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Translation invariant mappings on KPC-hypergroups

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Abstract

In this paper, we give an extension of the Wendel's theorem on KPC-hypergroups. We also show that every translation invariant mapping is corresponding with a unique positive measure on the KPC-hypergroup.

Keywords: DJS-hypergroup; KPC-hypergroup; Translation Invariant Mapping; Wendel's Theorem.
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1. Introduction

Locally compact hypergroups, as extensions of locally compact groups, were introduced in a series of papers by Dunkl [2], Jewett [4], and Spector [10] in 70's (we refer to this definition of hypergroup as DJS-hypergroup). For more details about DJS-hypergroups we refer to [1] and [9]. In 2010, Kalyuzhnyi, Podkolzin, and Chapovsky [5] have introduced new axioms for hypergroups. This new concept is an extension of DJS-hypergroups, and generalizes a normal hypercomplex system with a basis unity to the nonunimodular case. We refer to this notion as KPC-hypergroup. They show that there is an example of a compact KPC-hypergroup related to the generalized Tchebycheff polynomials, which is not a DJS-hypergroup [5]. Kalyuzhnyi et al, study harmonic analysis on KPC-hypergroups in [5] (see also [11]). In this paper, for a KPC–hypergroup Q, we give an extension of Wendel's theorem which presents some equivalence conditions for bounded linear operators on $L^1(Q)$ commute with translation operators. This theorem was proved for locally compact abelian groups by Larsen in 1971 [8]. Then, Lasser extended this theorem on locally compact commutative DJS-hypergroups in 1982 [7]. In 2010, Youmbi proved it for not necessarily commutative DJS-hypergroups [12]. In this paper, we give an extension of this theorem for cocommutative KPC-hypergroups. Also, we show that any translation invariant mapping on a cocommutative KPC-hypergroup corresponds with a unique positive measure.

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2. Preliminaries

Let Q be a locally compact Hausdorff space. We denote by $M(Q)$ the space of all complex Radon measures on Q, by $M_b(Q)$ the set of all bounded measures in $M(Q)$, and by $M^+(Q)$ the set of all positive measures in $M(Q)$. The spaces of complex-valued functions that are continuous, continuous and bounded, continuous with compact support, continuous and equal to zero at infinity are denoted by $C(Q)$, $C_b(Q)$, $C_c(Q)$, and $C_0(Q)$, respectively. The support of a function f is denoted by $\mathrm{supp}(f)$.

First, we recall the definition and some properties of the locally compact cocommutative KPC-hypergroups. For more details we refer to [5].

Definition 2.1. Let Q be a locally compact second countable Hausdorff space with an involutive homeomorphism $\star : Q \longrightarrow Q$ satisfying the following conditions:

1. there is an element $e \in Q$ such that $e^\star = e$;
2. there is a $\mathbb{C}$-linear mapping $\Delta : C(Q) \to C(Q \times Q)$ such that
   i. Δ is co-associative, that is,
   $$(\Delta \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \Delta) \circ \Delta;$$
   ii. Δ is positive, that is, $\Delta f \geq 0$ for all $f \in C(Q)$ such that $f \geq 0$;
   iii. Δ preserves the identity, that is, $(\Delta 1)(p, q) = 1$ for all $p, q \in Q$;
   iv. For all $f, g \in C_c(Q)$ we have $(1 \otimes f).(\Delta g) \in C_c(Q \times Q)$ and $(f \otimes 1).(\Delta g) \in C_c(Q \times Q)$.
3. the homomorphism $\epsilon : C(Q) \to \mathbb{C}$ defined by $\epsilon(f) = f(e)$, satisfies the counit property, that is,
   $$(\epsilon \times \mathrm{id}) \circ \Delta = (\mathrm{id} \times \epsilon) \circ \Delta = \mathrm{id},$$
   in other words, $(\Delta f)(e, p) = (\Delta f)(p, e) = f(p)$ for all $p \in Q$.
4. the function $\check{f}$ defined by $\check{f}(q) = f(q^\star)$ for $f \in C(Q)$ satisfies
   $$(\Delta \check{f})(p, q) = (\Delta f)(q^\star, p^\star).$$
5. there exists a positive measure m on Q, supp $m = Q$, such that
   $$\int_Q (\Delta f)(p, q) g(q) dm(q) = \int_Q f(q)(\Delta g)(p^\star, q) dm(q)$$
   for all $f \in C_b(Q)$ and $g \in C_c(Q)$, or $f \in C_c(Q)$ and $g \in C_b(Q), p \in Q$; such a measure m will be called a left Haar measure on Q.

