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Mazur-Ulam theorem in probabilistic normed groups

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Abstract

In this paper, we give a probabilistic counterpart of Mazur-Ulam theorem in probabilistic normed groups. We show, under some conditions, that every surjective isometry between two probabilistic normed groups is a homomorphism.

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1. Introduction and preliminaries

Mazur and Ulam showed that every bijective isometry between real normed spaces is affine [5]. Since then it has attracted the attention of some researchers in order to generalize this result (see e.g. [8]). In particular, the Mazur-Ulam theorem has been investigated in normed and metric groups [3, 10] and in probabilistic and random normed spaces [1, 6].

In this paper we give a probabilistic counterpart of the Mazu-Ulam theorem in probabilistic normed groups introduced by the authors in [7]. We begin with some basic notions which will be needed in this paper.

A distribution function is a function F from the extended real line $[-\infty, +\infty]$ to the interval $[0, 1]$ such that F is nondecreasing and left-continuous and satisfies $F(-\infty) = 0$, $F(+\infty) = 1\mathcal{A}$. We denote the set of all distribution functions by Δ. A subset of Δ consisting of all distribution functions F with $F(0) = 0$ will be denoted by Δ^+. The subset D^+ of Δ^+ is defined as follows:

$$D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\},$$

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where $l^- f(x)$ denotes the left limit of the function f at the point x. For $F, G \in \Delta^+$ we mean $F \leq G$ by $F(x) \leq G(x)$, for all $x \in \mathbb{R}$. The distribution function $\mathcal{H}_a$ is given by

$$\mathcal{H}_a(x) = \begin{cases} 0, & \text{if } x \leq a, \\ 1, & \text{if } x > a, \end{cases}$$

for all $a, x \in \mathbb{R}$. The maximal element for Δ^+ (and also for D^+) according to the presented order is the distribution function $\mathcal{H}_0$.

A triangular norm (briefly t-norm) is a binary function T from $[0,1] \times [0,1]$ to $[0,1]$ which is associative, commutative, nondecreasing in each place and $T(a,1) = a$, for all $a \in [0,1]$. A triangle function is a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ such that τ is associative, commutative, nondecreasing for all $F, G, H \in \Delta^+$ and it has $\mathcal{H}_0$ as unit [4]. A sequence $\{F_n\}$ in Δ^+ converges weakly to a distribution function F, written by $F_n \xrightarrow{w} F$, if and only if the sequence $\{F_n(x)\}$ converges to $F(x)$ at each continuity point x of F (see Definition 4.2.4. in [9]). A triangle function τ is said to be continuous if $F_n \xrightarrow{w} F$ and $G_n \xrightarrow{w} G$ in Δ^+ imply that $\tau(F_n, G_n) \to \tau(F, G)$. For example, if T is a continuous t-norm, then τ_T is a continuous triangle function, where τ_T is defined by

$$\tau_T(F,G)(x) = \sup_{s+t=x} T(F(s), G(t)), \tag{1.1}$$

for all $F, G \in \Delta^+$ and every $x, s, t \in \mathbb{R}$.

Definition 1.1. [7] A triple (G, F, τ) is called a probabilistic normed group, where G is a group with identity element e, τ is a continuous triangle function and F is a mapping from G into Δ^+ satisfying the following conditions:

(PGN1) $F_x = \mathcal{H}_0$ if and only if $x = e$,

(PGN2) $F_{xy} \geq \tau(F_x, F_y)$, whenever $x, y \in G$,

(PGN3) $F_{x^{-1}} = F_x$, where x^{-1} is the inverse element of x.

Then F is called a probabilistic group-norm on G. The probabilistic group-norm F is called abelian if $F_{xy} = F_{yx}$, for each $x, y \in G$.

In a probabilistic normed group (G, F, τ), for each x in G and $\lambda > 0$, the strong λ-neighborhood of x is the set

$$N_x(\lambda) \;=\; \{y \in G : F_{xy^{-1}}(\lambda) > 1 - \lambda\}.$$

The strong neighborhood system for G is the union $\bigcup_{x \in G} \mathcal{N}_x$ where $\mathcal{N}_x = \{N_x(\lambda) : \lambda > 0\}$. Note that the strong neighborhood system for G determines a Hausdorff topology for G (see Theorem 12.1.2 in [9]).

2. Main theorem

Definition 2.1. [2] A group G is called divisible if for every $g \in G$, and every positive integer n there exists $y \in G$ such that $y^n = g$. We say that group G is 2-divisible if for each $g \in G$ there exists $y \in G$ such that $y^2 = g$. The algebraic center of points $x, y \in G$ is an element $z \in G$, denoted by $\sqrt{xy}$, such that $z^2 = xy$.

