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On the shadowing property of nonautonomous discrete systems

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Abstract

In this paper we study shadowing property for sequences of mappings on compact metric spaces, i.e. nonautonomous discrete dynamical systems. We investigate the relation of weak contractions with shadowing and h-shadowing property.

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1. Introduction and preliminaries

Let (X, d) be a compact metric space, and f be a continuous map on X. We consider the associated autonomous difference equation of the following form:

$$x_{i+1} = f(x_i) \tag{1.1}$$

A finite or infinite sequence $\{x_0, x_1, \dots\}$ of points in X is called a δ-pseudo-orbit ($\delta > 0$) of (1.1) if $d(f(x_{i-1}), x_i) < \delta$ for all $i \geq 1$. We say that equation (1.1) , (or f) has usual shadowing property if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, \dots\}$, there exists $y \in X$ with $d(f^i(y), x_i) < \varepsilon$ for all $i \geq 0$. The notion of pseudo-orbits appeared in several branches of dynamical systems theory, and various types of the shadowing property were presented and investigated extensively, see [1, 4, 10, 11].

In this paper we study shadowing property of nonautonomous discrete systems. We consider the compact metric space X and a sequence $f_{1,\infty} = \{f_i\}_{i=1}^{\infty}$ in which each $f_i : X \to X$ is continuous. We

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call the pair $(X, f_{1,\infty})$ a nonautonomous discrete system (on X). For further simplicity we use only $f_{1,\infty}$ in the sequel. The associated nonautonomous difference equation has the following form:

$$x_{i+1} = f_i(x_i) \tag{1.2}$$

For every $n \geq 1$, we write $f_i^n = f_n \circ f_{n-1} \circ ... \circ f_i$. Orbit of a nonautonomous system $f_{1,\infty}$ in a point x is the following sequence:

$$O(x) = \{x, f_1(x), f_2 \circ f_1(x), \ldots, f_n \circ \ldots \circ f_1(x), \ldots\}$$

On the other hand a pseudo-orbit of the system is as follows:

Definition 1.1. A finite or infinite sequence $\{x_0, x_1, \ldots\}$ of points in X is called a δ-pseudo-orbit ($\delta > 0$) of (1.2), if $d(f_i(x_{i-1}), x_i) < \delta$ for all $i \geq 1$.

In the nonautonomous case the standard definition of shadowing has the following form, see [10].

Definition 1.2. We say that $f_{1,\infty}$ has shadowing property if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, \ldots\}$, there exists $y \in X$ with $d(y, x_0) < \varepsilon$ and $d(f_1^i(y), x_i) < \varepsilon$, for all $i \geq 1$.

In this work we study various shadowing properties of sequences of mappings and their relations with contractions and weak contractions. At the end of the paper we give an example for further illustration.

2. Shadowing and h-shadowing

First we prove the following simple lemma.

Lemma 2.1. *The sequence $f_{1,\infty}$ has shadowing property if and only if for every $\varepsilon > 0$ there exists $\delta > 0$ such that every finite δ-pseudo-orbit is ε-shadowed.*

Proof . Let $\varepsilon > 0$ and $\delta > 0$ be such that every finite δ-pseudo-orbit, $\frac{\varepsilon}{2}$-shadowed. Let $\{x_i\}_{i=1}^{\infty}$ be a δ-pseudo-orbit. For every $n \geq 1$, $\{x_0, x_1, \ldots, x_n\}$, $\frac{\varepsilon}{2}$-shadowed by $y_n \in X$ and there is a subsequence $\{y_{n_k}\}_{k \geq 0}$ and a point $y \in X$ such that $y_{n_k} \to y$ as $k \to \infty$. Now for each $i \geq 1$, there is a $n_k > i$ such that $d(f_1^i(y_{n_k}), f_1^i(y)) < \frac{\varepsilon}{2}$. Therefore

$$d(f_1^i(y), x_i) \leq d(f_1^i(y), f_1^i(y_{n_k})) + d(f_1^i(y_{n_k}), x_i) < \varepsilon$$

and hence $f_{1,\infty}$ has the shadowing property. □

There are several variants of shadowing property, we define a stronger form which is called h-shadowing, see [2, 8, 9].

