Int. J. Nonlinear Anal. Appl. 8 (2017) No. 2, 65-70
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2017.1078.1226

On the stability of linear differential equations of second order

Abbas Najati[a,*], Mohammad Reza Abdollahpour[a], Choonkil Park[b]

[a]Department of Mathematics, Faculty of Sciences, University of Mohaghegh Ardabili, Ardabil 56199-11367, Iran
[b]Department of Mathematics, Hanyang University, Seoul, 133–791, South Korea

(Communicated by H. Khodaei)

Abstract

The aim of this paper is to investigate the Hyers-Ulam stability of the linear differential equation

$$y''(x) + \alpha y'(x) + \beta y(x) = f(x)$$

in general case, where $y \in C^2[a,b]$, $f \in C[a,b]$ and $-\infty < a < b < +\infty$. The result of this paper improves a result of Li and Shen [*Hyers-Ulam stability of linear differential equations of second order,* Appl. Math. Lett. 23 (2010) 306–309].

Keywords: Hyers-Ulam stability; linear differential equation of second order.
2010 MSC: Primary 34K20, 26D10; Secondary 34K06, 39B52, 39B82.

1. Introduction and preliminaries

Let X be a normed space and I be an open interval in $\mathbb{R}$, the set of all real numbers. We say that the differential equation

$$a_n(x)y^{(n)}(x) + a_{n-1}(x)y^{(n-1)}(x) + \cdots + a_1(x)y'(x) + a_0y(x) + h(x) = 0, \quad x \in I, \tag{1.1}$$

has the Hyers-Ulam stability if, for any mapping $f : I \to X$ satisfying the differential inequality

$$\|a_n(x)f^{(n)}(x) + a_{n-1}(x)f^{(n-1)}(x) + \cdots + a_1(x)f'(x) + a_0f(x) + h(x)\| \leqslant \varepsilon$$

*Corresponding author
Email addresses: a.nejati@yahoo.com (Abbas Najati), mrabdollahpour@yahoo.com (Mohammad Reza Abdollahpour), baak@hanyang.ac.kr (Choonkil Park)

Received: December 2015 *Revised:* April 2017

for all $x \in I$ and $\varepsilon \geq 0$, there exists a solution $g : I \to X$ of (1.1) such that $\|f(x) - g(x)\| \leqslant K(\varepsilon)$ for all $x \in I$, where $K(\varepsilon)$ is an expression for ε only.

The question concerning the stability of group homomorphisms was posed by Ulam [27]. More precisely, He proposed the following problem: Given a group $(G_1, \cdot)$, a metric group $(G_2, *, d)$ and a positive number ϵ, does there exist a $\delta > 0$ such that if a function $f : G_1 \to G_2$ satisfies the inequality $d(f(x.y), f(x)*f(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $T : G_1 \to G_2$ such that $d(f(x), T(x)) < \epsilon$ for all $x \in G_1$? If this problem has a solution, we say that the homomorphisms from G_1 to G_2 are stable. In 1941, Hyers [7] gave a partial solution of Ulam's problem for the case of approximate additive mappings under the assumption that G_1 and G_2 are Banach spaces. Aoki [3] and Th.M. Rassias [24] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded. In fact an answer has been given in the following way.

Theorem 1.1. Suppose X and Y are two real Banach spaces and $f : X \to Y$ is a mapping such that $f(tx)$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in X$. If there exist $\theta \geqslant 0$ and $p \in \mathbb{R} \setminus \{1\}$ such that

$$\|f(x+y) - f(x) - f(y)\| \leqslant \theta(\|x\|^p + \|y\|^p), \quad x, y \in X,$$

then there is a unique linear mapping $T : X \to Y$ such that

$$\|f(x) - T(x)\| \leqslant \frac{2\theta}{|2 - 2^p|}\|x\|^p, \quad x \in X.$$

In 1941, Hyers [7] obtained the result for $p = 0$. And then, Aoki [3] and Th. M. Rassias [24] generalized the above result of Hyers to the case where $0 \leqslant p < 1$. Moreover, Th. M. Rassias noticed in [24] that the proof also works for $p < 0$. A similar result was obtained by Gajda [8] for $p > 1$. In the same paper, Gajda showed that a similar result does not hold for $p = 1$ (see also [25].) Since then, the stability problems of various functional equations have been investigated by many authors. We refer the interested reader to [15] and papers [6, 5, 9, 23] for the stability problems of functional equations in details.