Definition 2.2. Let (G, F, μ) and (G', F', τ) be two probabilistic normed groups. A mapping $T : (G, F, \mu) \to (G', F', \tau)$ is called an isometry if for each $x, y \in G$,

$$F'_{T(x)T(y)^{-1}} = F_{xy^{-1}}.$$

Let (G, F, τ) be a probabilistic normed group. Consider the following conditions:
(C1) There exists a constant $c > 1$ such that $F_{x^2}(t) \leq F_x(\frac{t}{c})$, for all $x \in G$ and $t > 0$.
(C2) $F_x \in D^+$, for all $x \in G$.
(C3) $\tau(D^+ \times D^+) \subseteq D^+$.

The following example gives a probabilistic normed group satisfying the conditions (C1),(C2) and (C3).

Example 2.3. Consider the probabilistic normed group $(\mathbb{R}, F, \tau_T)$, where $\mathbb{R}$ is the additive group of real numbers and $F_x = \mathcal{H}_{|x|}$, for all $x \in \mathbb{R}$. We have $F_{x^n} = \mathcal{H}_{n|x|}$, for each $n \in \mathbb{N}$ and each $x \in \mathbb{R}$. Therefore

$$F_{x^n}(t) = \begin{cases} 0, & \text{if } t \leq n \mid x \mid \\ 1, & \text{if } t > n \mid x \mid \end{cases} = \begin{cases} 0, & \text{if } \frac{t}{n} \leq \mid x \mid \\ 1, & \text{if } \frac{t}{n} \quad \mid x \mid \end{cases} = F_x(\frac{t}{n}),$$

for each $x, t \in \mathbb{R}$ and every $n \in \mathbb{N}$. Now for $n \geq 2$, choosing $1 < c \leq n$ we get

$$F_{x^n}(t) = F_x(\frac{t}{n}) \leq F_x(\frac{t}{c}),$$

for each $x, t \in \mathbb{R}$. Particularly, for $n = 2$ putting $1 < c \leq 2$, we get

$$F_{x^2}(t) \leq F_x(\frac{t}{c}),$$

for all $x, t \in \mathbb{R}$. It is obvious that for every $x \in \mathbb{R}$, $F_x = \mathcal{H}_{|x|} \in D^+$. Since $\tau_T(\mathcal{H}_{|x|}, \mathcal{H}_{|y|}) = \mathcal{H}_{|x|+|y|}$, for all $x, y \in \mathbb{R}$, we get

$$\tau_T(F_x, F_y) \in D^+.$$

Now consider the probabilistic normed group $(\mathbb{R}_+, F, \tau_T)$, where $\mathbb{R}_+$ is the multiplicative group with $e = 1$. Let $F_h = \mathcal{H}_{|\log(h)|}$, for all $h \in \mathbb{R}_+$. We have

$$F_{h^2}(t) = \mathcal{H}_{|\log h^2|}(t) = \mathcal{H}_{2|\log h|}(t) = \mathcal{H}_{|\log h|}(\frac{t}{2}),$$

for each $t, h \in \mathbb{R}_+$. Putting $1 < c \leq 2$, we have $F_{h^2}(t) \leq F_h(\frac{t}{c})$.

Theorem 2.4. *Let (G, F, μ) and (G', F', τ) be two probabilistic normed groups such that both G, G' are uniquely 2-divisible abelian groups, and conditions $(C1), (C2)$ and $(C3)$ hold for both (G', F', τ) and (G, F, μ). If $T : G \to G'$ is a surjective isometry, then*

$$F'_{T(\sqrt{xy})(\sqrt{T(x)T(y)})^{-1}} = \mathcal{H}_0,$$

for all $x, y \in G$.

Proof . Let $x, y \in G$ and set

$$a = \sqrt{xy}, \quad b = \sqrt{T(x)T(y)}, \quad E = F'_{\sqrt{T(x)T(y)^{-1}}}.$$

Let $\{q_n\}$ be a sequence of maps form G' to itself, defined for each $z \in G'$ by