Definition 2.2. The sequence $f_{1,\infty}$ has h-shadowing property if for every $\varepsilon > 0$ there exists $\delta > 0$ such that for every δ-pseudo-orbit $\{x_0, x_1, \ldots, x_n\} \subseteq X$ there is $y \in X$ with $d(y, x_0) < \varepsilon$ and,

$$d(f_1^i(y), x_i) < \varepsilon \ \textit{for all} \ 1 \leq i < n \qquad \textit{and} \qquad f_1^n(y) = x_n.$$

Definition 2.3. The sequence $f_{1,\infty}$ is called strongly equicontinuous if for each $x \in X$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies, $d(f_m^n(x), f_m^n(y)) < \varepsilon$, for all $n \geq m \geq 1$.

We study some relations between shadowing and h-shadowing.

Theorem 2.4. *Suppose that for each $i \in \mathbb{N}$, $f_i : X \to X$ is a homeomorphism, then the following conditions are equivalent:*

(1) The sequence $f_{1,\infty}$ has h-shadowing property.

(2) The sequence $f_{1,\infty}$ has shadowing property and the sequence $\{f_i^{-1}\}_{i=1}^{\infty}$ is strongly equicontinuous.

Proof . At first we prove (1) $\Rightarrow$ (2). By Lemma 2.1 it is trivial that $f_{1,\infty}$ has shadowing property. It is enough to prove that $\{f_i^{-1}\}_{i=1}^{\infty}$ is strongly equicontinuous. Let $\varepsilon > 0$ and $\delta < \varepsilon$ be provided by h-shadowing. Fix $x, y \in X$ and suppose that $d(x, y) < \delta$. For $n > 1$ the sequence:

$$\{f_1^{-1} o \ldots o f_n^{-1}(x), f_2^{-1} o \ldots o f_n^{-1}(x), \ldots, f_{n-1}^{-1} o f_n^{-1}(x), f_n^{-1}(x), y\}$$

is a δ-pseudo-orbit, so by h-shadowing of $f_{1,\infty}$ there is $z \in X$ such that for any $n > m > 1$:

$$d(f_1^{m-1}(z), f_m^{-1} o f_{m+1}^{-1} o \ldots o f_n^{-1}(x) < \varepsilon, \qquad f_1^n(z) = y$$

hence

$$d((f_m^n)^{-1}(x), (f_m^n)^{-1}(y)) = d(f_1^{m-1}(z), f_m^{-1} o f_{m+1}^{-1} o \ldots o f_n^{-1}(x)) < \varepsilon$$

and therefore $f_{1,\infty}$ is strongly equicontinuous.

To prove (2) $\Rightarrow$ (1), let $\varepsilon > 0$. There exists $0 < \eta < \frac{\varepsilon}{2}$ as in the definition of strong equicontinuity for $\frac{\varepsilon}{2}$. Let $0 < \delta < \eta$ be such that every δ-pseudo-orbit is η-shadowed. Suppose that $\{x_0, x_1, \ldots, x_n\}$ is a δ-pseudo-orbit. We set $y = f_1^{-1} o f_2^{-1} o \ldots o f_n^{-1}(x_n)$ which implies $f_1^n(y) = x_n$ and for each $1 \le i < n$:

$$d(f_1^i(y), x_i) \le d(f_1^i(y), f_1^i(z)) + d(f_1^i(z), x_i) = d(f_{i+1}^{-1} o f_{i+2}^{-1} o \ldots o f_n^{-1}(x_n), f_1^i(z)) + d(f_1^i(z), x_i)$$

Since $f_{1,\infty}$ has shadowing property, there is $z \in X$ such that for each $1 \le i \le n$:

$$d(f_1^i(z), x_i) < \eta < \frac{\varepsilon}{2}$$

hence we have

$$d(f_{i+1}^{-1} o f_{i+2}^{-1} o \ldots o f_n^{-1}(x_n), f_1^i(z)) =$$

$$d(f_{i+1}^{-1} o f_{i+2}^{-1} o \ldots o f_n^{-1}(x_n), f_{i+1}^{-1} o f_{i+2}^{-1} o \ldots o f_n^{-1} o f_n o \ldots o f_1(z)) < \frac{\varepsilon}{2}.$$

Therefore for each $1 \le i < n$, $d(f_1^i(y), x_i) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ Hence $f_{1,\infty}$ has h-shadowing property. $\square$

Now we prove the following useful and technical result, which is previously proved in the case of Lipschitz mappings, by [6]. We prove this result under the assumption of equicontinuity.