Then $(Q, \star, e, \Delta, m)$, or simply Q, is called a locally compact KPC-hypergroup.

Notation. In the above definition, we have used the following notations:

$$[(\Delta \times \mathrm{id}) \circ \Delta(f)](p, q, r) := \Delta(\Delta f(p, \cdot))(q, r),$$
$$[(\mathrm{id} \times \Delta) \circ \Delta(f)](p, q, r) := \Delta(\Delta f(\cdot, q))(p, r),$$
$$[(\epsilon \times \mathrm{id}) \circ \Delta(f)](p) := \epsilon(\Delta f(p, \cdot)) = \Delta f(p, e),$$
$$[(\mathrm{id} \times \epsilon) \circ \Delta(f)](p) := \epsilon(\Delta f(\cdot, p)) = \Delta f(e, p),$$
$$(f \otimes 1)(p, q) \cdot (\Delta g)(p, q) = f(p)1(q) \cdot \Delta g(p, q),$$
$$(1 \otimes f)(p, q) \cdot (\Delta g)(p, q) := 1(p)f(q) \cdot \Delta g(p, q),$$

where $f \in C(Q)$ and $p, q, r \in Q$.

A KPC-hypergroup Q is called cocommutative if $\Delta f(p, q) = \Delta f(q, p)$, for all $f \in C_b(Q)$ and all $p, q \in Q$.

Throughout this paper Q is a locally compact cocommutative KPC-hypergroup and m is a left Haar measure on Q.

Definition 2.2. Let $\mu, \nu \in M(Q)$ be such that the linear functional $\mu * \nu$ defined by

$$(\mu * \nu)(f) = \int_Q \int_Q \Delta(f)(p,q) d\mu(p) d\nu(q), \qquad (f \in C_c(Q))$$

is a measure. Then the measures μ and ν are called convolvable. Specially, we have $(\delta_p * \delta_q)(f) = (\Delta f)(p,q)$, where $p, q \in Q$.

If $\mu, \nu \in M(Q)$ are bounded, then μ and ν are convolvable ([5], Lemma 3.3).

Definition 2.3. Let m be a left Haar measure on Q. The convolution of complex-valued Borel measurable functions f and g on Q is denoted by $f * g$ and is defined by

$$(f * g)(q) = \int_Q f(p)(\Delta g)(p^\star, q) dm(p),$$

where $q \in Q$.

Definition 2.4. Let A be a $C^\star$–algebra and $B \subseteq A$ be a $C^\star$-subalgebra of A. A bounded linear map $P : A \longrightarrow B$ is called a conditional expectation if it satisfies in the following properties:

(i) $P^2 = P$ and $||P|| = 1$;
(ii) P is positive, that is $P(a^\star a) \geq 0$ for any $a \in A$;
(iii) $P(b_1 a b_2) = b_1 P(a) b_2$ for any $a \in A$ and $b_1, b_2 \in B$;
(iv) $P(a^\star)P(a) \leq P(a^\star a)$ for all $a \in A$.
It follows from (ii) and the polarization identity that
(v) $P(a^\star) = P(a)^\star$ for all $a \in A$.