In connection with the stability of exponential functions, the Hyers-Ulam stability of differential equations has been investigated by Alsina and Ger [4] (see also [20, 22]): If $\varepsilon > 0$ and a differentiable function $f : I \to \mathbb{R}$ satisfies the differential inequality $|y'(x) - y(x)| \leqslant \varepsilon$, where I is an open interval of $\mathbb{R}$, then there exists a differentiable function $f_0 : I \to \mathbb{R}$ satisfying $f_0'(x) = f_0(x)$ such that $|f(x) - f_0(x)| \leqslant 3\varepsilon$ for all $x \in I$.

The above result by Alsina and Ger was generalized by Miura, Takahasi and Choda [19], by Miura [16], also by Takahasi, Miura and Miyajima [26]. Indeed, they dealt with the Hyers-Ulam stability of the differential equation $y'(t) = \lambda y(t)$, while Alsina and Ger investigated the differential equation $y'(t) = y(t)$. Miura et al [18] proved the Hyers-Ulam stability of the first-order linear differential equations $y'(t) + g(t)y(t) = 0$, where $g(t)$ is a continuous function, while Jung [12] proved the Hyers-Ulam stability of differential equations of the form $\varphi(t)y'(t) = y(t)$.

Theorem 1.2. [12] Let $\varphi : I = (a, b) \to \mathbb{R}$ be a given function for which the integral $\int_a^t dx/\varphi(x)$ exists for any $t \in I$. If either $\varphi(t) > 0$ holds for all $t \in I$ or $\varphi(t) < 0$ holds for all $t \in I$, and if a differentiable function $y : I \to \mathbb{R}$ satisfies inequality

$$|\varphi(t)y'(t) - y(t)| \leqslant \varepsilon, \quad t \in I,$$

then there exists a real number c such that

$$\left| y(t) - c\exp\left\{ \int_a^t \frac{dx}{\varphi(x)} \right\} \right| \leqslant \varepsilon, \quad t \in I.$$

The above result has been generalized by some authors. We refer the interested readers to [1, 2, 11, 12, 14, 16, 17].

Recently, Li and Shen [21] have investigated the Hyers-Ulam stability of the following linear differential equations of second order

$$y''(x) + \alpha y'(x) + \beta y(x) = f(x), \tag{1.2}$$

where $y \in C^2[a,b]$, $f \in C[a,b]$ and $-\infty < a < b < +\infty$. Indeed, they proved that, if the characteristic equation $\lambda^2 + \alpha\lambda + \beta = 0$ has two different positive roots, then the differential equation $y''(x) + \alpha y'(x) + \beta y(x) = f(x)$ has the Hyers-Ulam stability.

The aim of this paper is to investigate the Hyers-Ulam stability of the linear differential equation (1.2) in general case. More precisely, we prove that the equation $y''(x)+\alpha y'(x)+\beta y(x) = f(x)$ always has the Hyers-Ulam stability and the proof methods are different from those of [21] and others.

2. Hyers-Ulam stability of the differential equation $y'' + \alpha y' + \beta y = f(x)$

In the following theorem, we prove the Hyers-Ulam stability of the differential equation (1.2), which obviously improves a result of Li and Shen [21]. Throughout this section, a and b are real numbers with $-\infty < a < b < +\infty$.

