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Common fixed point theorem for nonexpansive type single valued mappings

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Abstract

The aim of this paper is to prove a common fixed point theorem for nonexpansive type single valued mappings which include both continuous and discontinuous mappings by relaxing the condition of continuity by weak reciprocally continuous mapping. Our result is generalize and extends the corresponding result of Jhade et al. [P.K. Jhade, A.S. Saluja and R. Kushwah, Coincidence and fixed points of nonexpansive type multivalued and single valued maps, European J. Pure Appl. Math., 4 (2011) 330-339].

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1. Introduction and preliminaries

Fixed point theory plays a basic role in applications of many branches of mathematics. The term metric fixed point theory refers to those fixed point theoretic results in which geometric conditions on the underlying spaces and/or mappings play a crucial role. For the past twenty five years metric fixed point theory has been a flourishing area of research for many mathematicians. Although a substantial number of definitive results now has been discussed, a few questions lying at the heart of the theory remain open and there are many unanswered questions regarding the limit to which theory may extended.

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In 1922, a Polish mathematician Banach [2] proved a very important result regarding contraction mapping, known as the famous Banach contraction principle. This theorem provides a technique for solving a variety of applied problems in mathematical science and engineering. Then after many authors generalizes and extends the Banach contraction principle in different ways.

In 1982, Sessa [12] introduced the notion of weak commutativity condition for a pair of single valued maps. Later, Jungck [7] generalized the concept of weak commutativity by introducing the notion of compatibility of maps. Pant [10] introduced point wise R-weakly commutativity of maps for noncompatible maps. Recently, Al-Thagafi and Shahzad [1] introduced the notion of occasionally weakly compatible mappings and proved some fixed point theorems using this new type of mapping. Here important to mention that weak commutativity implies compatibility but the converse is not true. Weak commutativity implies R-weak commutativity but R-weak commutativity implies weak commutativity only when $R \leq 1$.

Two self mappings f and g of a metric space (X, d) are called R-weakly commuting of type -(A_g)[8], if there exists some positive real number R such that $d(ffx, gfx) \leq Rd(fx, gx)$ for all $x \in X$. Similarly, two self mappings f and g of a metric space (X, d) are called R-weakly commuting of type-(A_f)[8], if there exists some positive real number R such that $d(fgx, ggx) \leq Rd(fx, gx)$ for all $x \in X$. It seems important to note that compatible and non-compatible mappings can be R-weakly commuting of type-(A_g) or type-(A_f).

In 1998, Pant [11] introduced the concept of reciprocal continuity (r.c) for the pair of single valued maps as follows:

Definition 1.1. [11] Two self maps f and g of a metric space (X, d) are called reciprocal continuous if and only if $\lim_{n\to\infty} gfx_n = gt$ and $\lim_{n\to\infty} fgx_n = ft$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Note that a pair of mappings which is reciprocal continuous need not be continuous even on their common fixed point (see for example [11]).

Recently, Pant et al. [9] generalized reciprocal continuity by introducing the notion of weakly reciprocal continuity (w.r.c) for a pair of single valued maps as follows:

Definition 1.2. [9] Two self maps f and g of a metric space (X, d) are called weakly reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ or $\lim_{n\to\infty} gfx_n = gt$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

It seems important to note that reciprocal continuity implies weak reciprocal continuity but the converse is not true as shown below.

Example 1.3. [9] Let $X = [2, 20]$ and d be a usual metric in X. Define $f, g : X \to X$ as follows:

$$
\begin{aligned}
&fx = 2 \quad \text{if} \quad x = 2 \text{ or } x > 5;\\
&fx = 6 \quad \text{if} \quad 2 < x \leq 5;\\
&gx = 12 \text{ if } 2 < x \leq 5;\\
&g2 = 2,\ gx = 12 \text{ if } 2 < x \leq 5;\\
&gx = (x+1)/3 \text{ if } x > 5.
\end{aligned}
$$

Then clearly f and g are weakly reciprocally continuous but not reciprocally continuous. Some common fixed point theorems for the w.r.c. pairs of maps was also obtained by Pant et al. [9].

It seems to be noted that only w.r.c. does not guarantee the existence of common fixed point or even coincidence point. The following example illustrated this fact.

