	0	4	-1	1	5	0

Figure 2 shows the zero, 0.5 and one level sets.

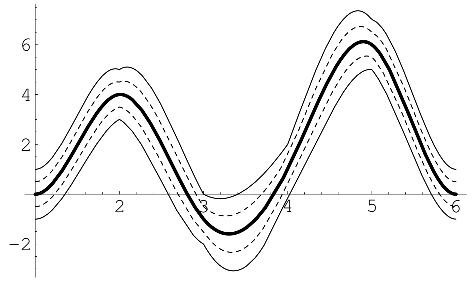

Fig. 2. The solid line represent the support and the dashed line represent 0.5-level set and the thick line represent 1-level set of $F(x)$.

Example 3. Suppose we have the data (x_i, u_i) and $l = 3$, where

x_i	1	1.1	1.2	1.3	1.4	1.5
m_i	0	5	1	4	0	1
l_i	2	1	0	4	3	1
r_i	1	2	3	3	2	1

Figure 3 shows the zero, 0.5 and one level sets of $F(x)$. To compare the results of $F(x)$ and other studied in [1, 2, 3, 4], see Figs. 3-7.

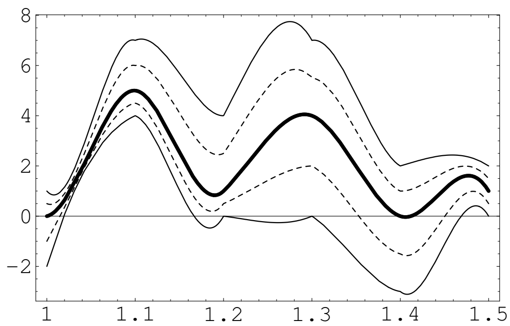

Fig. 3. The solid line represent the support and the dashed line represent 0.5-level set and the thick line represent 1-level set of $F(x)$.

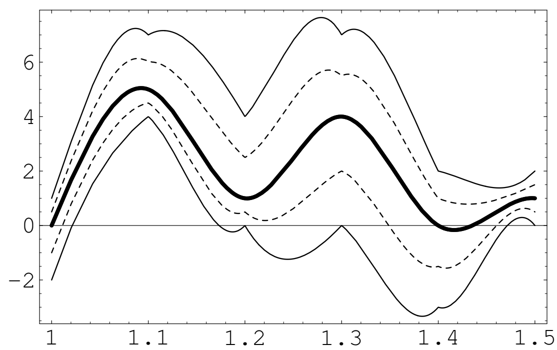

Fig. 4. The solid line represent the support and the dashed line represent 0.5-level set and the thick line represent 1-level set of cubic natural spline.

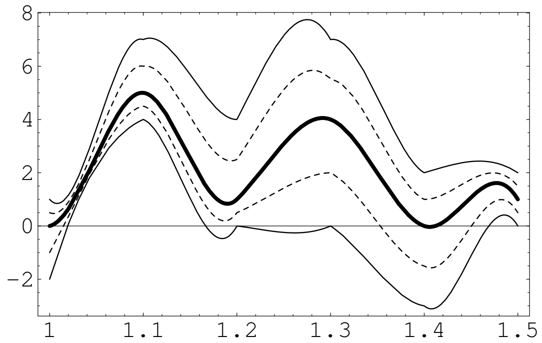

Fig. 5. The solid line represent the support and the dashed line represent 0.5-level set and the thick line represent 1-level set of *complete* cubic fuzzy spline.

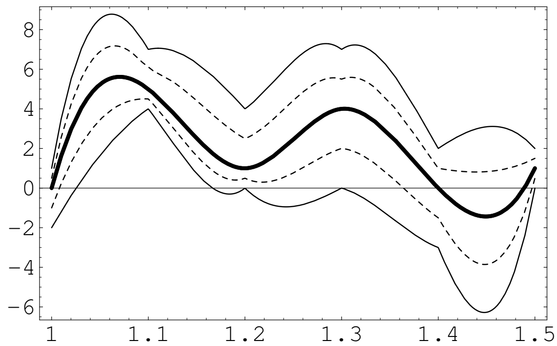

Fig. 6. The solid line represent the support and the dashed line represent 0.5-level set and the thick line represent 1-level set of $not - a - knot$ cubic fuzzy spline.

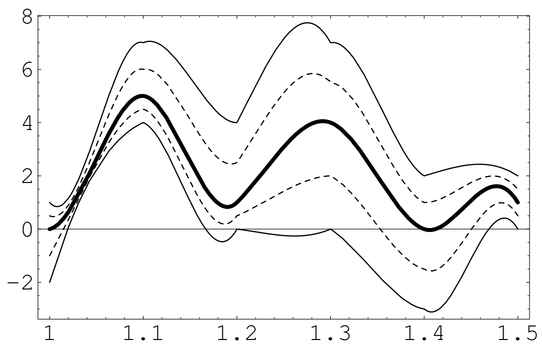

Fig. 7. The solid line represent the support and the dashed line represent 0.5-level set and the thick line represent 1-level set of $E(3)$ cubic fuzzy spline.

5. Conclusions

In this paper, we defined a new set of spline functions called "Flat End Fuzzy Spline" to interpolate given fuzzy data. To avoid complexity of fuzzy multiplication or fuzzy division in construction of full fuzzy interpolation, we proved that the sign of defined splines, s_i, dose not change on subintervals. Also, we presented numerical examples to illustrate the differences between of using our spline and other interpolations that have been studied before.

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