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Fixed point theorems for generalized quasi-contractions in cone b-metric spaces over Banach algebras without the assumption of normality with applications

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Abstract

In this paper, we introduce the concept of generalized quasi-contractions in the setting of cone b-metric spaces over Banach algebras. By omitting the assumption of normality we establish common fixed point theorems for the generalized quasi-contractions with the spectral radius $r(\lambda)$ of the quasi-contractive constant vector λ satisfying $r(\lambda) \in [0, \frac{1}{s})$ in the setting of cone b-metric spaces over Banach algebras, where the coefficient s satisfies $s \geq 1$. As consequences, we obtain common fixed point theorems for the generalized g-quasi-contractions in the setting of such spaces. The main results generalize, extend and unify several well-known comparable results in the literature. Moreover, we apply our main results to some nonlinear equations, which shows that these results are more general than corresponding ones in the setting of b-metric or metric spaces.

Keywords: cone b-metric spaces over Banach algebras; non-normal cones; c-sequences; generalized quasi-contractions; fixed point theorem.
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1. Introduction

Huang and Zhang [1] introduced the concept of cone metric space, proved the properties of sequences on cone metric spaces and obtained various fixed point theorems for contractive mappings. The existence of a common fixed point on cone metric space was considered in [2, 3]. Also, Ilić and Rakočević [8] introduced quasi-contraction on cone metric space when the underlying cone is normal. Later on, Kadelburg et al. [5] obtained a fixed point result without the normality of the underlying cone, but only in the case of a quasi-contractive constant $\lambda \in [0, 1/2)$ (see [5, Theorem 2.2]). However, Gajić and Rakočević [4] proved that result is true for $\lambda \in [0, 1)$ on cone metric spaces which answered the open question whether the result is true for $\lambda \in [0, 1)$. Recently, Hussain and Shah [9] introduced cone b-metric spaces, as a generalization of b-metric spaces and cone metric spaces, and established some important topological properties in such spaces. Following Hussain and Shah, Huang and Xu [8] obtained some interesting fixed point results for contractive mappings in cone b-metric spaces. Inspired by [4], Shi and Xu [20] presented a similar common fixed point result in the case of the contractive constant $\lambda \in [0, 1/s)$ in cone b-metric spaces without the assumption of normality (see [20]. Similar results can be seen in [21].

Let (X, d) be a complete metric space. Recall that a mapping $T : X \to X$ is called a quasi-contraction if, for some $k \in [0, 1)$ and for all $x, y \in X$, one has

$$d(Tx, Ty) \leqslant k \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Ćirić [12] introduced and studied quasi-contractions as one of the most general classes of contractive-type mappings. He proved the well-known theorem that any quasi-contraction T has a unique fixed point. Recently, scholars obtained various similar results on cone metric spaces. See, for instance, [4, 5, 6].

Recently, some authors investigated the problem of whether cone metric spaces are equivalent to metric spaces in terms of the existence of the fixed points of the mappings involved. They used to establish the equivalence between some fixed point results in metric and in (topological vector spaces valued) cone metric spaces (see [10, 16, 22]. Very recently, Liu and Xu [13] introduced the concept of cone metric spaces with Banach algebras, replacing Banach spaces by Banach algebras as the underlying spaces of cone metric spaces. Although they proved some fixed point theorems of quasi-contractions, the proof relied strongly on the assumption that the underlying cone is normal. We may state that it is significant to introduce the concept of cone metric spaces with Banach algebras (which we call in this paper cone metric spaces over Banach algebras). This is because there are examples to show that one is unable to conclude that the cone metric space (X, d) with a Banach algebra $\mathcal{A}$ discussed is equivalent to the metric space (X, d^*), where the metric d^* is defined by $d^* = \xi_e \circ d$, here the nonlinear scalarization function $\xi_e : A \to \mathbb{R}$ ($e \in \text{int}P$) is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}.$$

See [10, 13, 16, 17] for more details.

In the present paper we introduce the concept of generalized quasi-contractions (g-quasi-contractions) in cone b-metric spaces over Banach algebras and obtain fixed point theorems (as a result, common fixed point theorems) for a self-mapping (two weakly compatible self-mappings) satisfying the quasi-contractive (g-quasi-contractive) condition in the case of the quasi-contractive (g-quasi-contractive) constant vector with $r(\lambda) \in [0, 1/s)$ in cone b-metric spaces without the assumption of normality, where the coefficient s satisfies $s \geq 1$. As consequences, our main results not only extend the fixed point theorem of quasi-contractions in cone b-metric spaces to the case in cone b-metric spaces over Banach algebras, but also yield new corresponding results concerning the generalized quasi-contractions in cone metric spaces over Banach algebras. Our main results generalize and extend the relevant results in the literature (see, for example, [14, 15, 24]).

2. Preliminaries

Let $\mathcal{A}$ always be a real Banach algebra. That is, $\mathcal{A}$ is a real Banach space in which an operation of multiplication is defined, subject to the following properties (for all x, y, $z \in \mathcal{A}$, $\alpha \in \mathbb{R}$):

1. $(xy)z = x(yz)$;
2. $x(y + z) = xy + xz$ and $(x + y)z = xz + yz$;
3. $\alpha(xy) = (\alpha x)y = x(\alpha y)$;
4. $\|xy\| \leqslant \|x\| \, \|y\|$.

Throughout this paper, we shall assume that a Banach algebra $\mathcal{A}$ has a unit (i.e., a multiplicative identity) e such that $ex = xe = x$ for all $x \in \mathcal{A}$. An element $x \in \mathcal{A}$ is said to be invertible if there is an inverse element $y \in \mathcal{A}$ such that $xy = yx = e$. The inverse of x is denoted by x^{-1}. For more details, we refer to [11].

The following proposition is well known (see [11]).

Proposition 2.1. Let $\mathcal{A}$ be a Banach algebra with a unit e, and $x \in \mathcal{A}$. If the spectral radius $r(x)$ of x is less than 1, i.e.,

$$r(x) = \lim_{n\to\infty} \|x^n\|^{\frac{1}{n}} = \inf_{n\geqslant 1} \|x^n\|^{\frac{1}{n}} < 1,$$

then $e - x$ is invertible. Actually,

$$(e - x)^{-1} = \sum_{i=0}^{\infty} x^i.$$

Now let us recall the concepts of cone and semi-order for a Banach algebra $\mathcal{A}$. A subset P of $\mathcal{A}$ is called a cone if

1. P is non-empty closed and $\{\theta, e\} \subset P$;
2. $\alpha P + \beta P \subset P$ for all non-negative real numbers α, β;
3. $P^2 = PP \subset P$;
4. $P \cap (-P) = \{\theta\}$,

where θ denotes the null of the Banach algebra $\mathcal{A}$. For a given cone $P \subset \mathcal{A}$, we can define a partial ordering $\preceq$ with respect to P by $x \preceq y$ if and only if $y - x \in P$. $x \prec y$ will stand for $x \preceq y$ and $x \neq y$, while $x \ll y$ will stand for $y - x \in \mathrm{int}P$, where $\mathrm{int}P$ denotes the interior of P.