Example 1.4. [5] Let $X = [0, \infty)$ and d be a usual metric in X. Define

$$T(x) = \begin{cases} 0 \text{ if } x \leq 2; \\ \frac{3}{2} \text{ if } 2 < x \leq 3; \\ 3 \text{ if } 3 < x \leq 4; \\ x \text{ if } x > 4 \end{cases}$$

$$f(x) = \begin{cases} \frac{3}{2} \text{ if } x \leq 2; \\ x+1 \text{ if } 2 < x < 3; \\ x \text{ if } 3 \leq x < 4; \\ x+6 \text{ if } x \geq 4 \end{cases}$$

If we take $\{x_n\} = (3 + \frac{1}{n})$.

$$\lim_{n\to\infty} T(3 + \tfrac{1}{n}) = 3$$

$$\lim_{n\to\infty} f(3 + \tfrac{1}{n}) = \lim_{n\to\infty}(3 + \tfrac{1}{n}) = 3$$

$$\lim_{n\to\infty} fT(3 + \tfrac{1}{n}) = \lim_{n\to\infty} f\{3\} = f(3)$$

$$\lim_{n\to\infty} Tf(3 + \tfrac{1}{n}) = \lim_{n\to\infty} T(3 + \tfrac{1}{n}) = 3 \neq T(3)$$

Since $\lim_{n\to\infty} fTx_n = ft$ but $\lim_{n\to\infty} Tfx_n \neq Tt$, the pair (T, f) is not reciprocally continuous but w.r.c and compatible. Also, T and f do not have any coincidence point.

A map $T : X \to X$ is said to be nonexpansive if $d(Tx, Ty) \leq d(x, y)$ for all$x, y \in X$. Ciric [3] investigated a class of nonexpansive type self maps T of X and established some fixed point theorems for such type of mappings.

Recently, Jhade et al. [6] gave the following nonexpansive type condition. Let $T, f : X \to X$ and

$$\begin{aligned} d(Tx, Ty) \leq {} & a(x, y)d(fx, fy) + b(x, y)\max\{d(fx, Tx), d(fy, Ty)\} \\ & + c(x, y)\max\{d(fx, fy), d(fx, Tx), d(fy, Ty)\} \\ & + e(x, y)\max\{d(fx, fy), d(fx, Tx), d(fy, Ty), d(fx, Ty)\} \end{aligned} \tag{1.1}$$

where $a(x, y), b(x, y), c(x, y), e(x, y) \geq 0$ and $\beta = \inf_{x,y\in X} e(x, y) > 0$, $\gamma = \inf_{x,y\in X}[1 + b(x, y) + e(x, y)] > 0$ with

$$\sup_{x,y\in X}\{a(x, y) + b(x, y) + c(x, y) + 2e(x, y)\} = 1.$$

Jhade et al. [6] proved that a compatible pair of maps on the complete metric space satisfying (1.1) will have a coincidence point if f is surjective or continuous.

In this paper, we extend the scope of the study of nonexpansive type condition to the class of mappings which include both continuous and discontinuous mappings by dropping the condition of continuity by w.r.c.

2. Main Result

Theorem 2.1. *Let T and f be two weakly reciprocally continuous self maps of a complete metric space (X, d) satisfying (1.1) with $T(X) \subseteq f(X)$, then T and f have a common fixed point in X if either*

(a) *T and f are compatible or*

(b) *T and f are R-weakly commuting of type-(A_f) or*

(c) *T and f are R-weakly commuting of type-(A_T).*

Proof . We prove the result in three cases.

Case (a): Let T and f are compatible. Choose $x_0 \in X$. We construct a sequence $\{x_n\}$ in X such that$fx_1 = Tx_0$. In general, choose x_{n+1} such that$fx_{n+1} = Tx_n$. As proved in Theorem 2.1 in [6], we get $\{fx_n\}$ and $\{Tx_n\}$ are Cauchy sequences in X and completeness of the space implies $\lim_{n\to\infty} fx_{n+1} = \lim_{n\to\infty} Tx_n = t$ for some $t \in X$. Since f and T are weakly reciprocally continuous, hence either $\lim_{n\to\infty} fTx_n \to ft$ or $\lim_{n\to\infty} Tfx_n \to Tt$.