Example 2.5. Let $(Q, \star, e, \Delta, m)$ be a KPC-hypergroup, A denote the $C^\star$-algebra $C_b(Q)$, A_0 its $C^\star$-subalgebra $C_0(Q)$, and let I be the ideal of A consisting of functions with compact support. Let $P : A \longrightarrow A$ be a conditional expectation such that $B := P(A_0)$ is a $C^\star$-algebra, $P(I) \subseteq I$, and the following hold:

$$((P \times id) \circ \Delta \circ P)(f) = ((id \times P) \circ \Delta \circ P)(f) = ((P \times P) \circ \Delta)(f),$$

$$P(\check{f}) = (P(f))\check{},$$

for all $f \in A$. Denote by $\tilde{Q}$ the spectrum of the commutative algebra B, which is a Hausdorff locally compact space. For each $g \in B \subset A$, let

$$\tilde{\Delta}(g) = ((P \times P) \circ \Delta)(g).$$

If $\tilde{q} \in \tilde{Q}$ and $g \in B$, then we set

$$\tilde{q}^\star(g) = \check{g}(q), \quad \tilde{e} = \epsilon,$$

and $\tilde{\mu}$ is defined by

$$\tilde{m} = m \circ P.$$

Then $(\tilde{Q}, \star, \tilde{e}, \tilde{\Delta}, \tilde{m})$ is a KPC-hypergroup [6].

Definition 2.6. We denote the convolution of the function $f \in C_c(Q)$ and the measure $\mu \in M_b(Q)$ by $\mu * f$ and define as following

$$(\mu * f)(q) := \int_Q \Delta f(p^\star, q) d\mu(p). \qquad (q \in Q)$$

Proposition 2.7. *Let $\mu \in M_b(Q)$ and $f \in C_c(Q)$. Then $\mu * f$ is an element of $L^1(Q)$.*

Proof . Let $\mu \in M_b(Q)$ and $f \in C_c(Q)$. We have $|\Delta f(p,q)| = |\int f(t) d(\delta_p * \delta_q)(t)| \leq \int |f(t)| d(\delta_p * \delta_q)(t) = \Delta|f|(p,q)$, where $p, q \in Q$. Thus, by Definition 2.1, we have

$$\begin{aligned}
\|\mu * f\|_1 &= \int_Q |(\mu * f)(q)| dm(q) \\
&\leq \int_Q \left(\int_Q |\Delta f(p^\star, q)| d|\mu|(p) \right) dm(q) \\
&\leq \int_Q \left(\int_Q \Delta|f|(p^\star, q) dm(q) \right) d|\mu|(p) \\
&= \int_Q \left(\int_Q 1(q) \Delta|f|(p^\star, q) dm(q) \right) d|\mu|(p) \\
&= \int_Q \left(\int_Q \Delta 1(p, q) |f|(q) dm(q) \right) d|\mu|(p) \\
&= \int_Q \int_Q |f|(q) dm(q) d|\mu|(p) \\
&= \|f\|_1 \int_Q d|\mu|(p) = \|f\|_1 \|\mu\| < \infty.
\end{aligned}$$

Then $\mu * f \in L^1(Q)$. $\square$

Corollary 2.8. *Let $\mu \in M_b(Q)$ and $f \in L^1(Q)$. Let $(f_n)_{n=1}^\infty \subseteq C_c(Q)$ such that $f_n \to f$ in $L^1(Q)$ as $n \to \infty$. Then $\lim_{n\to\infty}(\mu * f_n)$ exists, is an element of $L^1(Q)$, and is independent from the choice of $(f_n)_{n=1}^\infty$.*

Proof . Under the hypothesis, by Proposition 2.7, the function $\mu * f$ is an element of $L^1(Q)$ and for $m, n \in \mathbb{N}$ we have

$$\|\mu * f_n - \mu * f_m\|_1 = \|\mu * [f_n - f_m]\|_1 \leq \|\mu\| \cdot \|f_n - f_m\|_1.$$

