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Nonexpansive mappings on complex C*-algebras and their fixed points

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Abstract

A normed space $\mathfrak{X}$ is said to have the fixed point property, if for each nonexpansive mapping $T : E \longrightarrow E$ on a nonempty bounded closed convex subset E of $\mathfrak{X}$ has a fixed point. In this paper, we first show that if X is a locally compact Hausdorff space then the following are equivalent: (i) X is infinite set, (ii) $C_0(X)$ is infinite dimensional, (iii) $C_0(X)$ does not have the fixed point property. We also show that if A is a commutative complex C*–algebra with nonempty carrier space, then the following statements are equivalent: (i) Carrier space of A is infinite, (ii) A is infinite dimensional, (iii) A does not have the fixed point property. Moreover, we show that if A is an infinite dimensional complex C*–algebra (not necessarily commutative), then A does not have the fixed point property.

Keywords: Banach space, C*–algebra, fixed point property, nonexpansive mapping, normed linear space.
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1. Introduction and Preliminaries

Let $T : E \longrightarrow E$ be a self-map on the nonempty set E. We denote $\{x \in E : T(x) = x\}$ by $\mathrm{Fix}(T)$ and call the *fixed points set* of T. The symbol $\mathbb{K}$ denote a field that can be either $\mathbb{C}$ or $\mathbb{R}$. Let $(X, \|\cdot\|)$ be a normed linear space over $\mathbb{K}$. A mapping $T : E \subseteq \mathfrak{X} \longrightarrow \mathfrak{X}$ is *nonexpansive* if $\|T(x)-T(y)\| \leqslant \|x-y\|$ for all $x, y \in E$. We say that the normed linear space $(X, \|\cdot\|)$ over $\mathbb{K}$ has the *fixed point property* (or *weak fixed point property*) if for every nonempty bounded closed convex (or weakly compact convex, respectively) subset E of X and every nonexpansive mapping $T : E \longrightarrow E$ we have $\mathrm{Fix}(T) \neq \emptyset$.

One of the central goals in fixed point theory is to find which normed linear spaces over $\mathbb{K}$ have the (weak) fixed point property.

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Theorem 1.1. *Let $(X_1, \|\cdot\|_1)$ be a Banach space, $(X_2, \|\cdot\|_2)$ be a normed linear space and there exist a linear isometry from $(X_1, \|\cdot\|_1)$ into $(X_2, \|\cdot\|_2)$ over $\mathbb{K}$. If $(X_1, \|\cdot\|_1)$ does not have the fixed point property then $(X_2, \|\cdot\|_2)$ does not have the fixed point property.*

Proof . Let $(X_1, \|\cdot\|_1)$ does not have the fixed point property. Then there exist a nonempty bounded closed convex subset E of X_1 and a nonexpansive mapping $T : E \longrightarrow E$ such that $\mathrm{Fix}(T) = \emptyset$. Let $\Psi : X_1 \longrightarrow X_2$ be a linear isometry from $(X_1, \|\cdot\|_1)$ into $(X_2, \|\cdot\|_2)$ over $\mathbb{K}$. Then $\Psi(E)$ is a nonempty convex subset of X_2 and $\Psi(E)$ is bounded in $(X_2, \|\cdot\|_2)$. Moreover, since E is a closed subset of X_1 in the Banach space $(X_1, \|\cdot\|_1)$ and $\Psi : X_1 \longrightarrow X_2$ is a linear isometry from $(X_1, \|\cdot\|_1)$ into $(X_2, \|\cdot\|_2)$, we deduce that $\Psi(E)$ is a closed subset of X_2 in the normed linear space $(X_2, \|\cdot\|_2)$. We define the mapping $S : \Psi(E) \longrightarrow \Psi(E)$ by

$$S(\Psi(x)) = \Psi(T(x)) \qquad (x \in E).$$

Since for all x and y in X_1 we have

$$\begin{aligned} \|S(\Psi(x)) - S(\Psi(y))\|_2 &= \|\Psi(T(x)) - \Psi(T(y))\|_2 \\ &= \|T(x) - T(y)\|_1 \\ &\leqslant \|x - y\|_1 \\ &= \|\Psi(x) - \Psi(y)\|_2, \end{aligned}$$