$$q_0(z) = T(a^2(T^{-1}(z))^{-1}), \quad q_1(z) = b^2 z^{-1},$$

and for $n \in \mathbb{N}$,

$$q_{n+1} = q_{n-1} \circ q_n \circ q_{n-1}^{-1}.$$

For $n \in \mathbb{N}$ define $\{p_n\}$, a sequence of points in G', by

$$p_1 = b, \quad p_{n+1} = q_{n-1}(p_n).$$

By induction, one can see that for all $n \in \mathbb{N}_0$ we have

$$q_n(T(x)) = T(y), \qquad q_n(T(y)) = T(x). \tag{2.1}$$

We show that for each $u, v \in G'$ and all $n \in \mathbb{N}_0$,

$$F'_{q_n(u)q_n(v)^{-1}} = F'_{uv^{-1}}.$$

For $n = 0$,

$$\begin{aligned}
F'_{q_0(u)q_0(v)^{-1}} &= F'_{T(a^2(T^{-1}(u))^{-1})(T(a^2(T^{-1}(v)^{-1})))^{-1}} \\
&= F_{a^2(T^{-1}(u))^{-1}(a^2(T^{-1}(v))^{-1})^{-1}} = F_{a^2a^{-2}(T^{-1}(u))^{-1}T^{-1}(v)} \\
&= F_{T^{-1}(u)^{-1}T^{-1}(v)} = F_{(T^{-1}(u)^{-1}T^{-1}(v))^{-1}} = F_{T^{-1}(u)T^{-1}(v)^{-1}} \\
&= F'_{(TT^{-1}(u))(TT^{-1}(v)^{-1})} \\
&= F'_{uv^{-1}}.
\end{aligned}$$

Suppose that the statement holds for some $n \in \mathbb{N}$. Then we get

$$\begin{aligned}
F'_{q_{n+1}(u)q_{n+1}(v)^{-1}} &= F'_{q_{n-1}\circ q_n\circ q_{n-1}^{-1}(u)(q_{n-1}\circ q_n\circ q_{n-1}^{-1}(v))^{-1}} \\
&= F'_{q_n\circ q_{n-1}^{-1}(u)(q_n\circ q_{n-1}^{-1}(v))^{-1}} \\
&= F'_{q_{n-1}^{-1}(u)(q_{n-1}^{-1}(v))^{-1}} \\
&= F'_{q_{n-1}\circ q_{n-1}^{-1}(u)(q_{n-1}\circ q_{n-1}^{-1}(v))^{-1}} \\
&= F'_{uv^{-1}}.
\end{aligned}$$

So

$$F'_{q_n(u)q_n(v)^{-1}} = F'_{uv^{-1}},$$

for each $u, v \in G'$ and all $n \in \mathbb{N}_0$. Now by induction we are going to show that

$$F'_{p_nT(x)^{-1}} = E, \qquad F'_{p_nT(y)^{-1}} = E, \tag{2.2}$$

for $n \in \mathbb{N}$. For $n = 1$, we have

$$F'_{p_1T(x)^{-1}} = F'_{\sqrt{T(x)T(y)}T(x)^{-1}} = F'_{\sqrt{T(y)T(x)^{-1}}} = E.$$

(Note that in the above equation we use the fact that if $s^2 = tr$ and $v^2 = mn$, then $s^2v^2 = (sv)^2$ and $sv = \sqrt{trmn} = \sqrt{tr}\sqrt{mn}$, for all $s, v, r, t, m, n \in G'$.)
Likewise,

$$F'_{p_1T(y)^{-1}} = F'_{\sqrt{T(y)T(x)^{-1}}} = E.$$

Hence (2.2) holds for $n = 1$. Suppose that (2.2) holds for some $n \in \mathbb{N}$. Then by using (2.1) and the induction hypothesis we get

$$F'_{p_{n+1}T(x)^{-1}} = F'_{q_{n-1}(p_n)(q_{n-1}(T(y)))^{-1}} = F'_{p_nT(y)^{-1}} = E.$$

Similarly we have

$$F'_{p_{n+1}T(y)^{-1}} = E.$$

Now by (2.2) for $n \geq 2$,

$$F'_{p_np_{n-1}^{-1}} = F'_{p_nT(x)^{-1}T(x)p_{n-1}^{-1}} \geq \tau(F'_{p_nT(x)^{-1}}, F'_{T(x)p_{n-1}^{-1}}) = \tau(E, E). \tag{2.3}$$

Again, by induction we prove that there is constant $c > 1$ such that

$$F'_{q_n(z)z^{-1}}(t) \leq F'_{p_nz^{-1}}(\frac{t}{c}), \tag{2.4}$$

for each $z \in G'$, $t > 0$ and $n \in \mathbb{N}$. For n=1, we have

$$F'_{q_1(z)z^{-1}} = F'_{b^2z^{-1}z^{-1}} = F'_{b^2(z^{-1})^2} = F'_{(bz^{-1})^2}.$$