Theorem 2.5. *Suppose that the sequence $f_{1,\infty}$ is equicontinuous, then the following conditions are equivalent:*

(1) The sequence $f_{1,\infty}$ has shadowing property.

(2) The sequence $\{f_{in+1}^{in+n}\}_{i=0}^{\infty}$ has shadowing property for all $n \ge 1$.

(3) The sequence $\{f_{in+1}^{in+n}\}_{i=0}^{\infty}$ has shadowing property for some $n \ge 1$.

Proof . First we prove (1) $\Rightarrow$ (2). By the shadowing of $f_{1,\infty}$, for $\varepsilon > 0$ there exists $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed by a point in X. Let $\{x_0, x_1, x_2, \ldots, x_m\}$ be a δ-pseudo-orbit for $\{f_{in+1}^{in+n}\}_{i=0}^{\infty}$, then the following sequence:

$$\{x_0, f_1(x_0), f_1^2(x_0), \ldots, f_1^{n-1}(x_0), x_1, f_{n+1}(x_1), f_{n+1}^{n+2}(x_1), \ldots, f_{n+1}^{2n-1}(x_1), x_2,$$

$$\ldots, x_{m-1}, f_{(m-1)n+1}(x_{m-1}), \ldots, f_{(m-1)n+1}^{mn-1}(x_{m-1}), x_m\}$$

is a δ-pseudo-orbit for $f_{1,\infty}$. Hence there exist $y \in X$ such that for all $0 \leq i \leq m-1$, $d(f_1^{in+n}(y), x_{i+1}) < \varepsilon$. Therefore $\{f_{in+1}^{in+n}\}_{i=0}^{\infty}$ has shadowing property for all $n \geq 1$.

The proof of (2) $\Rightarrow$ (3) is trivial. To prove (3) $\Rightarrow$ (1), let $\varepsilon > 0$. Since $f_{1,\infty}$ is equicontinuous and X is compact, there exists $\eta > 0$ such that $d(x, y) < \eta$ implies $d(f_k^{k+i}(x), f_k^{k+i}(y)) < \frac{\varepsilon}{2}$ for every $k \geq 1$ and $0 \leq i \leq n$.

By the shadowing of $\{f_{in+1}^{in+n}\}_{i=0}^{\infty}$ there exists $0 < \delta < \frac{\varepsilon}{2}$ such that each δ-pseudo-orbit is η-shadowed by a point in X. Since $f_{1,\infty}$ is equicontinuous and X is compact, there exists $0 < \gamma < \frac{\delta}{n}$ such that $d(x, y) < \gamma$ implies $d(f_k^{k+i}(x), f_k^{k+i}(y)) < \frac{\delta}{n}$ for every $k \geq 1$ and $0 \leq i \leq n$. Let $\{x_0, x_1, x_2, \ldots, x_m\}$ be a γ-pseudo-orbit for $f_{1,\infty}$. We have $m = sn + r$ such that $s, r \in N, 0 \leq r < n$. We claim that $\{x_0, x_n, \ldots, x_{sn}\}$ is a δ-pseudo-orbit for $\{f_{in+1}^{in+n}\}_{i=0}^{\infty}$. Indeed,

$$d(f_1^n(x_0), x_n) \leq d(x_n, f_n(x_{n-1})) + d(f_n(x_{n-1}), f_{n-1}^n(x_{n-2})) +$$

$$\cdots + d(f_2^n(x_1), f_1^n(x_0)) < \gamma + \frac{\delta}{n} + \cdots + \frac{\delta}{n} \leq \delta.$$