Let $\lim_{n\to\infty} fTx_n \to ft$. Now using the compatibility of f and T, we get $\lim_{n\to\infty} d(fTx_n, Tfx_n) = 0$. Letting $n \to \infty$, we get $\lim_{n\to\infty} Tfx_n \to ft$ and $Tfx_{n+1} = TTx_n \to ft$.

Now using (1.1),

$$\begin{aligned} d(Tt, TTx_n) \leq\ & a(x,y)d(ft, fTx_n) + b(x,y)\max\{d(ft,Tt), d(fTx_n, TTx_n)\} \\ & + c(x,y)\max\{d(ft, fTx_n), d(ft,Tt), d(fTx_n, TTx_n)\} \\ & + e(x,y)\max\{d(ft, fTx_n), d(ft,Tt), d(fTx_n, TTx_n), d(ft, TTx_n)\}. \end{aligned}$$

On letting $n \to \infty$, we get $d(Tt, ft) \leq \{b(x,y) + c(x,y) + e(x,y)\}d(Tt, ft)$. Since $\sup_{x,y\in X}\{a(x,y) + b(x,y) + c(x,y) + 2e(x,y)\} = 1$ and $a(x,y) > 0$; $\inf_{x,y\in X} e(x,y) > 0$ implies that $ft = Tt$. Compatibility of f and T implies commutativity at coincidence point, hence $fTt = Tft = fft = TTt$.

Again using (1.1),

$$\begin{aligned} d(Tt, TTt) \leq\ & a(x,y)d(ft, fTt) + b(x,y)\max\{d(ft,Tt), d(fTt, TTt)\} \\ & + c(x,y)\max\{d(ft, fTt), d(ft,Tt), d(fTt, TTt)\} \\ & + e(x,y)\max\{d(ft, fTt), d(ft,Tt), d(fTt, TTt), d(ft, TTt)\} \\ & = [a(x,y) + c(x,y) + e(x,y)]d(Tt, TTt). \end{aligned}$$

Since $\beta > 0$ implies that $\sup_{x,y\in X}[a(x,y) + c(x,y) + e(x,y)] < 1$. Hence $Tt = TTt = fTt$, i.e., Tt is a common fixed point of $f$and T.

Next, suppose that$\lim_{n\to\infty} Tfx_n \to Tt$. Since $T(X) \subseteq f(X)$ implies that $Tt = fz$ for some $z \in X$ and $\lim_{n\to\infty} Tfx_n \to fz$. Compatibility of f and T implies $\lim_{n\to\infty} fTx_n \to fz$. Since $Tfx_{n+1} = TTx_n$ and$Tfx_{n+1} \to fz$, it follows that$TTx_n \to fz$.

Now using (1.1),

$$\begin{aligned} d(Tz, TTx_n) \leq\ & a(x,y)d(fz, fTx_n) + b(x,y)\max\{d(fz,Tz), d(fTx_n, TTx_n)\} \\ & + c(x,y)\max\{d(fz, fTx_n), d(fz,Tz), d(fTx_n, TTx_n)\} \\ & + e(x,y)\max\{d(fz, fTx_n), d(fz,Tz), d(fTx_n, TTx_n), d(fz, TTx_n)\}. \end{aligned}$$

On letting $n \to \infty$, we get $d(fz, Tz) \leq [b(x,y) + c(x,y) + e(x,y)]d(fz, Tz)$ which implies that$fz = Tz$. Compatibility of f and T implies commutativity at coincidence point, hence$fTz = Tfz = TTz = ffz$.

Again from (1.1), we get

$$\begin{aligned} d(Tz,TTz) &\le a(x,y)d(fz,fTz) + b(x,y)\max\{d(fz,Tz),d(fTz,TTz)\} \\ &+ c(x,y)\max\{d(fz,fTz),d(fz,Tz),d(fTz,TTz)\} \\ &+ e(x,y)\max\{d(fz,fTz),d(fz,Tz),d(fTz,TTz),d(fz,TTz)\} \\ &\le [a(x,y)+c(x,y)+e(x,y)]d(Tz,TTz). \end{aligned}$$

This implies that $Tz = TTz = fTz$, i.e., Tz is a common fixed point of f and T.