The cone P is called normal if there is a number $M > 0$ such that for all x, $y \in \mathcal{A}$,

$$\theta \preceq x \preceq y \Rightarrow \|x\| \leqslant M\|y\|.$$

The least positive number satisfying above is called the normal constant of P.

In the following we always assume that P is a cone in Banach algebra $\mathcal{A}$ with $\mathrm{int}P \neq \emptyset$ and $\preceq$ is the partial ordering with respect to P.

Now, let us recall the basic concepts concerning cone metric spaces over Banach algebras, as is indicated in the following two definitions.

Definition 2.2. (See [1], [13] and [14]) Let X be a non-empty set. Suppose the mapping $d : X \times X \to \mathcal{A}$ satisfies

1. $0 \preceq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \preceq d(x, z) + d(z, x)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space over a Banach algebra $\mathcal{A}$.

Definition 2.3. (See [1], [13] and [14]) Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ a sequence in X. Then

1. $\{x_n\}$ converges to x whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geqslant N$. We denote this by $\lim_{n\to\infty} x_n = x$ or $x_n \to x$.
2. $\{x_n\}$ is a Cauchy sequence whenever for each $c \in \mathcal{A}$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geqslant N$.
3. (X, d) is a complete cone metric space if every Cauchy sequence is convergent.

Now, we shall appeal to the following lemmas in the sequel.

Lemma 2.4. If E is a real Banach space with a cone P and if $a \preceq \lambda a$ with $a \in P$ and $0 \leq \lambda < 1$, then $a = \theta$.

Lemma 2.5. [15] If E is a real Banach space with a solid cone P and if $\theta < u \ll c$ for each $\theta \ll c$, then $u = \theta$.

Lemma 2.6. [15] If E is a real Banach space with a solid cone P and if $\|x_n\| \to 0 (n \to \infty)$, then for any $\theta \ll \epsilon$, there exists $N \in \mathbb{N}$ such that for any $n > N$, we have $x_n \ll \epsilon$.

Finally, let us recall the concept of generalized quasi-contraction defining on the cone metric spaces over Banach algebras, which is introduced in [14]. Note that it is called quasi-contraction in [14].

Definition 2.7. [14] Let (X, d) be a cone metric space over a Banach algebra $\mathcal{A}$. A mapping $T : X \to X$ is called a generalized quasi-contraction if for some $k \in P$ with $r(k) < 1$ and for all $x, y \in X$, one has

$$d(Tx, Ty) \preceq ku,$$

where

$$u \in \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}.$$

Remark 2.8. If $r(k) < 1$, then $\|k^m\| \to 0 \ (m \to \infty)$.

Similarly, the basic concepts concerning cone b-metric spaces over Banach algebras are necessary, as is indicated in the following definitions.

Definition 2.9. [9] Let X be a nonempty set and $s \geq 1$ a given real number. A mapping $d : X \times X \to \mathcal{A}$ is said to be a cone b-metric if and only if for all $x, y, z \in X$ the following conditions are satisfied:

(i) $\theta \prec d(x,y)$ with $x \neq y$ and $d(x,y) = \theta$ if and only if $x = y$;

(ii) $d(x,y) = d(y,x)$;

(iii) $d(x,y) \preceq s[d(x,z) + d(z,y)]$.

The pair (X,d) is called a cone b-metric space over a Banach algebra $\mathcal{A}$.

Example 2.10. Denote by $L_p (0 < p < 1)$ the set of all real measurable functions $x(t) (t \in [0,1])$ such that $\int_0^1 |x(t)|^p \mathrm{dt} < \infty$. Let $X = L_p$, $E = \mathbb{R}^2$, $P = \{(x,y) \in E \mid x, y \geq 0\} \subset \mathbb{R}^2$ and $d : X \times X \to E$ such that

$$d(x,y) = \left(\alpha\left\{\int_0^1 |x(t) - y(t)|^p \mathrm{dt}\right\}^{\frac{1}{p}}, \beta\left\{\int_0^1 |x(t) - y(t)|^p \mathrm{dt}\right\}^{\frac{1}{p}}\right),$$

where $\alpha, \beta \geq 0$ are constants. Then (X,d) is a cone b-metric space over a Banach algebra with the coefficient $s = 2^{\frac{1}{p}-1}$ (see the subsequent Example 3.2 for details).

Example 2.11. Let $X = \mathbb{R}$, $E = C^1_{\mathbb{R}}[0,1]$ and $P = \{f \in E : f \geq 0\}$. Define $d : X \times X \to E$ by $d(x,y) = |x - y|^{1.5}\varphi(t)$ where $\varphi : [0,1] \to \mathbb{R}$ is a function such that $\varphi(t) = e^t$. It is easy to see that (X,d) is a cone b-metric space over a Banach algebra with the coefficient $s = 2^{0.5}$, but it is not a cone metric space.

Definition 2.12. [9] Let (X,d) be a cone b-metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X. We say

(i) $\{x_n\}$ converges to x whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x) \ll c$ for all $n \geq N$. We denote this by $\lim\limits_{n\to\infty} x_n = x$ or $x_n \to x (n \to \infty)$.

(ii) $\{x_n\}$ is a Cauchy sequence whenever for every $c \in E$ with $\theta \ll c$ there is a natural number N such that $d(x_n, x_m) \ll c$ for all $n, m \geq N$.

(iii) (X,d) is a complete cone b-metric space over a Banach algebra $\mathcal{A}$ if every Cauchy sequence is convergent.

Lemma 2.13. [9] Let $\preceq$ be the partial ordering with respect to P, where P is the given cone P of the Banach algebra $\mathcal{A}$. The following properties are often used while dealing with cone b-metric spaces where the underlying cone is not necessarily normal.

(1) If $u \ll v$ and $v \preceq w$, then $u \ll w$.

(2) If $\theta \preceq u \ll c$ for each $c \in \mathrm{int}P$, then $u = \theta$.

(3) If $a \preceq b + c$ for each $c \in \mathrm{int}P$, then $a \preceq b$.

(4) If $c \in \mathrm{int}P$ and $a_n \to \theta$, then there exists $n_0 \in \mathbb{N}$ such that $a_n \ll c$ for all $n > n_0$.

(5) Let (X,d) be a cone b-metric space over a Banach algebra $\mathcal{A}$, $x \in X$ and $\{x_n\}$ be a sequence in X. If $d(x_n, x) \preceq b_n$ and $b_n \to \theta$, then $x_n \to x$.

Lemma 2.14. The limit of a convergent sequence in cone b-metric space is unique.