Similarly for $1 \leq i \leq s$, we have $d(f_{(i-1)n+1}^{in}(x_{(i-1)n+1}), x_{in}) < \delta$. By the shadowing of $\{f_{in+1}^{in+n}\}_{i=0}^{\infty}$ there exists $y \in X$ such that for $0 \leq i \leq s$, $d(f_1^{in}(y), x_{in}) < \eta$. For every $0 \leq j \leq n-1$ we have:

$$d(f_1^{in+j}(y), f_{in+1}^{in+j}(x_{in})) < \frac{\varepsilon}{2}$$

and

$$d(f_{in+1}^{in+j}(x_{in}), x_{in+j}) \leq d(x_{in+j}, f_{in+j}(x_{in+j-1})) +$$

$$d(f_{in+j}(x_{in+j-1}), f_{in+j-1}^{in+j}(x_{in+j-2})) + \cdots + d(f_{in+2}^{in+j}(x_{in+1}), f_{in+1}^{in+j}(x_{in}))$$

$$< \gamma + \frac{\delta}{n} + \cdots + \frac{\delta}{n} \leq \delta < \frac{\varepsilon}{2}.$$

We conclude $d(f_1^{in+j}(y), x_{in+j}) < \varepsilon$. Hence the point y, ε-shadows $\{x_0, x_1, \ldots, x_m\}$. $\square$

Here we prove the preservation of h-shadowing property under topological equivalence.

Theorem 2.6. *Suppose that (X, d_1) and (Y, d_2) are compact metric spaces, $g_i : X \to Y$ are homeomorphisms, for all $i \in \mathbb{N}$, and both $\{g_i\}_{i=1}^{\infty}$, $\{g_i^{-1}\}_{i=1}^{\infty}$ are equicontinuous. Then $f_{1,\infty}$ has h-shadowing property if and only if $\{g_{i+1} o f_i o g_i^{-1}\}_{i=1}^{\infty}$ has h-shadowing property.*

Proof . Suppose that $f_{1,\infty}$ has h-shadowing property and let $\varepsilon > 0$. Since $\{g_i\}_{i=1}^{\infty}$ is equicontinuous and X is compact, there exists $\eta > 0$ such that $d_1(x, y) < \eta$ implies $d_2(g_i(x), g_i(y)) < \varepsilon$, for all $i \in \mathbb{N}$. By the h-shadowing of $f_{1,\infty}$, there exists $\gamma > 0$ such that each γ-pseudo-orbit is η-shadowed by a point in X. Let $\delta > 0$ be such that $d_2(x, y) < \delta$ implies $d_1(g_i^{-1}(x), g_i^{-1}(y)) < \gamma$, for all $i \in \mathbb{N}$. Suppose that $\{y_0, y_1, y_2, \ldots, y_m\}$ is a δ-pseudo-orbit for $\{g_{i+1} o f_i o g_i^{-1}\}_{i=1}^{\infty}$, then for all $1 \leq i \leq m$, $d_2(g_{i+1} o f_i o g_i^{-1}(y_{i-1}), x_i) < \delta$. Let $x_i = g_{i+1}^{-1}(y_i)$ for $1 \leq i \leq m$, we have $d_2(g_{i+1} o f_i(x_{i-1}), g_{i+1}(x_i)) < \delta$, which implies $d_1(f_i(x_{i-1}), x_i) < \gamma$. So $\{x_0, x_1, x_2, \dots, x_m\}$ is a γ-pseudo-orbit for $f_{1,\infty}$, and there is $z \in X$ such that:

$$d_1(f_1^i(z), x_i) < \eta \ for\ all\ 1 \le i \le m-1 \qquad and \qquad f_1^m(z) = x_m$$

Hence $d_1(f_i o g_i^{-1} o ... o g_3 o f_2 o g_2^{-1} o g_2 o f_1 g_1^{-1}(g_1(z)), g_{i+1}^{-1}(y_i)) < \eta$ which implies that:

$$d_2(g_{i+1} o f_i o g_i^{-1} o \cdots o g_3 o f_2 o g_2^{-1} o g_2 o f_1 o g_1^{-1}(g_1(z)), y_i) < \varepsilon \ for\ all\ 1 \le i \le m-1$$

and

$$f_m o g_m^{-1} o \cdots o g_2 o f_1 o g_1^{-1}(g_1(z)) = g_{m+1}^{-1}(y_m)$$

which implies that $g_{m+1} o f_m o g_m^{-1} o ... o g_2 o f_1 o g_1^{-1}(g_1(z)) = y_m$. □

3. Shadowing and contractions

Now we investigate the relation of contractions and weak contractions with the shadowing property for the sequences of mappings, the autonomous case studied in [3].