we conclude that $S : \Psi(E) \longrightarrow \Psi(E)$ is a nonexpansive mapping. We claim that $\mathrm{Fix}(S) = \emptyset$. Suppose that $\Psi(x_1) \in \mathrm{Fix}(S)$ where $x_1 \in E$. Then

$$0 = S(\Psi(x_1)) - \Psi(x_1) = \Psi(T(x_1)) - \Psi(x_1) = \Psi(T(x_1) - x_1),$$

and so $0 = T(x_1) - x_1$. This implies that $x_1 \in \mathrm{Fix}(T)$ contadicting to $\mathrm{Fix}(T) = \emptyset$. Hence, our claim is justified. Therefore, $(X_2, \|\cdot\|_2)$ does not have the fixed point property. $\square$

Corollary 1.2. Let $(X, \|\cdot\|)$ be a Banach space and Y be a closed linear subspace of $\mathfrak{X}$ over $\mathbb{K}$. If $(Y, \|\cdot\|)$ does not have the fixed point property, then $(\mathfrak{X}, \|\cdot\|)$ does not have the fixed point property.

Let A be a complex algebra and let $A_e := A \times \mathbb{C}$. Then A_e is a complex algebra with unit $e = (0, 1)$ whenever algebra operations are defined by

$$(f, \lambda) + (g, \mu) = (f + g, \lambda + \mu), \quad \alpha(f, \lambda) = (\alpha f, \alpha\lambda), \quad (f + \lambda)(g, \mu) = (fg + \mu f + \lambda g, \lambda\mu),$$

for $f, g \in A$, $\lambda, \mu, \alpha \in \mathbb{C}$. We say that A_e is the unitisation of A. Clearly, A_e is commutative if A is commutative. Moreover, if $\|\cdot\|$ is an algebra norm on A then A_e is a normed algebra under the norm $\|\cdot\|$ defined by

$$\|(f, \lambda)\| = \|f\| + |\lambda| \qquad (f \in A, \lambda \in \mathbb{C}).$$

Note that $(A_e, \|\cdot\|)$ is a unital Banach algebra if $(A, \|\cdot\|)$ is a Banach algebra.

Let A be a complex algebra with unit e and let $G(A)$ be the set of all invertible elements of A. We define the *spectrum* of an element $f \in A$ to be the set $\{\lambda \in \mathbb{C} : \lambda e - f \notin G(A)\}$ and denote it by $\sigma_A(f)$.

Let A be a complex algebra and A_e be the unitisation of A. For $f \in A$, the set $\sigma_{A_e}(f, 0)$ is called the *spectrum* of f and denoted by $\sigma_A(f)$.

Let A be a complex normed algebra and let $f \in A$. The *spectral radius* of f is denoted by $r_A(f)$ and defined by

$$r_A(f) = \inf\left\{\|f^n\|^{\frac{1}{n}} \;:\; n \in \mathbb{N}\right\}.$$

It is known [5, Lemma 1.2.5 and Theorem 1.2.8] that

$$r_A(f) = \lim_{n\to\infty} \|f^n\|^{\frac{1}{n}} = \sup\{|\lambda| \;:\; \lambda \in \sigma_A(f)\}.$$

Let A be a complex algebra. A *character* on A is a nonzero multiplicative linear functional on A. We denote by $\Delta(A)$ the set of all characters on A. If A has the unite e, then $\phi(e) = 1$ for all $\phi \in \Delta(A)$. Note that each $\phi \in \Delta(A)$ has a unique extension $\tilde{\phi} \in \Delta(A_e)$ given by

$$\tilde{\phi}(f + \lambda e) = \phi(f) + \lambda \qquad (f \in A,\ \lambda \in \mathbb{C}).$$

It is known that if A is complex Banach algebra and $\phi \in \Delta(A)$, then ϕ is bounded and $\|\phi\| \leqslant 1$. In particular, $\|\phi\| = 1$ if A is unital. Moreover, if A is a unital commutative complex Banach algebra, then $\Delta(A) \neq \emptyset$ and $\sigma_A(f) = \{\phi(f) \;:\; \phi \in \Delta(A)\}$ for all $f \in A$. If A is without unit, it is possible $\Delta(A) = \emptyset$. (See [5, Example 2.1.6 and Example 2.1.7])

Let A be a commutative complex Banach algebra with $\Delta(A) \neq \emptyset$. For each $f \in A$, we define $\hat{f} : \Delta(A) \longrightarrow \mathbb{C}$ by $\hat{f}(\phi) = \phi(f)$ and say that $\hat{f}$ is the *Gelfand transform* of f. We denote $\{\hat{f} : f \in A\}$ by $\hat{A}$. Then $\hat{A}$ strongly separates the points of $\Delta(A)$. Moreover, the following statements are equivalent:

(i) $\hat{A}$ is self-adjoint.