Now let us introduce the concepts of generalized quasi-contraction and generalized g-quasi-contraction defined on a cone b-metric space over a Banach algebra, which is necessary in the subsequent discussions.

Definition 2.15. Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$ over a Banach algebra $\mathcal{A}$. A mapping $f : X \to X$ is called a generalized quasi-contraction on X, if for some $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and for all x, $y \in X$, one has

$$d(fx, fy) \preceq \lambda u,$$

where

$$u \in C(x, y) = \{d(x, y), d(x, fx), d(y, fy), d(x, fy), d(y, fx)\}. \tag{2.1}$$

Definition 2.16. Let (X, d) be a cone b-metric space with the coefficient $s \geq 1$ over a Banach algebra $\mathcal{A}$. A mapping $f : X \to X$ is called a generalized g-quasi-contraction on X where $g : X \to X$, $f(X) \subset g(X)$, if for some $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and for all x, $y \in X$, one has

$$d(fx, fy) \preceq \lambda u,$$

where

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}. \tag{2.2}$$

The concept of c-sequence in a solid cone in a Banach space is crucial to the arguments when one copes with the case without the assumption of normality.

Definition 2.17. (See [18] and [19]) Let P be a solid cone in a Banach space $\mathcal{A}$. A sequence $\{u_n\} \subset P$ is a c-sequence if for each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \ll c$ for $n \geq n_0$.

It is easy to show the following propositions.

Proposition 2.18. [18] Let P be a solid cone in a Banach space $\mathcal{A}$ and let $\{u_n\}$ and $\{v_n\}$ be sequences in P. If $\{u_n\}$ and $\{v_n\}$ are c-sequences and $\alpha, \beta > 0$, then $\{\alpha u_n + \beta v_n\}$ is a c-sequence.

In addition to Proposition 2.18 above, the following propositions are crucial to the proof of our main result.

Proposition 2.19. [18] Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Then the following conditions are equivalent.

(1) $\{u_n\}$ is a c-sequence.
(2) For each $c \gg \theta$ there exists $n_0 \in \mathbb{N}$ such that $u_n \prec c$ for $n \geq n_0$.
(3) For each $c \gg \theta$ there exists $n_1 \in \mathbb{N}$ such that $u_n \preceq c$ for $n \geq n_1$.

Proposition 2.20. [23] Let P be a solid cone in a Banach algebra $\mathcal{A}$ and let $\{u_n\}$ be a sequence in P. Suppose that $k \in P$ is an arbitrarily given vector and $\{u_n\}$ is a c-sequence in P. Then $\{ku_n\}$ is a c-sequence.

Proposition 2.21. [23] Let $\mathcal{A}$ be a Banach algebra with a unit e, P be a cone in $\mathcal{A}$ and $\preceq$ be the semi-order be yielded by the cone P. Let $\lambda \in P$. If the spectral radius $r(\lambda)$ of λ is less than 1, then the following assertions hold true.

(i) If $\lambda \succeq \theta$, then we have $(e - \lambda)^{-1} \succeq \theta$. In addition, we have $\theta \preceq (e - \lambda)^{-1}\lambda^n \preceq (e - \lambda)^{-1}\lambda$ for any integer $n \geq 1$.
(ii) For any $u \succ \theta$, we have $u \npreceq \lambda u$. Moreover, we have $u \npreceq \lambda^n u$ for any integer $n \geq 1$.

Proposition 2.22. [23] Let (X, d) be a complete cone metric space with a Banach algebra $\mathcal{A}$ and let P be the underlying solid cone in Banach algebra $\mathcal{A}$. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to $x \in X$, then we have
(i) $\{d(x_n, x)\}$ is a c-sequence;
(ii) for any $p \in \mathbb{N}$, $\{d(x_n, x_{n+p})\}$ is a c-sequence.

3. Main results

In this section, we give some fixed point theorems for a self-mapping satisfying the quasi-contractive (g-quasi-contractive) condition in the case of the quasi-contractive (g-quasi-contractive) constant vector with $r(\lambda) \in [0, 1/s)$ in cone b-metric spaces without the assumption of normality, where the coefficient s satisfies $s \geq 1$. Consequently, common fixed point results for two weakly compatible self-mappings concerning generalized g-quasi-contraction in the setting of such spaces are obtained as its corollaries.

Theorem 3.1. *Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ be the generalized quasi-contraction with the quasi-contractive constant vector satisfying $r(\lambda) \in [0, 1/s)$. If $f(X)$ is a complete subspace of X, then f has a unique common fixed point in X.*

We begin the proof of Theorem 3.1 with a useful lemma. For each $x_0 \in X$, set $x_1 = fx_0$ and $x_{n+1} = fx_n$. We will prove that $\{x_n\}$ is a Cauchy sequence. First, we shall show the following lemmas. Note that for these lemmas, we suppose that all the conditions in Theorem 3.1 are satisfied.

Lemma 3.2. *For any $N \geq 2$ and $1 \leq m \leq N-1$, one has that*

$$d(x_N, x_m) \preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0). \tag{3.1}$$

Proof . We now prove Lemma 3.2 by induction. When $N = 2, m = 1$, since $f : X \to X$ is a quasi-contraction, there exists

$$u_1 \in C(x_1, x_0) = \{d(x_1, x_0), d(x_1, x_2), d(x_0, x_1), d(x_1, x_1), d(x_0, x_2)\}$$

such that

$$d(x_2, x_1) \preceq \lambda u_1.$$

Hence, $u_1 = d(x_1, x_0)$ or $u_1 = d(x_0, x_2)$. (Note that it is obvious that $u_1 \neq d(x_1, x_2)$ since $d(x_2, x_1) \npreceq \lambda d(x_1, x_2)$ and $u_1 \neq d(x_1, x_1)$ since $d(x_1, x_2) \neq \theta$.)

When $u_1 = d(x_1, x_0)$, then we have

$$\begin{aligned} d(x_2, x_1) &\preceq \lambda d(x_0, x_1) \\ &\preceq s\lambda d(x_0, x_1) \preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0). \end{aligned}$$

When $u_1 = d(x_2, x_0)$, then we have

$$d(x_2, x_1) \preceq \lambda d(x_2, x_0) \preceq s\lambda[d(x_2, x_1) + d(x_1, x_0)].$$

So we get

$$(e - s\lambda)d(x_2, x_1) \preceq s\lambda d(x_1, x_0),$$

which implies that

$$d(x_2, x_1) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0).$$

Hence, (3.1) holds for $N = 2$ and $m = 1$.