Proposition 3.1. *Suppose that X is metric space, and $f_i : X \to X$ is a contraction with constant $L \in (0, 1)$, for all $i \ge 1$, then the sequence $f_{1,\infty}$ has shadowing property.*

Proof . Let $\varepsilon > 0$, $\delta = (1 - L)\varepsilon$ and $B_n = B(x_n, \varepsilon)$ for $n \ge 1$. Suppose that $z \in B_n$, then

$$d(f_n(z), x_n) \le d(f_n(z), f_n(x_{n-1}) + d(f_n(x_{n-1}), x_n) < d(z, x_n) + \delta < L\varepsilon + \delta = \varepsilon \tag{3.1}$$

Hence for each $n \ge 1$, $f_n(B_{n-1}) \subseteq B_n$ and therefore:

$$f_1^n(B_0) \subseteq B_n \ for\ all\ n \ge 1$$

thus for every $y \in B_0$,

$$d(f_1^n(y), x_n) < \varepsilon \ for\ all\ n \ge 1$$

which proves shadowing property. □

Theorem 3.2. *Suppose that $f_i : X \to X$ are weak contractions, for all $i \ge 1$, and $f_n \to f$ point wise, in which f is also a weak contraction. Then $f_{1,\infty}$ has shadowing property.*

Proof . Let $\varepsilon > 0$ and denote

$$\eta(\varepsilon) := \sup\{d(f_i(x), f_i(y)) \quad : \quad 0 < d(x, y) < \varepsilon \quad , \quad i \ge 1\}.$$

It is easy to see that $\eta(\varepsilon) \le \varepsilon$. We claim that $\eta(\varepsilon) < \varepsilon$. Indeed, if $\eta(\varepsilon) = \varepsilon$, then there exist sequences $\{d(x_i, y_i)\}_{i=1}^{\infty}$ and $\{k(i)\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that for all $i \ge 1$, $0 \le d(x_i, y_i) < \varepsilon$ and we have:

$$\lim_{i \longrightarrow \infty} d(f_{k(i)}(x_i), f_{k(i)}(y_i)) = \eta(\varepsilon) = \varepsilon.$$

Since X is compact there is a subsequence $\{n_i\}_{i=1}^{\infty} \subseteq \mathbb{N}$ such that $x_{n_i} \to x_0$ and $y_{n_i} \to y_0$ and therefore:

$$\varepsilon = \lim_{i \longrightarrow \infty} d(f_{k(i)}(x_i), f_{k(i)}(y_i)) = \eta(\varepsilon) = \varepsilon = d(f(x_0), f(y_0)) < d(x_0, y_0) =$$

$$\lim_{i \longrightarrow \infty} d(x_{n_i}, y_{n_i}) \le \varepsilon$$

which is impossible. We put $\delta = \varepsilon - \eta(\varepsilon)$. Let $\{x_i\}_{i=1}^{\infty}$ be a δ-pseudo orbit and $B_n = B(x_n, \varepsilon)$. If $z \in B_{n-1}$ then as in the proof of the previous theorem inequality It follows 3.1 holds. Hence for each $n \ge 1$, $f_n(B_{n-1}) \subseteq B_n$ which implies, $d(f_1^n(y), x_n) < \varepsilon$ for every $y \in B_0$ and $n \ge 1$. Thus $f_{1,\infty}$ has shadowing property. □

Definition 3.3. The sequence $\{x_i\}_{i=1}^{\infty}$ is called an asymptotic pseudo-orbit if $d(f_i(x_{i-1}), x_i) \to 0$ as $i \to \infty$. Further more $\{x_i\}_{i=1}^{\infty}$ is called an asymptotic δ-pseudo-orbit if it is both an asymptotic pseudo-orbit and a δ-pseudo-orbit.