Case (b): Now suppose that T and f are R-weakly commuting of type-(A_f). Since f and T are weakly reciprocally continuous, hence either $\lim_{n\to\infty} fTx_n \to ft$ or $\lim_{n\to\infty} Tfx_n \to Tt$. Then R-weakly commutativity of type-(A_f) of f and T gives $d(TTx_n, fTx_n) \le Rd(Tx_n, fx_n)$. Making $n \to \infty$, we get $TTx_n \to ft$ for some $t \in X$.

Now using (1.1), we get

$$\begin{aligned} d(Tt,TTx_n) &\le a(x,y)d(ft,fTx_n) + b(x,y)\max\{d(ft,Tt),d(fTx_n,TTx_n)\} \\ &+ c(x,y)\max\{d(ft,fTx_n),d(ft,Tt),d(fTx_n,TTx_n)\} \\ &+ e(x,y)\max\{d(ft,fTx_n),d(ft,Tt),d(fTx_n,TTx_n),d(ft,TTx_n)\}. \end{aligned}$$

On letting $n \to \infty$, we get $d(ft,Tt) \le [b(x,y)+c(x,y)+e(x,y)]d(ft,Tt)$, i.e., $Tt = ft$.

Again by R-weak commutativity of type-(A_f) , $d(TTt, fTt) \le Rd(ft, Tt)$. This gives $TTt = fTt$ or $TTt = Tft = fTt = fft$.

Using (1.1) again

$$\begin{aligned} d(Tt,TTt) &\le a(x,y)d(ft,fTt) + b(x,y)\max\{d(ft,Tt),d(fTt,TTt)\} \\ &+ c(x,y)\max\{d(ft,fTt),d(ft,Tt),d(fTt,TTt)\} \\ &+ e(x,y)\max\{d(ft,fTt),d(ft,Tt),d(fTt,TTt),d(ft,TTt)\} \\ &= [a(x,y)+c(x,y)+e(x,y)]d(Tt,TTt) \end{aligned}$$

which implies that $Tt = TTt = fTt$, i.e., Tt is a common fixed point of f and T.

Now suppose that $\lim_{n\to\infty} Tfx_n \to Tt$. Since $T(X) \subseteq f(X)$ implies that $Tt = fz$ for some $z \in X$ and $\lim_{n\to\infty} Tfx_n \to fz$. Since $Tfx_{n+1} = TTx_n$ and $Tfx_{n+1} \to fz$, it follows that $TTx_n \to fz$. Then R-weak commutativity of type-(A_f) of f and T gives $d(TTx_n, fTx_n) \le Rd(Tx_n, fx_n)$. On letting $n \to \infty$, we get $fTx_n \to fz$.

Now using (1.1), we have

$$\begin{aligned} d\,(Tz,TTx_n) &\le a\,(x,y)\,d\,(fz,fTx_n) + b\,(x,y)\max\{d\,(fz,Tz)\,,d\,(fTx_n,TTx_n)\} \\ &+ c\,(x,y)\max\{d\,(fz,fTx_n)\,,d\,(fz,Tz)\,,d\,(fTx_n,TTx_n)\} \\ &+ e\,(x,y)\max\{d\,(fz,fTx_n)\,,d\,(fz,Tz)\,,d\,(fTx_n,TTx_n)\,,d\,(fz,TTx_n)\}. \end{aligned}$$

On letting $n \to \infty$, we get $d\,(Tz,fz) \le [b\,(x,y)+c\,(x,y)+e\,(x,y)]\,d\,(Tz,fz)$ which implies that $fz = Tz$.

Again by R-weak commutativity of type-(A_f) implies that $d\,(TTz,fTz) \le Rd\,(fz,Tz)$. This gives $TTz = fTz$ or $TTz = Tfz = fTz = ffz$.

Again from (1.1),

$$\begin{aligned} d\,(Tz,TTz) &\le a\,(x,y)\,d\,(fz,fTz) + b\,(x,y)\max\{d\,(fz,Tz)\,,d\,(fTz,TTz)\} \\ &+ c\,(x,y)\max\{d\,(fz,fTz)\,,d\,(fz,Tz)\,,d\,(fTz,TTz)\} \\ &+ e\,(x,y)\max\{d\,(fz,fTz)\,,d\,(fz,Tz)\,,d\,(fTz,TTz)\,,d\,(fz,TTz)\} \\ &= [a\,(x,y)+c\,(x,y)+e\,(x,y)]\,d\,(Tz,TTz). \end{aligned}$$

This implies that $Tz = TTz = fTz$, i.e., Tz is a common fixed point of f and T.