Then, since $f_n \to f$ in $L^1(Q)$ as $n \to \infty$, $(\mu * f_n)_{n=1}^\infty$ is a cauchy (and so convergrnce) sequence in $L^1(Q)$. Let $f \in L^1(Q)$, and $(f_n)_{n=1}^\infty, (h_n)_{n=1}^\infty \subseteq C_c(Q)$ such that $f_n \to f$ and $h_n \to f$ in $L^1(Q)$ as $n \to \infty$. Then by Proposition 2.7, we have

$$\begin{aligned}
\|\mu * f_n - \mu * h_n\|_1 &= \|\mu * (f_n - h_n)\|_1 \\
&\leq \|\mu\| \|f_n - h_n\|_1 \\
&= \|\mu\| \|f_n - h_n + f - f\|_1 \\
&\leq \|\mu\| \left(\|f_n - f\|_1 + \|h_n - f\|_1 \right).
\end{aligned}$$

So $\|\mu * f_n - \mu * h_n\|_1 \to 0$, and hence, $\lim_{n\to\infty} \mu * f_n = \lim_{n\to\infty} \mu * h_n$. $\square$

Definition 2.9. Let μ, f, and $(f_n)_{n=1}^\infty$ be as in Corollary 2.8. We call the function $\lim_{n\to\infty}(\mu * f_n)$, the convolution of μ and f, and denote it by $\mu * f$.

3. Translation invariant mappings on KPC-hypergroups

Definition 3.1. A positive linear mapping $T : C_c(Q) \longrightarrow C(Q)$ is called translation invariant if for any $p \in Q$ and $f \in C_c(Q), T(\delta_p * f) = \delta_p * Tf$.

Theorem 3.2. *A mapping $T : C_c(Q) \longrightarrow C(Q)$ is translation invariant if and only if there exists a unique positive measure $\mu \in M_b(Q)$ such that $Tf = \mu * f$ for any f in $C_c(Q)$.*

Proof . Suppose that there exists a unique positive measure $\mu \in M_b(Q)$ such that $Tf = \mu * f$, for all $f \in C_c(Q)$. Since Δ is a positive mapping, $\mu * f$, T is positive too. Clearly, T is linear, and since Q is cocommutative, we have

$$\begin{aligned}(\mu * \nu)(f) &= \int_Q \int_Q \Delta f(p, q) d\mu(p) d\nu(q) \\ &= \int_Q \int_Q \Delta f(q, p) d\nu(q) d\mu(p) \\ &= (\nu * \mu)(f).\end{aligned}$$

In particular, $\mu * \delta_a = \delta_a * \mu$, for all $a \in Q$. Therefore,

$$T(\delta_p * f) = \mu * (\delta_p * f) = (\mu * \delta_p) * f = (\delta_p * \mu) * f = \delta_p * (\mu * f) = \delta_p * Tf,$$

where $p \in Q$, i.e. T is translation invariant. Conversely, let T be a translation invariant mapping. Then, the mapping $f \mapsto T(\check{f})(e)$ is bounded, linear and positive. By Riesz representation theorem, there is a measure $\mu \in M(Q)$ such that $T(\check{f})(e) = \int f(p) d\mu(p)$, for all $f \in C_c(Q)$. Also, if $f \in C_c(Q)$ and $p \in Q$, we have

$$\begin{aligned}(\mu * f)(p) &= \int_Q \Delta f(q^\star, p) d\mu(q) = \int_Q \Delta f(p, q^\star) d\mu(q) = \int_Q \int_Q \Delta f(t, q^\star) d\delta_p(t) d\mu(q) \\ &= \int_Q \int_Q \Delta f(t^\star, q^\star) d\delta_p(t^\star) d\mu(q) = \int_Q \int_Q \Delta f(t^\star, q^\star) d\delta_{p^\star}(t) d\mu(q) = \int_Q (\delta_{p^\star} * f)(q^\star) d\mu(q) \\ &= T(\delta_{p^\star} * f)(e) = \delta_{p^\star} * Tf(e) = \int_Q \Delta Tf(q^\star, e) d\delta_{p^\star}(q) = \Delta Tf(p, e) = Tf(p)\end{aligned}$$

and the proof is completed. □

Definition 3.3. A function $\chi \in C_b(Q)$ is called a character of a cocommutative KPC-hypergroup Q if $(\Delta\chi)(p, q) = \chi(p)\chi(q)$ and $\chi(p^\star) = \overline{\chi(p)}$, for all $p, q \in Q$.