Suppose that for some $N \geq 2$ and for any $2 \leq p \leq N$ and $1 \leq n \leq p$, one has

$$d(x_p, x_n) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0). \tag{3.2}$$

That is,

$$d(x_p, x_1) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0), \tag{3.3}$$

$$d(x_p, x_2) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0), \tag{3.4}$$

$$\vdots$$

$$d(x_p, x_{p-1}) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0). \tag{3.5}$$

Then, we need to prove that for $N + 1 \geq 2$ and any $1 \leq n < N + 1$, one has

$$d(x_{N+1}, x_n) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0). \tag{3.6}$$

That is,

$$d(x_{N+1}, x_1) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0), \tag{3.7}$$

$$d(x_{N+1}, x_2) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0), \tag{3.8}$$

$$\vdots$$

$$d(x_{N+1}, x_{N-1}) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0), \tag{3.9}$$

$$d(x_{N+1}, x_N) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0). \tag{3.10}$$

In fact, since $f : X \to X$ is a quasi-contraction, there exists

$$u_1 \in C(x_N, x_0) = \{d(x_N, x_0), d(x_N, x_{N+1}), d(x_0, x_1), d(x_N, x_1), d(x_0, x_{N+1})\}$$

such that

$$d(x_{N+1}, x_1) \preceq \lambda u_1.$$

If $u_1 = d(x_N, x_1)$, then by (3.3) we have

$$d(x_{N+1}, x_1) \preceq s\lambda^2(e - s\lambda)^{-1}d(x_1, x_0) \preceq (s\lambda)^2(e - s\lambda)^{-1}d(x_1, x_0) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0).$$

If $u_1 = d(x_0, x_1)$, then we have

$$d(x_{N+1}, x_1) \preceq \lambda d(x_1, x_0) \preceq s\lambda d(x_1, x_0) \preceq s\lambda(e - s\lambda)^{-1}d(x_1, x_0).$$

If $u_1 = d(x_N, x_0)$, then by (3.3) we have

$$\begin{aligned} d(x_{N+1}, x_1) &\preceq \lambda d(x_N, x_0) \preceq s\lambda(d(x_N, x_1) + d(x_1, x_0)) \\ &\preceq s\lambda(s\lambda(e - s\lambda)^{-1}d(x_1, x_0) + d(x_1, x_0)) \\ &= s\lambda((s\lambda)^{-1} + e)d(x_1, x_0) \\ &= s\lambda(e - s\lambda)^{-1}d(x_1, x_0). \end{aligned}$$

If $u_1 = d(x_0, x_{N+1})$, then we have

$$d(x_{N+1}, x_1) \preceq \lambda d(x_0, x_{N+1}) \preceq s\lambda(d(x_0, x_1) + d(x_1, x_{N+1})).$$

Hence, we see

$$(e - s\lambda)d(x_{N+1}, x_1) \preceq s\lambda d(x_0, x_1),$$

which implies that

$$d(x_{N+1}, x_1) \preceq (e - s\lambda)^{-1} s\lambda d(x_0, x_1).$$

Without loss of generality, suppose that $u_1 = d(x_N, x_{N+1})$. Since $f : X \to X$ is a quasi-contraction, there exists $u_2 \in C(x_{N-1}, x_N)$ such that

$$u_1 = d(x_N, x_{N+1}) \preceq \lambda u_2,$$

where

$$C(x_{N-1}, x_N) = \{d(x_{N-1}, x_N), d(x_{N-1}, x_N), d(x_N, x_{N+1}), d(x_{N-1}, x_{N+1}), d(x_N, x_N)\}.$$

So, we get

$$d(x_{N+1}, x_1) \preceq \lambda u_1 \preceq \lambda^2 u_2.$$

Similarly, it is easy to see that $u_2 \neq d(x_N, x_N)$ since $u_2 \neq \theta$ and $u_2 \neq d(x_N, x_{N+1})$ since $d(x_N, x_{N+1}) \npreceq \lambda^2 d(x_N, x_{N+1})$.

If $u_2 = d(x_{N-1}, x_N)$, then by the induction assumption (3.2) we have

$$\begin{aligned} d(x_{N+1}, x_1) &\preceq \lambda^2 u_2 \preceq s\lambda^3 (e - s\lambda)^{-1} d(x_1, x_0) \\ &\preceq (s\lambda)^3 (e - s\lambda)^{-1} d(x_1, x_0) \\ &\preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0). \end{aligned}$$

Without loss of generality, suppose that $u_2 = d(x_{N-1}, x_{N+1})$. There exists $u_3 \in C(x_{N-2}, x_N)$ such that

$$u_2 = d(x_{N-1}, x_{N+1}) \preceq \lambda u_3,$$

where

$$C(x_{N-2}, x_N) = \{d(x_{N-2}, x_N), d(x_{N-2}, x_{N-1}), d(x_N, x_{N+1}), d(x_{N-2}, x_{N+1}), d(x_N, x_{N-1})\}.$$

In general, suppose that $u_{i-1} = d(x_{N-i+2}, x_{N+1})$. Since $f : X \to X$ is a quasi-contraction, by the similar arguments above, there exists $u_i \in C(x_{N-i+1}, x_N)$ such that

$$u_{i-1} = d(x_{N-i+2}, x_{N+1}) \preceq \lambda u_i,$$

for which we obtain

$$d(x_{N+1}, x_1) \preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^i u_i,$$

where

$$\begin{aligned} C(x_{N-i+1}, x_N) = \{&d(x_{N-i+1}, x_N), d(x_{N-i+1}, x_{N-i+2}), d(x_N, x_{N+1}), \\ &d(x_{N-i+1}, x_{N+1}), d(x_N, x_{N-i+2})\}. \end{aligned}$$

Similarly, it is easy to see that $u_i \neq d(x_N, x_{N+1})$. This is because by Proposition 2.21(iii) we have

$$u_1 = d(x_N, x_{N+1}) \not\preceq \lambda^{i-1} d(x_N, x_{N+1}).$$

So we know that if $u_i = d(x_{N-i+1}, x_N)$ or $u_i = d(x_{N-i+1}, x_{N-i+2})$ or $u_i = d(x_N, x_{N-i+2})$ then by the induction assumption (3.2) we have $u_i \preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0)$. Hence,

$$\begin{aligned} d(x_{N+1}, x_1) &\preceq \lambda^i u_i \preceq s\lambda^{i+1}(e - s\lambda)^{-1} d(x_1, x_0) \\ &\preceq (s\lambda)^{i+1}(e - s\lambda)^{-1} d(x_1, x_0) \\ &\preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0), \end{aligned}$$

which means that (3.7) holds true. Without loss of generality, suppose that $u_i = d(x_{N-i+1}, x_{N+1})$. Then by the similar arguments as above we have $u_i \preceq \lambda u_{i+1}$, where $u_{i+1} \in C(x_{N-i}, x_N)$. Hence, there is a sequence $\{u_n\}$ such that

$$d(x_{N+1}, x_1) \preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^{N-1} u_{N-1} \preceq \lambda^N u_N,$$

where

$$u_{N-1} = d(x_2, x_{N+1}) \preceq \lambda u_N$$

and

$$u_N \in C(g; x_1, x_N) = \{d(x_1, x_N), d(x_1, x_2), d(x_N, x_{N+1}), d(x_N, x_2), d(x_1, x_{N+1})\}.$$