Definition 3.4. We say that the sequence $f_{1,\infty}$ has limit shadowing property if every asymptotic pseudo-orbit $\{x_i\}_{i=1}^{\infty}$ becomes asymptotic shadowed, by a point $y \in X$.

Theorem 3.5. *Suppose that $f_i : X \to X$ are surjective maps for all $i \in \mathbb{N}$, then $f_{1,\infty}$ has limit shadowing if,*

(1) f_is are contractions with the constant $L \in (0,1)$ for all $i \geq 1$.

(2) f_is are weak contractions, for all $i \geq 1$, and $f_n \to f$ point wise, in which f is weak contraction.

Proof . Assume that (1) holds. Let $\{x_n\}_{n=1}^{\infty}$ be an asymptotic pseudo orbit, since f_n is surjective, there is a sequence $\{y_n\}_{n=1}^{\infty} \subseteq X$ such that for every $n \geq 1$, $f_1^n(y_n) = x_n$. Let y be a limit point of $\{y_n\}_{n=1}^{\infty}$ and $\varepsilon > 0$. Put $\delta = (1-L)\varepsilon$, then there is $N \geq 0$ such that for each $n \geq N$ we have:

$$d(f_n(x_{n-1}), x_n) < \delta \quad and \quad d(f_1^N(y), f_1^N(y_N)) < \delta$$

and hence the sequence

$$y, f_1(y), f_1^2(y), \ldots, f_1^{N-1}(y), x_N, x_{N+1}, \ldots$$

is a δ-pseudo orbit. We denote the above sequence with $z_0, z_1, \ldots, z_{N-1}, z_N, \ldots$. Let $B_n = B(z_n, \varepsilon)$. We see that, for every $n \geq 1$, $f_n(B_{n-1}) \subseteq B_n$ which implies that:

$$d(f_1^n(y), x_n) < \varepsilon \text{ } for \text{ } all \text{ } n \geq N.$$

So we have $\lim_{n\longrightarrow\infty} d(f_1^n(y), x_n) = 0$, hence $f_{1,\infty}$ has limit shadowing property.
(2): For $\epsilon > 0$, let,

$$\eta(\varepsilon) := \sup\{d(f_i(x), f_i(y)) \quad : \quad 0 \leq d(x,y) < \varepsilon \quad , \quad i \geq 1\}.$$

As the proof of Theorem 3.2 , we have $\eta(\varepsilon) < \epsilon$. We put $\delta = \varepsilon - \eta(\varepsilon)$. Now as in the proof of the previous part, we conclude that $f_{1,\infty}$ has limit shadowing property. □

Example 3.6. Consider the sequence of mappings $f_n : [0,\infty) \to [0,\infty)$ defined by:

$$f_n(x) = \frac{n}{n+1}x.$$

For each $n \in \mathbb{N}$, f_n is a contraction mapping, therefore by [3], f_n has shadowing property in autonomous sense. We prove that this sequence dose not have shadowing property in nonautonomous sense. Suppose that $\{f_n\}$ has shadowing property, for $\varepsilon > 0$ there exists an appropriate $\delta > 0$. Now we consider the sequence $\{x_n\}_{n=0}^{\infty}$ defined by,

$$x_0 \in [0,\infty), x_{n+1} = f_{n+1}(x_n) + \frac{\delta}{2}$$

which is a δ-pseudo orbit. Computation shows that,

$$x_n = \frac{1}{n+1}x_0 + \frac{2}{n+1}\frac{\delta}{2} + \cdots + \frac{n+1}{n+1}\frac{\delta}{2} = \frac{1}{n+1}x_0 + \frac{n^2+3n}{2(n+1)}\frac{\delta}{2}.$$

Let $\{x_n\}_{n=0}^{\infty}$ be shadowed by z, since $f_1^n(z) = \frac{1}{n+1}z$ we have:

$$\left| \frac{1}{n+1}z - \frac{1}{n+1}x_0 - \frac{n^2+3n}{2(n+1)}\frac{\delta}{2} \right| < \varepsilon, \forall n$$

which is impossible. Further more $f_n \to id_{[0,\infty)}$ pointwise on $[0,\infty)$ and it is easy to see that $id_{[0,\infty)}$ has not shadowing property.

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