Definition 3.4. For any $f \in L^1(Q)$ and $\mu \in M(Q)$, the Fourier-Stieltjes transform $\hat{\mu}$ of μ and the Fourier transform $\hat{f}$ of f are defined by

$$\hat{\mu}(\xi) = \int_Q \overline{\xi(t)} d\mu(t) \text{ and } \hat{f}(\xi) = \int_Q f(t) \overline{\xi(t)} dm(t),$$

respectively, where $\xi \in \hat{Q}$. For definition of $\hat{Q}$ we refer to [5].

Lemma 3.5. *Let $\mu, \nu \in M_b(Q)$. Then $\widehat{(\mu * \nu)} = \hat{\mu}\hat{\nu}$. In particular, $\widehat{(f * g)} = \hat{f}\hat{g}$, for all $f, g \in C_c(Q)$.*

Proof .

$$\begin{aligned}
\hat{\mu}(\xi)\hat{\nu}(\xi) &= \int_Q \bar{\xi}(p)d\mu(p)\int_Q \bar{\xi}(q)d\nu(q) = \int_Q\int_Q \xi(p^\star)\xi(q^\star)d\mu(p)d\nu(q)\\
&= \int_Q\int_Q \Delta\xi(p^\star,q^\star)d\mu(p)d\nu(q) = \int_Q\int_Q \Delta\xi(q^\star,p^\star)d\mu(p)d\nu(q)\\
&= \int_Q\int_Q \Delta\check{\xi}(p,q)d\mu(p)d\nu(q) = \int_Q \bar{\xi}(t)d(\mu*\nu)(t) = (\mu*\nu)\hat{}(\xi).
\end{aligned}$$

Similarly, one can see that $(f*g)\hat{} = \hat{f}\hat{g}$, for all $f,g\in C_c(Q)$. $\square$

Lemma 3.6. *For any $\mu\in M_b(Q)$ and $f\in C_c(Q)$, we have $(\mu*f)\hat{} = \hat{\mu}\hat{f}$.*

Proof . Let $\mu\in M_b(Q)$ and $f\in C_c(Q)$. By Definition 2.1, for any $\xi\in\hat{Q}$ we have

$$\begin{aligned}
(\mu*f)\hat{}(\xi) &= \int(\mu*f)(p)\bar{\xi}(p)dm(p) = \int(\mu*f)(p)\xi(p^\star)dm(p)\\
&= \int\int \Delta f(q^\star,p)\xi(p^\star)d\mu(q)dm(p) = \int\int \Delta f(q^\star,p)\check{\xi}(p)dm(p)d\mu(q) \quad (H_4)\\
&= \int\int \Delta\check{\xi}(q,p)f(p)dm(p)d\mu(q) \quad (H_3) = \int\int \Delta\xi(p^\star,q^\star)f(p)dm(p)d\mu(q)\\
&= \int\int \xi(p^\star)\xi(q^\star)f(p)dm(p)d\mu(q) = \int\xi(p^\star)f(p)dm(p)\int\xi(q^\star)d\mu(q)\\
&= \hat{\mu}(\xi)\hat{f}(\xi).
\end{aligned}$$

Thus, $(\mu*f)\hat{} = \hat{\mu}\hat{f}$. $\square$

For each $f\in C_b(Q)$ and $a,p\in Q$, put $f^a(p) := \Delta f(a,p)$ and $f_a(p) := \Delta f(p,a)$.