Obviously, $u_N \neq d(x_1, x_{N+1})$ and $u_N \neq d(x_N, x_{N+1})$. On the contrary, if $u_N = d(x_1, x_{N+1})$, then $u_N \preceq \lambda^N u_N$, a contradiction. If $u_N = d(x_N, x_{N+1}) = u_1$, then we have

$$u_1 = d(x_N, x_{N+1}) \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^{N-1} u_{N-1} \preceq \lambda^{N-1} u_1,$$

a contradiction. Hence, it follows that $u_N = d(x_1, x_N)$, $u_N = d(x_1, x_2)$ or $u_N = d(x_N, x_2)$. By the induction assumption (3.2), in any case, we have

$$u_N \preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0). \tag{3.11}$$

Therefore, we get

$$\begin{aligned} d(x_{N+1}, x_1) &\preceq \lambda u_1 \preceq \lambda^2 u_2 \preceq \cdots \preceq \lambda^N u_N \\ &\preceq \lambda^N (e - s\lambda)^{-1} s\lambda d(x_1, x_0) \\ &\preceq (s\lambda)^{N+1}(e - s\lambda)^{-1} d(x_1, x_0) \\ &\preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0). \end{aligned} \tag{3.12}$$

That is to say, (3.7) is true. By (3.12), we have

$$u_1 \preceq \lambda^{N-1} s\lambda(e - s\lambda)^{-1} d(x_1, x_0).$$

Thus,

$$\begin{aligned} d(x_N, x_{N+1}) = u_1 &\preceq \lambda^{N-1} s\lambda(e - s\lambda)^{-1} d(x_1, x_0) \\ &\preceq (s\lambda)^N (e - s\lambda)^{-1} d(x_1, x_0) \\ &\preceq s\lambda(e - s\lambda)^{-1} d(x_1, x_0), \end{aligned}$$

which implies that (3.10) is true. Similarly, since

$$u_2 = d(x_{N-1}, x_{N+1}), \ldots, u_i = d(x_{N-i+1}, x_{N+1}), \ldots,$$

by (3.11) and (3.12) we get

$$u_i \preceq \lambda^{N-i} u_N \preceq s\lambda^{n-i+1}(e - s\lambda)^{-1} d(x_1, x_0). \tag{3.13}$$

Hence, it follows from (3.13) that (3.8)-(3.9) are all true. That is, (3.6) is true. Therefore, we conclude that Lemma 3.2 holds true. □

By Lemma 3.2, we immediately obtain the following result.

Lemma 3.3. We have that for all $i, j \in \mathbb{N}_+$

$$d(x_i, x_j) \preceq s\lambda(e - s\lambda)^{-1} d(x_0, x_1). \tag{3.14}$$

Now, we begin to prove Theorem 3.1. First, we need to show that $\{x_n\}$ is a Cauchy sequence. For all $n > m$, there exists

$$\begin{aligned} \nu_1 \in C(x_{n-1}, x_{m-1}) = \{&d(x_{n-1}, x_{m-1}), d(x_{n-1}, x_n), \\ &d(x_{m-1}, x_m), d(x_{n-1}, x_m), d(x_{m-1}, x_n)\} \end{aligned}$$

such that

$$d(fx_{n-1}, fx_{m-1}) \preceq \lambda \nu_1.$$

Using the quasi-contractive condition repeatedly, we easily show by induction that there must exist

$$\nu_k \in \{d(x_i, x_j) : 0 \le i < j \le n\} \ (k = 2, 3, \ldots, m)$$

such that

$$\nu_k \preceq \lambda \nu_{k+1} \ (k = 1, 2, \ldots, m - 1). \tag{3.15}$$

For convenience, we write $\nu_m = d(x_i, x_j)$ where $0 \le i < j \le n$.

Using the triangular inequality, we have

$$d(x_i, x_j) \preceq sd(x_i, x_0) + sd(x_0, x_j)(0 \le i, j \le n),$$

and by Lemma 3.3 we obtain

$$\begin{aligned} d(x_n, x_m) = d(fx_{n-1}, fx_{m-1}) &\preceq \lambda\nu_1 \preceq \lambda^2\nu_2 \preceq \cdots \preceq \lambda^m \nu_m \\ &\preceq \lambda^m d(x_i, x_j) \\ &= s\lambda^{m+1}(e - s\lambda)^{-1} d(x_1, x_0). \end{aligned}$$

Since $r(\lambda) < 1/s \le 1$, by Remark 2.8 we have that $s\lambda^{m+1}(e - s\lambda)^{-1} d(x_1, x_0) \to \theta$ as $m \to \infty$, so by Proposition 2.20, it is easy to see that for any $c \in \text{int}P$, there exists $n_0 \in \mathbb{N}$ such that for all $n > m > n_0$,

$$d(x_n, x_m) \preceq s\lambda^{m+1}(e - s\lambda)^{-1} d(x_1, x_0) \ll c.$$

So $\{x_n\}$ is a Cauchy sequence in X. If $f(X)$ is complete, there exist $q \in f(X) \subset X$ such that $x_n \to q$ as $n \to \infty$.

Now, from (2.1) we get

$$d(fx_n, fq) \preceq \lambda\nu$$

where

$$\nu \in C(x_n, q) = \{d(x_n, q), d(x_n, fx_n), d(q, fq), d(x_n, fq), d(fx_n, q)\}.$$

Clearly at least one of the following five cases holds for infinitely many n.

(1) $d(fx_n, fq) \preceq \lambda d(x_n, q) \preceq s\lambda d(x_{n+1}, q) + s\lambda d(x_{n+1}, x_n)$;

(2) $d(fx_n, fq) \preceq \lambda d(x_n, fx_n) = \lambda d(x_n, x_{n+1})$;

(3) $d(fx_n, fq) \preceq \lambda d(q, fq) \preceq s\lambda d(x_{n+1}, q) + s\lambda d(x_{n+1}, fq)$,
that is, $d(fx_n, fq) \preceq s\lambda(e - s\lambda)^{-1}d(x_{n+1}, q)$;

(4) $d(fx_n, fq) \preceq \lambda d(x_n, fq) \preceq s\lambda d(x_{n+1}, fq) + s\lambda d(x_{n+1}, x_n)$,
that is, $d(fx_n, fq) \preceq s\lambda(e - s\lambda)^{-1}d(x_{n+1}, x_n)$;

(5) $d(fx_n, fq) \preceq \lambda d(fx_n, fq) = \lambda d(x_{n+1}, q)$.