(ii) For each $f \in A$, there exists an element $g \in A$ such that $\phi(g) = \overline{\phi(f)}$ for all $\phi \in \Delta(A)$.

We endow $\Delta(A)$ with the Gelfand topology, the weakest topology on $\Delta(A)$ for which every $\hat{f} \in \hat{A}$ is continuous. $\Delta(A)$ with the Gelfand topology is called the *carrier space* of A. We know [5, Theorem 2.2.3] that

(i) $\Delta(A)$ is a locally compact Hausdorff space,

(ii) $\Delta(A_e)$ is one-point compactification of $\Delta(A)$,

(iii) $\Delta(A)$ is compact if A has the unit element.

Fupinwong studied the fixed point property of commutative complex Banach algebras in [3] and obtained the following result.

Theorem 1.3. (See [3, Theorem 3.1]) Let A be an infinite dimensional commutative complex Banach algebra with $\Delta(A) \neq \emptyset$ satisfying each of the following:

(i) $\hat{A}$ is self-adjoint,

(ii) if $f, g \in A$ such that $|\phi(f)| \leqslant |\phi(g)|$ for all $\phi \in \Delta(A)$, then $\|f\| \leqslant \|g\|$,

(iii) $\inf\{r_A(f) : f \in A, \|f\| = 1\} > 0$.

Then A does not have the fixed point property.

Fupinwong and Dhompongsa were obtained the mentioned result in [4] whenever A is unital. (See [4, Theorem 4.3])

Let X be a locally compact Hausdorff space. We denote by $C(X)$ and $C_0(X)$ the set of all complex-valued continuous functions on X and the set of all functions in $C(X)$ which vanish at infinity, respectively. Then $C(X)$ is a commutative complex algebra with unit 1_X and $C_0(X)$ is a complex subalgebra of $C(X)$. Moreover, $C_0(X) = C(X)$ if X is compact. It is known that $C_0(X)$ under the uniform norm on X defined by

$$\|f\|_X = \sup\{|f(x)| : x \in X\} \qquad (f \in C_0(X))$$

is a commutative complex Banach algebra. Moreover, $C_0(X)$ is without unit if X is not compact.

Applying the concept of peak points, it is shown [1] that certain uniformly closed subalgebras of $C(X)$ do not have the fixed point property, where X is a compact Hausdorff space.

Dhompongsa, Fupinwong and Lawton studied the fixed point property and weak fixed point property of complex $\mathsf{C}^\star$–algebras in [2].

In Section 2, applying Theorem 1.3 we study the fixed point property of $C_0(X)$ and certain its uniformly closed subalgebras.

In Section 3, we show that a commutative complex $\mathsf{C}^\star$–algebra A does not have the fixed point property if and only if A is infinite dimensional. We also prove that if A is an infinite dimensional complex $\mathsf{C}^\star$–algebra (not necessarily commutative), then A does not have the fixed point property.

2. The fixed point property of $C_0(X)$

Applying Urysohn's lemma [9, Theorem 2.12], Theorem 1.3, and Schauder–Tychonoff fixed point theorem [8, Theorem 5.28], we study the fixed point property of $C_0(X)$ whenever X is a locally compact Hausdorff space.

Theorem 2.1. *Let X be a locally compact Hausdorff space and $A = C_0(X)$. Then the following statements are equivalent:*

(i) X is infinite set.

(ii) A is infinite dimensional.

(iii) $(A, \|\cdot\|_X)$ does not have the fixed point property.