Proposition 3.7. *Let $a\in Q$ and $\gamma\in\hat{Q}$. Then for any $f\in L^1(Q)$ we have $\hat{f}_a(\gamma) = \gamma(a)\hat{f}(\gamma)$.*

Proof .

$$\begin{aligned}
\hat{f}_a(\gamma) &= \int_Q f_a(p)\bar{\gamma}(p)dm(p) = \int_Q f_a(p)\gamma(p^\star)dm(p)\\
&= \int_Q \Delta f(a,p)\gamma(p^\star)dm(p) = \int_Q \Delta f(a,p)\check{\gamma}(p)dm(p)\\
&= \int_Q f(p)\Delta\check{\gamma}(a^\star,p)dm(p) = \int_Q f(p)\Delta\gamma(p^\star,a)dm(p)\\
&= \int_Q f(p)\gamma(p^\star)\gamma(a)dm(p) = \gamma(a)\hat{f}(\gamma).
\end{aligned}$$

$\square$

Lemma 3.8. *If Q is a cocommutative KPC-hypergroup then,*

i. for any $p\in Q$ we have $\delta(p)=1$, where δ is the modular function of Q.

ii. *for each* $f, g \in C_c(Q)$, $f * g = g * f$.

Proof . i. Let $f \in C_c(Q)$ and $p \in Q$. Then by Definition 2.1,

$$\begin{aligned}\delta(p)m(f) = (m * \delta_{p^\star})(f) &= \int_Q \int_Q \Delta f(q,t)dm(q)d\delta_{p^\star}(t)\\ &= \int_Q \Delta f(q,p^\star)dm(q) = \int_Q \Delta f(p^\star,q)1(q)dm(q)\\ &= \int_Q \Delta 1(p,q)f(q)dm(q) = \int_Q f(q)dm(q) = m(f).\end{aligned}$$

So $\delta(p) = 1$ for any $p \in Q$.

ii. If $f, g \in C_c(Q)$, for any $q \in Q$ we have

$$\begin{aligned}(f * g)(q) &= \int_Q f(p)\Delta g(p^\star,q)dm(p) = \int_Q f(p)\Delta g^-(q^\star,p)dm(p)\\ &= \int_Q \Delta f(q,p)g^-(p)dm(p) = \int_Q \Delta f(p,q)g(p^\star)dm(p)\\ &= \int_Q \Delta f(p^\star,q)g(p)\delta(p^\star)dm(p) = \int_Q \Delta f(p^\star,q)g(p)dm(p)\\ &= (g * f)(q).\end{aligned}$$

□

The following theorem is called Wendel's Theorem.

Theorem 3.9. *Let* Q *be a locally compact cocommutative KPC-hypergroup. Suppose that* $T : L^1(Q) \to C_c(Q)$ *is a bounded linear mapping. Then the following statements are equivalent:*

i. T *commutes with right translation operators, that is* $T(f^p) = T(f)^p$, *for all* $p \in Q$ *and* $f \in C_c(Q)$.

ii. $T(f * g) = T(f) * g$, *for all* $f,\ g \in C_c(Q)$.

iii. *There exists a unique transformation* ϕ *on* $\hat{Q}$ *such that* $\widehat{T(f)} = \phi\hat{f}$, *for all* $f \in C_c(Q)$.

iv. *There exists a unique measure* $\mu \in M(Q)$ *such that* $\widehat{T(f)} = \hat{\mu}\hat{f}$, *for all* $f \in C_c(Q)$.

v. *There exists a unique measure* $\mu \in M(Q)$ *such that* $T(f) = f * \mu = \mu * f$, *for all* $f \in C_c(Q)$.

Proof . (i) implies (ii): Let T commute with right translation operators, and $k \in L^\infty(Q)$. We define the mapping ψ on $L^1(Q)$ by

$$\psi(f) = \int_Q T(f)(t)k(t^\star)dm(t) \quad (f \in L^1(Q)).$$

Then ψ is a bounded linear mapping on $L^1(Q)$, because by ([5], Proposition 5.5) we have

$$\begin{aligned}\left|\int T(f)(t)k(t^\star)dm(t)\right| &\le \|k\|_\infty \|T(f)\|_1\\ &\le \|k\|_\infty \|T\| \|f\|_1,\end{aligned}$$

where $\|T\|$ denotes the usual operator norm of T. Then by ([3], 20.20) there is a function $h \in L^\infty(Q)$ such that