As $s\lambda \preceq s\lambda(e - s\lambda)^{-1}$ (since $\theta \preceq s\lambda$ and $r(s\lambda) < 1$), we obtain that

$$d(x_{n+1}, fq) \preceq s\lambda(e - s\lambda)^{-1}[d(x_{n+1}, x_n) + d(x_{n+1}, q)].$$

Since $x_n \to q$ as $n \to \infty$, we get that for any $c \in \mathrm{int}P$, there exists $n_1 \in \mathbb{N}$ such that for all $n > n_1$, one has

$$d(x_{n+1}, fq) \ll c.$$

By Lemmas 2.13 and 2.14, we have $x_n \to fq$ as $n \to \infty$ and $q = fq$.

Now if u is another point such that $u = fu$, hence

$$d(u, q) = d(fu, fq) \preceq \lambda\nu,$$

where $r(\lambda) \in [0, 1/s)$ and

$$\nu \in C(u, q) = \{d(u, q), d(u, fu), d(q, fq), d(u, fq), d(fu, q)\}.$$

It is obvious that $d(u, q) = \theta$, i.e., $u = q$. Therefore, q is the unique fixed point of f in X.

Next, we obtain common fixed point results for two weakly compatible self-mappings concerning g-quasi-contraction as corollaries of Theorem 3.1.

Corollary 3.4. *Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ be the g-quasi-contraction with the g-quasi-contractive constant vector satisfying $r(\lambda) \in [0, 1/s)$. If the range of g contains the range of f and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point in X.*

Proof . Since the mapping $f : X \to X$ be the g-quasi-contraction with the g-quasi-contractive constant vector satisfying $r(\lambda) \in [0, 1/s)$, by Definition 2.7, for $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and for all x, $y \in X$, we have

$$d(fx, fy) \preceq \lambda u,$$

where

$$u \in C(g; x, y) = \{d(gx, gy), d(gx, fx), d(gy, fy), d(gx, fy), d(gy, fx)\}.$$

Let $Y = gX$. Define $H : Y \to Y$, by $H(gw) = fw$. Replacing fx and gx by $H(gx)$ and $H(gy)$, respectively, in the above inequality we get, $d(H(gx), H(gy)) \preceq \lambda u$ where,

$$u \in d(gx, gy), d(gx, H(gx)), d(gy, H(gy)), d(gx, H(gy)), d(gy, H(gx)).$$

Let $x_1 = gx,\ y_1 = gy$. By the above arguments, we have

$$d(Hx_1, Hy_1) \preceq \lambda u,$$

where

$$u \in C(x_1, y_1) = \{d(x_1, y_1), d(x_1, Hx_1), d(y_1, Hy_1), d(x_1, Hy_1), d(y_1, Hx_1)\},$$

which implies that $H : Y \to Y$ is a generalized quasi-contraction on Y. By Theorem 3.1, there exists a unique $x^* \in Y$ such that $Hx^* = x^*$. So, there is $u^* \in X$ such that $x^* = gu^*$. Hence, we get $Hgu^* = gu^*$. That is, $fu^* = gu^*$, i.e., f and g have the common fixed point $u^* \in X$.

As for the uniqueness of the common fixed point and the other fixed point results concerning point of coincidence, we omit their proofs since the the methods for proving the results are standard. □

Corollary 3.5. *Let $(X,\ d)$ be a complete cone b-metric space with a Banach algebra $\mathcal{A}$ and let P be the underlying cone with $k \in P$. If the mapping $T :\ X \to X$ is a generalized quasi-contraction, then T has a unique fixed point in X. And for any $x \in X$, the iterative sequence $\{T^n x\}$ converges to the fixed point.*

Proof . Set $g = I_X$, the identity mapping from X to X. It is obvious to see that Corollary 3.4 yields Corollary 3.5. □

Remark 3.6. Corollary 3.4 extends [20, Theorem 2.6] to the case of cone b-metric spaces over Banach algebras.

Remark 3.7. Corollary 3.5 does not need to require the assumption of normality of the cone P. So Corollary 3.5 improves and generalizes Theorem 9 in [14].

Remark 3.8. From the proof of Lemma 3.2, we note that the technique of induction appearing in Corollary 3.4 is somewhat different from that in [14, Theorem 9], and also different from that in [20, Theorem 2.6], which is more interesting and easy to understood.

Corollary 3.9. Taking $E = \mathbb{R}$, $P = [0, +\infty)$, $\lambda \in [0, 1/s)$ in Theorem 3.1, we get Das-Naik's result from [9], that is, if $g = I_X$ we get Ćirić's result from [12], both in the setting of b-metric spaces.

The following corollary is the Jungck's result in the setting of cone b-metric spaces.

Corollary 3.10. Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mappings $f,\ g : X \to X$ satisfy the condition that for $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and for every $x, y \in X$ holds $d(fx, fy) \preceq \lambda d(gx, gy)$. If $g(X) \subset f(X)$ and $g(X)$ or $f(X)$ is a complete subspace of X, then f and g have a unique point of coincidence in X. Moreover, if f and g are weakly compatible, then f and g have a unique common fixed point.

The next result is the Banach contraction principle in the setting of cone b-metric spaces.

Corollary 3.11. [23] Let (X, d) be a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s \geq 1$ and the underlying solid cone P. Let the mapping $f : X \to X$ satisfy the condition that for $\lambda \in P$ with $r(\lambda) \in [0, 1/s)$ and for every $x, y \in X$ holds $d(fx, fy) \preceq \lambda d(x, y)$ (namely, f is a generalized quasi-contraction). If $f(X)$ is a complete subspace of X, then f has a unique point in X.

We present some examples to to support the main results.

Example 3.12. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$ and define a norm on $\mathcal{A}$ by $\|x\| = \|x\|_\infty + \|x'\|_\infty$ for $x \in \mathcal{A}$. Define multiplication in $\mathcal{A}$ just as the pointwise multiplication. Then $\mathcal{A}$ is a real Banach algebra with the unit $e = 1$ ($e(t) = 1$ for all $t \in [0, 1]$). The set $P = \{x \in \mathcal{A} : x(t) \geq 0 \text{ for all } t \in [0, 1]\}$ is a cone in $\mathcal{A}$. Moreover P is not normal.

Let $X = \{0, 1, 3\}$. Define $d : X \times X \to \mathcal{A}$ by $d(0, 1)(t) = d(1, 0)(t) = e^t$, $d(0, 3)(t) = d(3, 0)(t) = 9e^t$, $d(3, 1)(t) = d(1, 3)(t) = 4e^t$ and $d(x, x)(t) = \theta$ for all $t \in [0, 1]$ and $x \in X$. It is clear that (X, d) is a solid cone b-metric space over Banach algebra $\mathcal{A}$ with $s = \frac{9}{5}$ without normality. Further, let $f : X \to X$ be a mapping defined with $f(0) = f(1) = 1$ and $f(3) = 0$ and $k \in P$ defined by $k(t) = \frac{11}{38}t + \frac{1}{4}$. By the careful calculations, one can get that all conditions of Theorem 3.1 are fulfilled. The point $x = 1$ is the unique fixed point of f.