Proof . (i) $\Longrightarrow$ (ii). Let X be infinite set. We can choose a sequence $\{x_n\}_{n=1}^\infty$ in X such that $x_i \neq x_j$ if $i, j \in \mathbb{N}$ and $i \neq j$. By Urysohn's lemma, we obtain a sequence $\{f_n\}_{n=1}^\infty$ of functions in $C_0(X)$ such that $f_1(x_1) = 1$ and

$$f_n(x_1) = \ldots = f_n(x_{n-1}) = 0, \quad f_n(x_n) = 1 \qquad (n \in \mathbb{N},\ n \geqslant 2).$$

To prove that $C_0(X)$ is infinite dimensional, it is sufficient we show that the set $\{f_1, \cdots, f_n\}$ is a linearly independent set in $C_0(X)$ for all $n \in \mathbb{N}$. Since $f_1(x_1) = 1$, we deduce that $\{f_1\}$ is a linearly independent set in $C_0(X)$. Suppose that $n \in \mathbb{N}$ with $n \geqslant 2$. Let

$$\alpha_1 f_1 + \cdots + \alpha_n f_n = 0 \tag{2.1}$$

where $\alpha_1, \cdots, \alpha_n \in \mathbb{C}$. Since $f_1(x_1) = 1$ and $f_2(x_1) = \cdots = f_n(x_1) = 0$, we conclude that $\alpha_1 = 0$ by (2.1). Suppose that $j \in \{1, \cdots, n\}$ such that $\alpha_1 = \cdots = \alpha_{j-1} = 0$. Then

$$\alpha_j f_j + \cdots + \alpha_n f_n = 0, \tag{2.2}$$

by (2.1). If $j = n$, then $\alpha_1 = \cdots = \alpha_{n-1} = 0$ and $\alpha_n f_n = 0$ by (2.2). Thus $\alpha_n = 0$ since $f_n(x_n) = 1$. If $j \in \{1, \cdots, n-1\}$, then by (2.2) we have $\alpha_j = 0$ since $f_j(x_j) = 1$ and $f_{j+1}(x_j) = \cdots = f_n(x_j) = 0$. Therefore, $\alpha_j = 0$ for all $j \in \{1, \cdots, n\}$ and so the set $\{f_1, \cdots, f_n\}$ is a linearly independent set in $C_0(X)$. Hence, (ii) holds.

(ii) $\Longrightarrow$ (iii). Let A be infinite dimensional. It is known [6, Theorem 2.3] that

$$\Delta(A) = \{e_x : x \in X\}, \tag{2.3}$$

where $e_x : A \longrightarrow \mathbb{C}$ is defined by $e_x(f) = f(x)$ for all $f \in A$. Let $F \in \hat{A}$. Then there exists $f \in A$ such that $F = \hat{f}$, the Gelfand transform of f. Since $\bar{f} \in A$ and

$$\overline{F}(e_x) = \overline{F(e_x)} = \overline{\hat{f}(e_x)} = \overline{e_x(f)} = \overline{f(x)} = \bar{f}(x) = e_x(\bar{f}) = \widehat{\bar{f}}(e_x),$$

for all $x \in X$, we deduce that $\overline{F} = \widehat{\bar{f}}$ by (2.3). Hence, $\overline{F} \in \hat{A}$ and so $\hat{A}$ is self-adjoint.

Let $f, g \in A$ such that $|\phi(f)| \leqslant |\phi(g)|$ for all $\phi \in \Delta(A)$. Since $e_x \in \Delta(A)$ for each $x \in X$, we have

$$|f(x)| = |e_x(f)| \leqslant |e_x(g)| = |g(x)|$$

for each $x \in X$. Therefore, $\|f\|_X \leqslant \|g\|_X$.

Let $f \in A$ such that $\|f\|_X = 1$. Since $\|g^n\|_X = (\|g\|_X)^n$ for all $g \in A$ and for each $n \in \mathbb{N}$, we have $\|f^n\|_X = 1$ for each $n \in \mathbb{N}$, and so

$$r_A(f) = \lim_{n\to\infty} (\|f^n\|_X)^{\frac{1}{n}} = 1.$$

Therefore,

$$\inf\{r_A(f) : f \in A, \|f\|_X = 1\} = \inf\{1\} = 1 > 0.$$

Hence, A does not have the fixed point property by Theorem 1.3.