Theorem 2.1. *The differential equation $y'' + \alpha y' + \beta y = f(x)$ has the Hyers-Ulam stability, where $y \in C^2[a,b]$ and $f \in C[a,b]$.*

Proof . Suppose that μ, λ are the (real or complex) roots of $z^2 + \alpha z + \beta = 0$. Let $\mu = p + ic$ and $\lambda = q + id$ for $p, c, q, d \in \mathbb{R}$. Let $\varepsilon > 0$ and $y \in C^2[a,b]$ with $|y'' + \alpha y' + \beta y - f(x)| \leqslant \varepsilon$. Let $g(x) = y'(x) - \lambda y(x)$ and $z(x) = e^{\mu(x-b)}g(b) - e^{\mu x}\int_x^b f(t)e^{-\mu t}\,dt$ for all $x \in [a,b]$. Then we have

$$z'(x) = \mu z(x) + f(x) \tag{2.1}$$

for all $x \in [a,b]$. It is clear that

$$|g'(x) - \mu g(x) - f(x)| = |y''(x) + \alpha y'(x) + \beta y(x) - f(x)| \leqslant \varepsilon$$

and

$$\begin{aligned}
|z(x) - g(x)| &= \left|e^{\mu(x-b)}g(b) - g(x) - e^{\mu x}\int_x^b f(t)e^{-\mu t}\,dt\right| \\
&= |e^{\mu x}|\left|e^{-\mu b}g(b) - e^{-\mu x}g(x) - \int_x^b f(t)e^{-\mu t}\,dt\right| \\
&= e^{px}\left|\int_x^b [e^{-\mu t}g(t)]'\,dt - \int_x^b f(t)e^{-\mu t}\,dt\right| \\
&= e^{px}\left|\int_x^b e^{-\mu t}[g'(t) - \mu g(t) - f(t)]\,dt\right| \\
&\leqslant e^{px}\int_x^b |e^{-\mu t}||g'(t) - \mu g(t) - f(t)|\,dt \\
&\leqslant e^{px}\int_x^b e^{-pt}|g'(t) - \mu g(t) - f(t)|\,dt \\
&\leqslant \varepsilon e^{px}\int_x^b e^{-pt}\,dt
\end{aligned}$$

for all $x \in [a,b]$. Therefore, it follows that

$$|z(x)-g(x)| \leqslant \begin{cases} \dfrac{1-e^{-p(b-a)}}{p}\varepsilon, & \text{if } p \neq 0; \\ \\ (b-a)\varepsilon, & \text{if } p = 0 \end{cases} \tag{2.2}$$

for all $x \in [a,b]$. Now, we define

$$u(x) = y(b)e^{\lambda(x-b)} - e^{\lambda x}\int_x^b z(t)e^{-\lambda t}\,dt$$

for all $x \in [a,b]$. It is clear that $u \in C^2[a,b]$ and $u'(x) = \lambda u(x) + z(x)$. Hence (2.1) implies that $u''(x) + \alpha u'(x) + \beta u(x) = f(x)$ for all $x \in [a,b]$. Also, u satisfies the following

$$\begin{aligned} |y(x)-u(x)| &= \left| y(x) - y(b)e^{\lambda(x-b)} + e^{\lambda x}\int_x^b z(t)e^{-\lambda t}\,dt \right| \\ &= |e^{\lambda x}| \left| y(x)e^{-\lambda x} - y(b)e^{-\lambda b} + \int_x^b z(t)e^{-\lambda t}\,dt \right| \\ &= e^{qx} \left| \int_x^b z(t)e^{-\lambda t}\,dt - \int_x^b [y'(t)-\lambda y(t)]e^{-\lambda t}\,dt \right| \\ &= e^{qx} \left| \int_x^b [z(t)-y'(t)+\lambda y(t)]e^{-\lambda t}\,dt \right| \\ &\leqslant e^{qx} \int_x^b |z(t)-y'(t)+\lambda y(t)||e^{-\lambda t}|\,dt \\ &\leqslant e^{qx} \int_x^b |z(t)-g(t)|e^{-qt}\,dt \end{aligned}$$