We present other examples to show that Corollary 3.11 has meaningful applications in nonlinear applications. Before presenting the next example, we recall a useful known lemma. For the completeness, we give its proof.

Lemma 3.13. *Let $p \geq 1$ be a given real number and $f(t)$ be a Lebesgue measurable function defined on $[0, 1]$. Then one has*

$$\left| \int_0^1 f(t)\,dt \right|^p \leq \int_0^1 |f(t)|^p\,dt.$$

Proof . Without loss of generality, suppose $p > 1$. Let $q \geq 0$ be an arbitrary real number satisfying $\frac{1}{p} + \frac{1}{q} = 1$. Then for any Lebesgue measurable function $g(t)$ defined on $[0, 1]$, by Hölder inequality, we see

$$\int_0^1 |f(t)g(t)|\mathrm{d}t \leq \left(\int_0^1 |f(t)|^p \mathrm{d}t \right)^{\frac{1}{p}} \left(\int_0^1 |g(t)|^p \mathrm{d}t \right)^{\frac{1}{q}}.$$

Now taking $g(t) = 1$, we get

$$\int_0^1 |f(t)|\mathrm{d}t \leq \left(\int_0^1 |f(t)|^p \mathrm{d}t \right)^{\frac{1}{p}}.$$

Then it follows that

$$\left(\int_0^1 |f(t)|\mathrm{d}t \right)^p \leq \int_0^1 |f(t)|^p \mathrm{d}t,$$

which implies that Lemma 3.13 holds. □

Example 3.14. Let $X = C^1_{\mathbb{R}}[0, 1]$ be the set of all real continuous differential functions defined on $[0, 1]$. Let $\mathcal{A} = C^1_{\mathbb{R}}[0, 1]$. Consider the following nonlinear integral equation

$$\int_0^1 F(t, f(s))\,\mathrm{d}s = f(t), \tag{3.16}$$

where F satisfies:

(a) $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ is a continuous function,

(b) the partial derivative F_y of F with respect to y exists and $|F_y(x,y)| \leq L$ for some constant $L \in [0, \frac{1}{2})$.

Theorem 3.15. *The equation* (3.16) *has a unique solution in* $C^1_{\mathbb{R}}[0,1]$.

Proof . Let $\mathcal{A} = C^1_{\mathbb{R}}[0,1]$ and $P = \{x \in C^1_{\mathbb{R}}[0,1] \mid x = x(t) \geq 0, \forall t \in [0,1]\}$. Then P is a non-normal cone of the real Banach algebra $\mathcal{A}$ with the unit element $e = 1$ (see [19]) and the operations as

$$(x+y)(t) = x(t) + y(t),$$

$$(cx)(t) = cx(t),$$

$$(xy)(t) = x(t)x(t),$$

for all $x = x(t), y = y(t) \in \mathcal{A}$ and $c \in \mathbb{R}$. Note that the norm on $\mathcal{A}$ is defined as

$$\|f\| = \|f\|_\infty + \|f'\|_\infty ,$$

where $f \in \mathcal{A}$, $\|f\|_\infty = \sup_{0\leq t\leq 1} |f(t)|$. Then (X,d) is a cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s = 2^p$ $(p > 1)$.

In fact, let $x, y, z \in X$, and $u = x - z, v = z - y$, then $x - y = u + v$. By the inequality

$$(a+b)^p \leq (2\max\{a,b\})^p \leq 2^p(a^p + b^p), \quad a, b > 0$$

we see

$$|x-y|^p = |u+v|^p \leq (|u| + |v|)^p \leq 2^p(|x-z|^p + |z-y|^p),$$
$$|x-y|^p e^t \leq 2^p(|x-z|^p e^t + |z-y|^p e^t),$$

thus

$$d(x,y) \preceq s[d(x,z) + d(z,y)],$$

where $s = 2^p > 1$, but (X,d) is not a cone metric space. Let T be a self map of X defined by

$$Tf(t) = \int_0^1 F(t, f(s))\,\mathrm{d}s.$$

We now prove that T is a generalized contraction with the contractive constant vector L^p satisfying $r(L^p) \leq \|L^p\| = L^p < \frac{1}{2^p} = \frac{1}{s}$. In fact, by Lemma 3.3, we have

$$\begin{aligned}
d(Tf, Tg) &= \|(Tf - Tg)^p\|_\infty e^t \\
&= e^t \max_{0\leq x\leq 1} \left| \int_0^1 \big(F(x, f(t)) - F(x, g(t))\big)\mathrm{d}t \right|^p \\
&\leq e^t \max_{0\leq x\leq 1} \left(\int_0^1 \big|F(x, f(t)) - F(x, g(t))\big|\mathrm{d}t \right)^p \\
&\leq e^t \left(\int_0^1 L\big|f(t) - g(t)\big|\mathrm{d}t \right)^p \\
&\leq e^t \int_0^1 \big(L\big|f(t) - g(t)\big|\big)^p \mathrm{d}t \\
&\leq L^p e^t \max_{0\leq t\leq 1} \big|f(t) - g(t)\big|^p \\
&= L^p d\big(f, g\big).
\end{aligned}$$

Then by Corollary 3.11, T has a unique fixed point in X. That is, the equation (3.16) has a unique solution in $C^1_{\mathbb{R}}[0,1]$. $\square$

Remark 3.16. Compared with [24, Theorem 3.1], Example 3.14 shows that under the same conditions the unique solution to the integral equation (3.9) is not only continuous but also differential, while [24, Theorem 3.1] only shows the continuity of the solution. In addition, the technique appearing in Example 3.14 is somewhat different from that in [24, Theorem 3.1], since in Example 3.14, the proof is made in the setting of complete non-normal cone b-metric space over a Banach algebra but in [24, Theorem 3.1], the proof is provided in the setting of cone metric space, depending strongly on the normality of the underlying cone. So our main results generalizes the comparable results in [24].

Example 3.17. Let $X = \mathbb{R}^2$, $f = f(s,t) : X \to \mathbb{R}$, $g = g(s,t) : X \to \mathbb{R}$. Consider the following group of nonlinear coupled equations

$$\text{(I)} \begin{cases} f(x,\, y) = x, \\ g(x,\, y) = y - px, \end{cases}$$

where $p \geq 0$. Suppose that there exists $0 < k < 1$ such that

$$\left|\frac{\partial f}{\partial s}\right| \leq k, \ \left|\frac{\partial g}{\partial t}\right| \leq k$$

for all $(s,t) \in X$.