(iii) $\Longrightarrow$ (i). Let X be a nonempty finite set. Let $x \in X$. Since $X \setminus \{x\}$ is a finite set and X is a Hausdorff space, we deduce that $X \setminus \{x\}$ is closed in X and so $\{x\}$ is an open set in X. Hence, the given topology on X is $\mathbb{P}(X)$, the power set of X. This implies that there exists a linear isometry from $(C_0(X), \|\cdot\|_X)$ onto $\mathbb{C}^n$ with the Euclidean norm, where n is the cardinal number of X. Hence, every bounded closed subset E of $C_0(X)$ is compact in $(C_0(X), \|\cdot\|_X)$. Let E be a nonempty bounded closed convex subset of $C_0(X)$ and $T : E \longrightarrow E$ be a nonexpansive mapping. Then E is a nonempty compact convex subset of the Banach space $(C_0(X), \|\cdot\|_X)$ and $T : E \longrightarrow E$ is continuous. Hence, T has a fixed point by Schauder–Tychonoff fixed point theorem [8, Theorem 5.28]. Therefore, $(C_0(X), \|\cdot\|_X)$ has the fixed point property and so (iii) does not hold. $\square$

Let X be a locally compact Hausdorff space and K be a compact subset of X. We denote by $C_0Z(X,K)$ the set of all $f \in C_0(X)$ such that $f|_K = 0$. It is easy to see that $C_0Z(X,K)$ is a self–adjoint uniformly closed subalgebra of $C_0(X)$. Moreover, $C_0Z(X,K) = C_0(X)$ if and only if $K = \emptyset$.

Theorem 2.2. *Let X be a locally compact Hausdorff space and K be a compact subset of X such that $X \setminus K \neq \emptyset$. Then the following statements hold.*

(i) If $f \in C_0Z(X,K)$ and $g = f|_{X\setminus K}$, then $g \in C_0(X \setminus K)$.

(ii) If $g \in C_0(X \setminus K)$ and the function $g_0 : X \longrightarrow \mathbb{C}$ defines by

$$g_0(x) = \begin{cases} g(x) & x \in X \setminus K, \\ 0 & x \in K. \end{cases}$$

then $g_0 \in C_0Z(X,K)$.

(iii) The map $\Phi : C_0Z(X,K) \longrightarrow C_0(X \setminus K)$ defined by $\Phi(f) = f|_{X\setminus K}$, is an isometrical isomorphism from $(C_0Z(X,K), \|\cdot\|_X)$ onto $\big(C_0(X \setminus K), \|\cdot\|_{X\setminus K}\big)$.

(iv) $C_0Z(X,K)$ strongly separates the points of $X \setminus K$.

(v) For each $x \in X \setminus K$, $e_x \in \Delta\,(C_0Z(X,K))$.

(vi) If $x, y \in X \setminus K$ with $x \neq y$, then $e_x \neq e_y$.

(vii) $\Delta\left(C_0Z(X,K)\right) = \{e_x : x \in X \setminus K\}$.

Proof . (i) Let $f \in C_0Z(X,K)$ and $g = f|_{X\setminus K}$. Clearly, $g \in C(X \setminus K)$. Let $\varepsilon > 0$ and

$$H = \{x \in X : |f(x)| \geqslant \varepsilon\}.$$

Then H is a closed set in X and $H \subseteq X \setminus K$. Since $f \in C_0(X)$, there exists a compact subset L of X such that

$$f\left(X \setminus L\right) \subseteq \{z \in \mathbb{C} : |z| < \varepsilon\}.$$

Set $E = H \cap L$. Then E is a compact set in X and $E \subseteq X \setminus K$. Thus E is a compact set in $X \setminus K$. Moreover,

$$|g(x)| = |f(x)| < \varepsilon,$$

for all $x \in (X \setminus K) \setminus E$. Therefore, $g \in C_0\left(X \setminus K\right)$ and so (i) holds.

(ii) Let $g \in C_0\left(X \setminus K\right)$ and the function $g_0 : X \longrightarrow \mathbb{C}$ defines as above. To prove $g_0 \in C_0Z\left(X, K\right)$, it is sufficient we show that $g_0 \in C_0(X)$ since $g_0|_K = 0$. Let $x_0 \in X \setminus K$ and choose $\varepsilon > 0$. The continuity of $g : X \setminus K \longrightarrow \mathbb{C}$ at x_0 implies that there exists a neighborhood U_0 of x_0 in $X \setminus K$ such that

$$|g(x) - g(x_0)| < \varepsilon \quad (\forall x \in U_0). \tag{2.4}$$

Since $X \setminus K$ is an open set in X, we deduce that U_0 is a neighborhood of x_0 in X. Since $g_0(x_0) = 0 = g(x_0)$ and $g_0|_{X\setminus K} = g$, we conclude that

$$|g_0(x) - g_0(x_0)| < \varepsilon \quad (\forall x \in U_0),$$

by (2.4). Therefore, g_0 is continuous at x_0.