$$\int T(f)(q,p)k(q^\star)dm(q) = \int f(q,p)h(q^\star)dm(q) \qquad (*).$$

For each $f, g \in C_c(Q)$, we have

$$\begin{aligned}
\int [T(f)*g](q)k(q^\star)dm(q) &= \int [\int T(f)(p)g^q(p^\star)dm(p)]k(q^\star)dm(q) \\
&= \int [\int T(f)^p(q)g(p^\star)dm(p)]k(q^\star)dm(q) \\
&= \int [\int T(f^p)(q)k(q^\star)dm(q)]\check{g}(p)dm(p) \\
&= \int [\int f^p(q)h(q^\star)dm(q)]\check{g}(p)dm(p) \\
&= \int [\int f^p(q)\check{g}(p)dm(p)]h(q^\star)dm(q) \\
&= \int [\int \check{g}^p(q^\star)f(p)dm(p)]h(q^\star)dm(q) \\
&= \int [\int g^q(p^\star)f(p)dm(p)]h(q^\star)dm(q) \\
&= \int [(f*g)(q)]h(q^\star)dm(q) \qquad \text{(by } (*)) \\
&= \int [T(f*g)(q)]k(q^\star)dm(q).
\end{aligned}$$

Since $k \in L^\infty(Q)$ is arbitrary, we have $T(f)*g = T(f*g)$ for all $f, g \in C_c(Q)$.

(ii) implies (iii): Let $T(f)*g = T(f*g)$, for all $f, g \in C_c(Q)$. By Lemma 3.8, we have $T(f*g) = T(g*f)$, for all $f, g \in C_c(Q)$, and so $T(g)*f = T(g*f) = T(f*g) = T(f)*g$. Now, by Lemma 3.5, for all $f, g \in C_c(Q)$ we have

$$\widehat{T(f)}\hat{g} = \widehat{T(g)}\hat{f}. \qquad (**)$$

For each $\xi \in \hat{Q}$, we can choose $g \in C_c(Q)$ such that $\hat{g}(\xi) \neq 0$. If ϕ is defined by $\phi(\xi) := \frac{\widehat{T(g)}(\xi)}{\hat{g}(\xi)}$, then by $(**)$, ϕ is independent from g. For each $\xi \in \hat{Q}$, we have

$$\widehat{T(f)}(\xi) = \hat{f}(\xi)(\frac{\widehat{T(g)}}{\hat{g}})(\xi) = \phi(\xi)\hat{f}(\xi) = (\phi\hat{f})(\xi).$$

Therefore, $\widehat{T(f)} = \phi\hat{f}$.

(iii) implies (iv): Let $\widehat{T(f)} = \phi\hat{f}$, for all $f \in L^1(Q)$. Then $\phi\hat{f} \in \widehat{C_c(Q)}$. By ([5], Remark 10.7) the Fourier transform maps $L^1(Q)$ into $C_0(\hat{Q})$. Therefore, $\phi\hat{f}$ is continuous and $\phi \in C(\hat{Q})$. Thus, by ([7], Theorem 2.1), there exists $\mu \in M(Q)$ such that $\phi = \hat{\mu}$, and so $\widehat{T(f)} = \hat{\mu}\hat{f}$.

(iv) implies (v): By (iv) and Lemma 3.6, for all $\xi \in \hat{Q}$ we have $\widehat{(Tf)} = (\hat{\mu}\hat{f})(\xi) = (\mu * f)\hat{}(\xi) = (f*\mu)\hat{}(\xi)$. So, since $(T(f) - \mu * f)\hat{} = 0$, we have $T(f) = \mu * f = f * \mu$.

(v) implies (i): Let $f \in C_c(Q)$ and $p \in Q$. By hypothesis,

$$T(f^p) = \mu * f^p = \mu * (f * \delta_{p^\star}) = (\mu * f) * \delta_{p^\star} = (\mu * f)^p = T(f)^p.$$

Therefore, $T(f^p) = T(f)^p$. $\square$

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