for all $x \in [a,b]$. It follows from (2.2) that

$$|y(x)-u(x)| \leqslant \begin{cases} \dfrac{[1-e^{-p(b-a)}][1-e^{-q(b-a)}]}{pq}\varepsilon, & \text{if } p, q \neq 0; \\ \\ \dfrac{[1-e^{-p(b-a)}](b-a)}{p}\varepsilon, & \text{if } p \neq 0,\ q = 0; \\ \\ \dfrac{[1-e^{-q(b-a)}](b-a)}{q}\varepsilon, & \text{if } p = 0,\ q \neq 0; \\ \\ (b-a)^2\varepsilon, & \text{if } p, q = 0 \end{cases}$$

for all $x \in [a,b]$. This completes the proof. □

In the following corollaries we assume that $\alpha, \beta \in \mathbb{R}$.

Corollary 2.2. Let $y \in C^2[a,b]$ and $f \in C[a,b]$. If $\varepsilon \geqslant 0$ and $|y''(x) + \alpha y'(x) - f(x)| \leqslant \varepsilon$ for all $x \in [a,b]$, then there exists $u \in C^2[a,b]$ satisfying $u''(x) + \alpha u'(x) = f(x)$ and

$$|y(x)-u(x)| \leqslant \begin{cases} \dfrac{[e^{\alpha(b-a)}-1](b-a)}{\alpha}\varepsilon, & \text{if } \alpha \neq 0; \\ \\ (b-a)^2\varepsilon, & \text{if } \alpha = 0 \end{cases}$$

for all $x \in [a, b]$

Corollary 2.3. Let $y \in C^2[a, b]$ and $f \in C[a, b]$. If $\varepsilon \geqslant 0$ and $|y''(x) + \beta y(x) - f(x)| \leqslant \varepsilon$ for all $x \in [a, b]$, then there exists $u \in C^2[a, b]$ satisfying $u''(x) + \beta u(x) = f(x)$ and

$$|y(x) - u(x)| \leqslant \begin{cases} \dfrac{[1 - e^{-\sqrt{-\beta}(b-a)}][1 - e^{\sqrt{-\beta}(b-a)}]}{\beta}\varepsilon, & \text{if } \beta < 0; \\ (b-a)^2\varepsilon, & \text{if } \beta \geqslant 0 \end{cases}$$

for all $x \in [a, b]$

Remark 2.4. The Hyers-Ulam stability the second order linear differential equation $y'' + \beta(x)y = 0$ with boundary conditions $y(a) = y(b) = 0$ or with initial conditions $y(a) = y'(a) = 0$ has been investegated in [10]. Indeed the following results have been proved.

Theorem 2.5. Let $a, b \in \mathbb{R}$, $a < b$, $y \in C^2([a, b])$ and $\beta \in C([a, b])$. If $\max|\beta(x)| < \frac{8}{(b-a)^2}$, then the second order linear differential equation $y'' + \beta(x)y = 0$ has the Hyers-Ulam stability with boundary conditions $y(a) = y(b) = 0$.

Theorem 2.6. Let $a, b \in \mathbb{R}$, $a < b$, $y \in C^2([a, b])$ and $\beta \in C([a, b])$. If $\max|\beta(x)| < \frac{2}{(b-a)^2}$, then the second order linear differential equation $y'' + \beta(x)y = 0$ has the Hyers-Ulam stability with boundary conditions $y'(a) = y(a) = 0$.

Acknowledgments

C. Park was supported by Basic Science Research Program through the National Research Foundation of Korea funded by the Ministry of Education, Science and Technology (NRF-2012R1A1A2004299).