Let $x_0 \in K$ and choose $\varepsilon > 0$. Since $g \in C_0\left(X \setminus K\right)$, there exists a compact subset H in $X \setminus K$ such that

$$g\left((X \setminus K) \setminus H\right) \subseteq \left\{z \in \mathbb{C} : |z| < \varepsilon\right\}. \tag{2.5}$$

The compactness of H in $X \setminus K$ implies that H is a compact set in X and so H is closed in X. Set $U = X \setminus K$. Then U is an open set in X and $K \subseteq U$. Hence, U is a neighborhood of x_0 in X. If $x \in K$, then

$$|g_0(x) - g_0(x_0)| = 0 < \varepsilon.$$

Suppose that $x \in U \setminus K$. Then $x \in (X \setminus H) \setminus K = (X \setminus K) \setminus H$ and so $|g(x)| < \varepsilon$ by (2.5). Hence,

$$|g_0(x) - g_0(x_0)| = |g(x) - 0| = |g(x)| < \varepsilon.$$

Therefore, g_0 is continuous at x_0. So, $g \in C(X)$.

Let $\varepsilon > 0$ be given. Since $g \in C_0(X \setminus K)$, there exists a compact set H of $X \setminus K$ such that

$$g\left((X \setminus K) \setminus H\right) \subseteq \{z \in \mathbb{C} : |z| < \varepsilon\}.$$

Clearly, H is a compact set in X. Set $L = K \cup H$. Then H is a compact set in X and $X \setminus L = (X \setminus K) \setminus H$. So

$$|g_0(x)| = |g(x)| < \varepsilon,$$

for all $x \in X \setminus L$. Therefore, $g_0 \in C_0(X)$ and so (ii) holds.

(iii) By (i), Φ is well–defined. Clearly, Φ is an algebra homomorphism. Let $f \in C_0Z(X,K)$. Then $f \in C_0(X)$ and $f|_K = 0$. Thus $\|f\|_X = \|f\|_{X\setminus K}$ and so

$$\|\Phi(f)\|_{X\setminus K} = \|f|_{X\setminus K}\|_{X\setminus K} = \|f\|_{X\setminus K} = \|f\|_X.$$

Therefore, Φ is an isometry.

Let $g \in C_0(X \setminus K)$. We define the function $g_0 : X \longrightarrow \mathbb{C}$ by

$$g_0(x) = \begin{cases} g(x) & x \in X \setminus K, \\ 0 & x \in K. \end{cases}$$

Then $g_0 \in C_0Z(X,K)$ by (ii) and $g_0|_K = g$. Therefore, Φ is surjective.

(iv) Let $x_1, x_2 \in X\setminus K$ such that $x_1 \neq x_2$. By Urysohn's lemma, there exists a function $f_0 \in C_0(X)$ such that $f_0(x_1) = 1$ and $f_0(x) = 0$ for all $x \in K \cup \{x_2\}$. So $f_0 \in C_0Z(X,K)$ and $f_0(x_1) \neq f_0(x_2)$. Therefore, $C_0Z(X,K)$ separates the points of $X \setminus K$.

Let $x \in X \setminus K$. By Urysohn's lemma, there exists a function $f_1 \in C_0(X)$ such that $f_1(x) = 1$ and $f_1(y) = 0$ for all $y \in K$. So $f_1 \in C_0Z(X,K)$ and $f_1(x) \neq 0$. Therefore, (iv) holds.

(v) Let $x \in X \setminus K$. Clearly, e_x is a multiplicative complex linear functional on $C_0Z(X,K)$. By (iv), there exists a function $f_1 \in C_0Z(X,K)$ such that $f_1(x) \neq 0$ and so $e_x(f_1) \neq 0$. Therefore, $e_x \in \Delta\left(C_0Z(X,K)\right)$.