Theorem 3.18. *The coupled equations in (I) have a unique common solution in X.*

Proof . Let $\mathcal{A} = \mathbb{R}^2$ with the norm defined as $\|(x,y)\| = |x| + |y|$ for each $(x,y) \in \mathcal{A}$. Then $\mathcal{A}$ is a Banach algebra with the operations as

$$(x_1,y_1) + (x_2,y_2) = (x_1 + x_2, y_1 + y_2),$$

$$c(x_1,y_1) = (cx_1, cy_1),$$

$$(x_1,y_1)(x_2,y_2) = (x_1x_2, x_1y_2 + x_2y_1),$$

for all $(x_1,y_1),(x_2,y_2) \in \mathcal{A}$ and $c \in \mathbb{R}$. Moreover, $\mathcal{A}$ owns the unit element $e = (1,0)$.

Let $P = \{(x,y) \in \mathbb{R}^2 \mid x,y \geq 0\}$. Then P is a cone of $\mathcal{A}$.

Let $X = \mathbb{R}^2$ and the metric $d : X \times X \to \mathcal{A}$ be defined by

$$d\left((x_1,y_1),\,(x_2,y_2)\right) = (|x_1 - x_2|, |y_1 - y_2|) \in P.$$

Then (X,d) is a complete cone b-metric space over a Banach algebra $\mathcal{A}$ with the coefficient $s = 1$.

Now define mapping $T : X \to X$ by

$$T(x,y) = \left(f(x,y), g(x,y) + px\right). \tag{3.17}$$

From Lagrange Mean Value Theorem, we have

$$\begin{aligned} d\big(T(x_1,y_1),T(x_2,y_2)\big) &= d\big(\left(f(x_1,y_1), g(x_1,y_1) + px_1\right), \left(f(x_2,y_2), g(x_2,y_2) + px_2\right)\big) \\ &= \big(\big|f(x_1,y_1) - f(x_2,y_2)\big|, \big|g(x_1,y_1) - g(x_2,y_2) + p(x_1 - x_2)\big|\big) \\ &\preceq (k|x_1 - x_2|, k|y_1 - y_2| + p|x_1 - x_2|) \\ &= (k,p)\left(\big|x_1 - x_2)\big|,\ \big|y_1 - y_2\big|\right) \\ &= (k,p)\, d\big((x_1,y_1),(x_2,y_2)\big), \end{aligned}$$

and

$$\|(k,p)^n\|^{\frac{1}{n}} = \left\|\left(k^n, pnk^{n-1}\right)\right\|^{\frac{1}{n}} = (k^n + pnk^{n-1})^{\frac{1}{n}} \to k < 1 \quad (n \to \infty),$$

which implies $r((k,p)) < \frac{1}{s}$. Then by Corollary 3.11, T has a unique fixed point in X. $\square$

The following example is a direct result of Theorem 3.18.

Example 3.19. Consider the following group of nonlinear coupled equations

$$\text{(II)} \left\{ \begin{array}{l} \log(m+x) = x, \\ \arctan(n+y) = y - px, \end{array} \right.$$

where $p \geq 0, m > 1$ and $n \geq \sqrt{m-1}$. The coupled equations in (II) have a unique common solution.

In fact, set $f(t,s) = \log(m+s), g(t,s) = \arctan(n+y)$. Then all the conditions of Theorem 3.18 are satisfied. Thus it follows from Theorem 3.18 that the coupled equations (II) have a unique positive common solution.

Remark 3.20. In Example 3.19, if $p > 1$, then $\|(k,p)\| = k + p > 1$, so T is not a contraction in the Euclidean metric on X. Hence, one is unable to directly use Banach contraction principle to show T has a unique fixed point in X.

Remark 3.21. We must emphasize that one is unable to use the techniques presented in [24] to show that the fixed point result in Example 3.19 obtained by the fixed point theorem in the setting of cone b-metric space $(X,\ d)$ with Banach algebra $\mathcal{A}$ can be derived from the existing results in the context of b-metric space or metric space. In fact, Example 3.19 can also show that one is unable to conclude that the cone b-metric space $(X,\ d)$ over a Banach algebra $\mathcal{A}$ discussed is equivalent to the b-metric space $(X,\ d^*)$, where the metric d^* is defined by $d^* = \xi_e \circ d$, here the nonlinear scalarization function $\xi_e : A \to \mathbb{R}$ $(e \in \text{int}P)$ is defined by

$$\xi_e(y) = \inf\{r \in \mathbb{R} : y \in re - P\}.$$

See [10, 16, 22] for more details. In the case of Example 3.19, we have

$$\text{int}P = \{(x,\ y) \in \mathbb{R}^2 \,|\, x, y > 0\}.$$

For $e = (e_1,\ e_2) \in \text{int}P$, and $a = (a_1,\ a_2) \in \mathcal{A}$,

$$\begin{aligned} \xi_e(a) &= \xi_e\big((a_1,\ a_2)\big) \\ &= \inf\{t \in \mathbb{R} \,|\, (a_1,\ a_2) \leq t(e_1,\ e_2)\} \\ &= \max\left\{\frac{a_1}{e_1},\ \frac{a_2}{e_2}\right\}, \end{aligned}$$

and for $x = (x_1, y_1),\ y = (x_2, y_2) \in X$,

$$d^*(x,\ y) = \big(\xi_e \circ d\big)(x,\ y) = \xi_e\big((|x_1 - x_2|,\ |y_1 - y_2|)\big) = \max\left\{\frac{|x_1 - x_2|}{e_1},\ \frac{|y_1 - y_2|}{e_2}\right\}.$$

Now let the mapping $T : X \to X$ be defined in (3.10), namely,

$$T(x,y) = (f(x,y), g(x,y) + px)\,,$$

where $f(x, y) = \log(m + x)$, $g(x, y) = \arctan(n + y)$ and $p > \frac{e_2}{e_1}$. Considering $u = (1, 0)$, $v = (0, 0)$, we have

$$Tu = (\log(m + 1), \arctan n + p), Tv = (\log m, \arctan n),$$

and so

$$d^*(Tu, Tv) = \max\left\{\frac{\log(m + 1) - \log m}{e_1}, \frac{p}{e_2}\right\} \geq \frac{p}{e_2} > \frac{1}{e_1} = d^*(x, y),$$

which implies that T is not a contraction in metric space (X, d^*).

Remark 3.22. Since $p > \frac{e_2}{e_1}$ is arbitrary in Remark 3.20, it is obviously seen that Example 3.19 shows that there exist lots of fixed point theorems of generalized quasi-contractions in the setting of cone b-metric spaces or cone metric spaces over Banach algebras which cannot be yielded by the known results in the setting of b-metric space or metric spaces.

In conclusion, based on the arguments above, we hold that the fixed point results presented in this paper in the setting of cone b-metric spaces over Banach algebras are more general than corresponding ones in the setting of b-metric or metric spaces.

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