Case (c): Let T and f are R-weakly commuting of type-(A_T). Since f and T are weakly reciprocally continuous, hence either $\lim_{n\to\infty} fTx_n \to ft$ or $\lim_{n\to\infty} Tfx_n \to Tt$. Then R-weakly commutativity of type-(A_T) of f and T gives $d(Tfx_n, ffx_n) \leq Rd(Tx_n, fx_n)$. Making $n \to \infty$, we get $Tfx_n \to ft$ for some $t \in X$.

Now using (1.1), we get

$$\begin{aligned} d(Tt, TTx_n) \leq\ & a(x,y)\, d(ft, fTx_n) + b(x,y) \max\{d(ft, Tt), d(fTx_n, TTx_n)\} \\ & + c(x,y) \max\{d(ft, fTx_n), d(ft, Tt), d(fTx_n, TTx_n)\} \\ & + e(x,y) \max\{d(ft, fTx_n), d(ft, Tt), d(fTx_n, TTx_n), d(ft, TTx_n)\}. \end{aligned}$$

On letting $n \to \infty$, we get $d(ft, Tt) \leq [b(x,y) + c(x,y) + e(x,y)]\, d(ft, Tt)$, i.e., $Tt = ft$.

Again by R-weak commutativity of type-(A_T) , $d(Tft, fft) \leq Rd(Tt, ft)$. This gives $Tft = fft$ or $TTt = Tft = fTt = fft$. Using (1.1) again

$$\begin{aligned} d(Tt, TTt) \leq\ & a(x,y)\, d(ft, fTt) + b(x,y) \max\{d(ft, Tt), d(fTt, TTt)\} \\ & + c(x,y) \max\{d(ft, fTt), d(ft, Tt), d(fTt, TTt)\} \\ & + e(x,y) \max\{d(ft, fTt), d(ft, Tt), d(fTt, TTt), d(ft, TTt)\} \\ =\ & [a(x,y) + c(x,y) + e(x,y)]\, d(Tt, TTt) \end{aligned}$$

which implies that $Tt = TTt = fTt$, i.e., Tt is a common fixed point of f and T.

Now suppose that $\lim_{n\to\infty} Tfx_n \to Tt$. Since $T(X) \subseteq f(X)$ implies that $Tt = fz$ for some $z \in X$ and $\lim_{n\to\infty} Tfx_n \to fz$. Since $Tfx_{n+1} = TTx_n$ and $Tfx_{n+1} \to fz$, it follows that $TTx_n \to fz$. Then R-weak commutativity of type-(A_T) of f and T gives $d(Tfx_n, ffx_n) \leq Rd(Tx_n, fx_n)$. On letting $n \to \infty$, we get $ffx_n \to fz$.

Now using (1.1), we have

$$\begin{aligned} d(Tz, TTx_n) \leq\ & a(x,y)\, d(fz, fTx_n) + b(x,y) \max\{d(fz, Tz), d(fTx_n, TTx_n)\} \\ & + c(x,y) \max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n)\} \\ & + e(x,y) \max\{d(fz, fTx_n), d(fz, Tz), d(fTx_n, TTx_n), d(fz, TTx_n)\}. \end{aligned}$$

On letting $n \to \infty$, we get $d(Tz, fz) \leq [b(x,y) + c(x,y) + e(x,y)]\, d(Tz, fz)$ which implies that $fz = Tz$.

Again by R-weak commutativity of type-(A_T) implies that $d(Tfz, ffz) \leq Rd(fz, Tz)$. This gives $Tfz = ffz$ or $TTz = Tfz = fTz = ffz$.

Again from (1.1)

$$\begin{aligned} d(Tz, TTz) \leq\ & a(x,y)\, d(fz, fTz) + b(x,y) \max\{d(fz, Tz), d(fTz, TTz)\} \\ & + c(x,y) \max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz)\} \\ & + e(x,y) \max\{d(fz, fTz), d(fz, Tz), d(fTz, TTz), d(fz, TTz)\} \\ =\ & [a(x,y) + c(x,y) + e(x,y)]\, d(Tz, TTz). \end{aligned}$$

This implies that $Tz = TTz = fTz$, i.e., Tz is a common fixed point of f and T. This completes the proof of the theorem. □

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