(vi) Let $x, y \in X \setminus K$ such that $x \neq y$. By (iv), there exists a function $f_0 \in C_0Z(X,K)$ such that $f_0(x) \neq f_0(y)$. Hence, $e_x(f_0) \neq e_y(f_0)$ and so $e_x \neq e_y$.

(vii) By (v), we have

$$\{e_x : x \in X \setminus K\} \subseteq \Delta\left(C_0Z(X,K)\right). \tag{2.6}$$

Let $\psi \in \Delta\left(C_0Z(X,K)\right)$. By (iii), the map $\Phi : C_0Z(X,K) \longrightarrow C_0(X \setminus K)$ defined by

$$\Phi(f) = f|_{X\setminus K} \qquad (f \in C_0Z(X,K))$$

is an isometrical algebra isomorphism from $(C_0Z(X,K), \|\cdot\|_X)$ onto $\big(C_0(X \setminus K), \|\cdot\|_{X\setminus K}\big)$. Therefore, $\psi \circ \Phi^{-1}$ is a multiplicative complex linear function on $C_0(X \setminus K)$. On the other hand, there exists a function $f_0 \in C_0Z(X,K)$ such that $\psi(f_0) \neq 0$. Set $g_0 = \Phi(f_0)$. Then $g_0 \in C_0(X \setminus K)$ and $f_0 = \Phi^{-1}(g_0)$. Thus $\left(\psi \circ \Phi^{-1}\right)(g_0) \neq 0$ and so $\psi \circ \Phi^{-1} \in \Delta\left(C_0(X \setminus K)\right)$. Since $\Delta(C_0(X \setminus K)) = \{e_x : x \in X \setminus K\}$, there exists $y \in X \setminus K$ such that $\psi \circ \Phi^{-1} = e_y$ on $C_0(X \setminus K)$. Let $f \in C_0Z(X,K)$. Then $\Phi(f) \in C_0(X \setminus K)$ and so

$$(\psi \circ \Phi^{-1})(\Phi(f)) = e_y(f).$$

This implies that $\psi(f) = e_y(f)$. Therefore, $\psi = e_y$ on $C_0Z(X,K)$ and so

$$\Delta(C_0Z(X,K)) \subseteq \{e_x : x \in X \setminus K\}. \tag{2.7}$$

From (2.6) and (2.7), we have

$$\Delta(C_0Z(X,K)) = \{e_x : x \in X \setminus K\}.$$

Therefore, (vii) holds. □

Theorem 2.3. *Let X be a locally compact Hausdorff space and K be a compact subset of X. If $X \setminus K$ is infinite set, then $(C_0Z(X,K), \|\cdot\|_X)$ does not have the fixed point property.*

Proof . Since $X \setminus K$ is an open subset of X, we deduce that $X \setminus K$ with the relative topology is a locally compact Hausdorff space. Since $X \setminus K$ is infinite set, $\big(C_0(X \setminus K), \|\cdot\|_{X\setminus K}\big)$ does not have the fixed point property by Theorem 2.1.

By part (iii) of Theorem 2.2, the map $\Phi : C_0Z(X, K) \longrightarrow C_0(X \setminus K)$ defined by

$$\Phi(f) = f|_{X\setminus K} \qquad (f \in C_0Z(X, K))$$

is a linear isometry from $(C_0Z(X, K), \|\cdot\|_X)$ onto $\big(C_0(X \setminus K), \|\cdot\|_{X\setminus K}\big)$. Hence, $\Phi^{-1} : C_0(X\setminus K) \longrightarrow C_0Z(X, K)$ is a linear isometry from $(C_0(X\setminus K), \|\cdot\|_{X\setminus K})$ onto $(C_0Z(X, K), \|\cdot\|_X)$. Therefore, $(C_0Z(X, K), \|\cdot\|_X)$ does not have the fixed point property by Theorem 1.1. □

Remark 2.4. Let X be a locally compact Hausdorff space and K be a compact subset of X such that $X \setminus K$ is an infinite set. By part (iii) and part (vii) of Theorem 2.2, we can show that the commutative complex Banach algebra $(C_0Z(X, K), \|\cdot\|_X)$ satisfies in all conditions of Theorem 1.3, and so does not have the fixed point property.

3. Fixed point property of $\mathsf{C}^\star$–algebras

Applying Gelfand–Naimark theorem [5, Theorem 2.4.5], Theorem 1.1 and Theorem 2.1, we study the fixed point property of commutative complex $\mathsf{C}^\star$–algebras.