By the condition (C1), there exists constant $c > 1$ such that

$$F'_{(bz^{-1})^2}(t) \leq F'_{bz^{-1}}(\frac{t}{c}),$$

for each $z \in G'$ and $t > 0$. Hence

$$F'_{q_1(z)z^{-1}}(t) \leq F'_{p_1z^{-1}}(\frac{t}{c}),$$

for each $z \in G'$ and $t > 0$. Now suppose that the statement holds for some natural number n. Then for each $z \in G'$ and $t > 0$,

$$\begin{aligned} F'_{q_{n+1}(z)z^{-1}}(t) &= F'_{q_{n-1}q_nq_{n-1}^{-1}(z)(q_{n-1}q_{n-1}^{-1}(z))^{-1}}(t) \\ &= F'_{q_nq_{n-1}^{-1}(z)(q_{n-1}^{-1}(z))^{-1}}(t) \\ &\leq F'_{p_n(q_{n-1}^{-1}(z))^{-1}}(\frac{t}{c}) \\ &= F'_{q_{n-1}^{-1}q_{n-1}(p_n)(q_{n-1}^{-1}(z))^{-1}}(\frac{t}{c}) \\ &= F'_{q_{n-1}(p_n)z^{-1}}(\frac{t}{c}) = F'_{p_{n+1}z^{-1}}(\frac{t}{c}). \end{aligned}$$

In the inequality (2.4) replace z by p_{n+1}. Then for $n \in \mathbb{N}$ and $t > 0$, we obtain

$$F'_{q_n(p_{n+1})p_{n+1}^{-1}}(t) \leq F'_{p_np_{n+1}^{-1}}(\frac{t}{c}) = F'_{(p_np_{n+1}^{-1})^{-1}}(\frac{t}{c}).$$

Therefore

$$F'_{p_{n+2}p_{n+1}^{-1}}(t) \leq F'_{p_{n+1}p_n^{-1}}(\frac{t}{c}),$$

and for $n \geq 3$ and each $t > 0$, we have

$$F'_{p_np_{n-1}^{-1}}(t) \leq F'_{p_{n-1}p_{n-2}^{-1}}(\frac{t}{c}) \leq \cdots \leq F'_{p_2p_1^{-1}}(\frac{t}{c^{n-2}}). \tag{2.5}$$

By (2.3) and (2.5) for $n \geq 3$ we get

$$\tau(E,E)(t) \leq F'_{p_2 p_1^{-1}}(\frac{t}{c^{n-2}}). \tag{2.6}$$

On the other hand, there is $c_1 > 1$ such that

$$\begin{aligned} F'_{p_2 p_1^{-1}}(t) &= F'_{T(a^2(T^{-1}(b))^{-1})(TT^{-1}(b))^{-1}}(t) \\ &= F_{a^2((T^{-1}(b))^{-1})^2}(t) = F_{(a(T^{-1}(b))^{-1})^2}(t) \\ &\leq F_{a(T^{-1}(b))^{-1}}(\frac{t}{c_1}) \\ &= F'_{T(a)(TT^{-1}(b))^{-1}}(\frac{t}{c_1}) \\ &= F'_{T(a)b^{-1}}(\frac{t}{c_1}). \end{aligned}$$

for each $t > 0$. Consequently,

$$\tau(E,E)(c_1 c^{n-2} t) \leq F'_{p_2 p_1^{-1}}(c_1 t) \leq F'_{T(a)b^{-1}}(t),$$

for each $t > 0$. Since $F'_z \in D^+$ for each $z \in G'$, and $\tau(D^+ \times D^+) \subseteq D^+$ we have

$$\lim_{n \to +\infty} \tau(E,E)(c_1 c^{n-2} t) = 1,$$

for each $t > 0$. But $\mathcal{H}_0$ is a maximal element of D^+, therefore

$$F'_{T(a)b^{-1}} = \mathcal{H}_0.$$

□

Theorem 2.5. *Suppose that (G, F, μ) and (G', F', τ) are two probabilistic normed groups such that both G, G' are uniquely 2-divisible abelian groups. Let the conditions $(C1), (C2)$ and $(C3)$ hold for both (G', F', τ) and (G, F, μ). If $U : (G, F, \mu) \to (G', F', \tau)$ is a surjective isometry with $U(e) = e$, then U is a homomorphism.*

Proof . We can apply Theorem 2.4 for surjective isometry U. For each $x, y \in G$ we have

$$F'_{U(\sqrt{xy})(\sqrt{U(x)U(y)})^{-1}} = \mathcal{H}_0.$$

Thus

$$U(\sqrt{xy})(\sqrt{U(x)U(y)})^{-1} = e,$$

for each $x, y \in G$. That is,

$$U(\sqrt{xy}) = \sqrt{U(x)U(y)}, \tag{2.7}$$

for each $x, y \in G$. In the equation (2.7), let $y = e$. Since $U(e) = e$, we have

$$U(\sqrt{x}) = \sqrt{U(x)},$$

for each $x \in G$. Now for arbitrary $x, y \in G$ we get

$$U(xy) = (U(\sqrt{xy}))^2 = (\sqrt{U(x)U(y)})^2 = U(x)U(y),$$

i.e., U is a homomorphism. □

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