References

[1] M.R. Abdollahpour, A. Najati, C. Park, Th.M. Rassias and D.Y. Shin, *Approximate perfect differential equations of second order*, Adv. Difference Equ. 2012, 2012: 225, 5 pages.
[2] M.R. Abdollahpour and A. Najati, *Stability of linear differential equations of third order*, Appl. Math. Lett. 24 (2011) 1827–1830.
[3] T. Aoki, *On the stability of the linear transformationin Banach spaces*, J. Math. Soc. Japan 2 (1950) 64–66.
[4] C. Alsina and R. Ger, *On some inequalities and stability results related to the exponential function*, J. Inequal. Appl. 2 (1998) 373–380.
[5] M. Eshaghi Gordji and A. Najati, *Approximately J^*-homomorphisms: a fixed point approach*, J. Geom. Phys. 60 (2010) 809–814.
[6] G.L. Forti, *Hyers-Ulam stability of functional equations in several variables*, Aeq. Math. 50 (1995) 143–190.
[7] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Natl. Acad. Sci. U.S.A. 27 (1941) 222–224.
[8] Z. Gajda, *On stability of additive mappings*, Internat. J. Math. Math. Sci. 14 (1991) 431–434.
[9] P. Găvruta and L. Găvruta, *A new method for the generalized Hyers-Ulam-Rassias stability,* Int. J. Nonlinear Anal. Appl. 2 (2010) 11–18.
[10] P. Găvruta, Soon-Mo Jung and Y. Li, *Hyers-Ulam stability for second order linear differential equations with boundary conditions,* Elect. J. Diff. Eq. 80 (2011) 5 pages.
[11] M.B. Ghaemi, M. Eshaghi Gordji, B. Alizadeh and C. Park, *Hyers-Ulam stability of exact second order linear differential equations*, Adv. Diff. Eq. 2012 Art. No. 36, 7 pages.

[12] S. Jung, *Hyers-Ulam stability of linear differential equations of first order*, Appl. Math. Lett. 17 (2004) 1135–1140.

[13] S. Jung, *Approximation of analytic functions by Laguerre functions*, Appl. Math. Comput. 218 (2011) 832–835.

[14] S. Jung, *Hyers-Ulam stability for Gegenbauer differential equations*, Electron. J. Diff. Eq. 2013, No. 156, 8 pages.

[15] S.-M. Jung, *Hyers-Ulam–Rassias Stability of Functional Equations in Mathematical Analysis*, Hadronic Press, Palm Harbor, 2001.

[16] T. Miura, *On the Hyers-Ulam stability of a differentiable map*, Sci. Math. Japon 55 (2002) 17–24.

[17] T. Miura, S. Jung and S.-E. Takahasi, *Hyers-Ulam-Rassias stability of the Banach space valued differential equations* $y' = \lambda y$, J. Korean Math. Soc. 41 (2004) 995–1005.

[18] T. Miura, S. Miyajima and S.-E. Takahasi, *A characterization of Hyers-Ulam stability of first order linear differential operators,* J. Math. Anal. Appl. 286 (2003) 136–146.

[19] T. Miura, S.-E. Takahasi and H. Choda, *On the Hyers-Ulam stability of real continuous function valued differentiable map,* Tokyo J. Math. 24 (2001) 467–476.

[20] M. Obłoza, *Hyers stability of the linear differential equation*, Rocznik Nauk.-Dydakt. Prace Mat. 13 (1993) 259–270.

[21] Y. Li and Y. Shen, *Hyers-Ulam stability of linear differential equations of second order*, Appl. Math. Lett. 23 (2010) 306–309.

[22] M. Obłoza, *Connections between Hyers and Lyapunov stability of the ordinary differential equations*, Rocznik Nauk.-Dydakt. Prace Mat. 14 (1997) 141–146.

[23] C. Park and A. Najati, *Generalized additive functional inequalities in Banach algebras,* Int. J. Nonlinear Anal. Appl. 2 (2010) 54–62.

[24] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces,* Proc. Amer. Math. Soc. 72 (1978) 297–300.

[25] Th. M. Rassias and P. Šemrl, *On the behavior of mappings which do not satisfy Hyers-Ulam stability,* Proc. Amer. Math. Soc. 114 (1992) 989993.

[26] S.-E. Takahasi, T. Miura and S. Miyajima, *On the Hyers-Ulam stability of the Banach space-valued differential equation* $y' = \lambda y$, B. Korean Math. Soc. 39 (2002) 309–315.

[27] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ., New York, 1960.