Theorem 3.1. *Let $(A, \|\cdot\|)$ be a commutative complex $\mathsf{C}^\star$–algebra with $\Delta(A) \neq \emptyset$. Then the following statements are equivalent:*

(i) $\Delta(A)$ is infinite set.

(ii) A is infinite dimensional.

(iii) $(A, \|\cdot\|)$ does not have the fixed point property.

Proof . By Gelfand–Naimark theorem, the Gelfand homomorphism $x \mapsto \hat{x} : A \longrightarrow C_0(\Delta(A))$ is an isometric $\star$–isomorphism from $(A, \|\cdot\|)$ onto $\big(C_0(\Delta(A)), \|\cdot\|_{\Delta(A)}\big)$. Hence, $(A, \|\cdot\|)$ does not have the fixed point property if and only if $\big(C_0(\Delta(A)), \|\cdot\|_{\Delta(A)}\big)$ does not have the fixed point property by Theorem 1.1. Therefore, the proof is complete by Theorem 2.1. □

Corollary 3.2. *Let X be a locally compact Hausdorff space such that X is an infinite set. If A is an infinite dimensional self–adjoint uniformly closed complex subalgebra of $C_0(X)$, then $(A, \|\cdot\|_X)$ does not have the fixed point property.*

Proof . By hypotheses, $(A, \|\cdot\|_X)$ is a commutative complex $\mathsf{C}^\star$–algebra under the natural involution $f \mapsto \bar{f} : A \longrightarrow A$. Since A is infinite dimensional, $(A, \|\cdot\|_X)$ does not have the fixed point property by Theorem 3.1. □

Example 3.3. Let $m \in \mathbb{N}$ and define the function $g_m : \mathbb{C} \longrightarrow \mathbb{C}$ by

$$g_m(z) = \exp(-m|z|).$$

Let A_m be the complex subalgebra of $C_0(\mathbb{C})$ generated by g_m and B_m be the uniform closure of A_m in $(C_0(\mathbb{C}), \|\cdot\|_{\mathbb{C}})$. Then B_m is a uniformly closed self–adjoint complex subalgebra of $C_0(\mathbb{C})$. Since for each $n \in \mathbb{N}$ the set $\big\{(g_m)^k : k \in \{1, \ldots, n\}\big\}$ is a linearly independent set in B_m, we deduce that B_m is infinite dimensional. Therefore, $(B_m, \|\cdot\|_{\mathbb{C}})$ does not have the fixed point property, by Corollary 3.2.

Remark 3.4. Let X be a locally compact Hausdorff space and K be a compact subset of X such that $X \setminus K$ is an infinite set. Then $C_0Z(X, K)$ satisfies in conditions of Corollary 3.2. Therefore, $(C_0Z(X, K), \|\cdot\|_X)$ does not have the fixed point property.

The following result was given by Ogasawara in [7].

Theorem 3.5. (See [7, Theorem 1]) Let $(A, \|\cdot\|)$ be an infinite dimensional complex $\mathsf{C}^\star$–algebra with the algebra involution $\star$. Then there exists a commutative infinite dimensional complex subalgebra B of A such that $x^\star \in B$ for each $x \in B$ and $(B, \|\cdot\|)$ is a complex $\mathsf{C}^\star$–algebra with the algebra involution $\star$.

Applying Ogasawara's theorem (Theorem 3.5), Theorem 3.1, and Corollary 1.2 we obtain the following result.

Theorem 3.6. *Let $(A, \|\cdot\|)$ be a complex $\mathsf{C}^\star$–algebra with the algebra involution $\star$. If A is infinite dimensional, then $(A, \|\cdot\|)$ does not have the fixed point property.*

Proof . Let A is infinite dimensional. By Theorem 3.5, there exists a commutative infinite dimensional complex subalgebra B of A such that $x^\star \in B$ for each $x \in B$ and $(B, \|\cdot\|)$ is a complex $\mathsf{C}^\star$–algebra with the algebra involution $\star$. Therefore, $(B, \|\cdot\|)$ does not have the fixed point property by Theorem 3.1, and so $(A, \|\cdot\|)$ does not have the fixed point property by Corollary 1.2. $\square$

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