Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 63-75
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.295

Approximate a quadratic mapping in multi-Banach spaces, a fixed point approach

Sattar Alizadeh[a,*], Fridoun Moradlou[b]

[a]Department of Mathematics, Marand Branch, Islamic Azad University, Marand, Iran
[b]Department of Mathematics, Sahand University of Technology, Tabriz, Iran

(Communicated by Themistocles M. Rassias)

Abstract

Using the fixed point method, we prove the generalized Hyers–Ulam–Rassias stability of the following functional equation in multi-Banach spaces:

$$\sum_{j=1}^{n} f\Big(-2x_j + \sum_{i=1, i\neq j}^{n} x_i\Big) = (n-6)f\Big(\sum_{i=1}^{n} x_i\Big) + 9\sum_{i=1}^{n} f(x_i).$$

Keywords: Fixed point method; Hyers–Ulam–Rassias stability; Multi - Banach spaces; Quadratic mapping.
2010 MSC: Primary 39B82; Secondary 39B52, 46B99.

1. Introduction

A classical question in the theory of functional equations is the following: "When is it true that a function, which approximately satisfies a functional equation $\mathcal{E}$ must be close to an exact solution of $\mathcal{E}$?" If the problem accepts a solution, we say that the equation $\mathcal{E}$ is stable. Such a problem was formulated by Ulam [40] in 1940 and solved in the next year for the Cauchy functional equation by Hyers [17]. It gave rise the stability theory for functional equations. The result of Hyers was extended by Aoki [1] in 1950, by considering the unbounded Cauchy differences. In 1978, Th. M. Rassias [36] proved that the additive mapping T, obtained by Hyers or Aoki, is linear if, in addition, for each $x \in E$ the mapping $f(tx)$ is continous in $t \in \mathbb{R}$. Găvruta [16] generalized the Rassias'

*Corresponding author
Email addresses: salizadeh@marandiau.ac.ir (Sattar Alizadeh), moradlou@sut.ac.ir & fridoun.moradlou@gmail.com (Fridoun Moradlou)

Received: October 2014 *Revised:* February 2015

result. Following the techniques of the proof of the corollary of Hyers [17] we observed that Hyers introduced (in 1941) the following Hyers continuity condition: about the continuity of the mapping for each fixed, and then he proved homogenouity of degree one and therefore the famous linearity. This condition has been assumed further till now, through the complete Hyers direct method, in order to prove linearity for generalized Hyers–Ulam stability problem forms (see [19]). Beginning around the year 1980 The stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors and there are many interesting results concerning this problem (see [3], [6], [11]–[15], [20]–[31], [37], [38]).

J.M. Rassias [34] following the spirit of the innovative approach of Hyers [17], Aoki [1] and Th. M. Rassias [36] for the unbounded Cauchy difference proved a similar stability theorem in which he replaced the factor $\|x\|^p + \|y\|^p$ by $\|x\|^p \cdot \|y\|^q$ for $p, q \in \mathbb{R}$ with $p + q \neq 1$ (see also [33] for a number of other new results).

In 2003 Cădariu and Radu applied the fixed point method to the investigation of the Jensen functional equation [4] (see also [5], [6], [18], [32]). They could present a short and a simple proof (different of the "*direct method*", initiated by Hyers in 1941) for the generalized Hyers–Ulam stability of Jensen functional equation [4], for Cauchy functional equation [6] and for quadratic functional equation [5].

The following functional equation

$$Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y), \tag{1.1}$$

is called a *quadratic functional equation* and every solution of equation (1.1) is said to be a *quadratic mapping.* F. Skof [39] proved the Hyers–Ulam stability of the quadratic functional equation (1.1) for mappings $f : E_1 \to E_2$, where E_1 is a normed space and E_2 is a Banach space. In [7], S. Czerwik proved the Hyers–Ulam stability of the quadratic functional equation (1.1). C. Borelli and G. L. Forti [2] generalized the stability result of the quadratic functional equation (1.1).

Recently, Dales and Polyakov [8] introduced the notion of multi-normed spaces. This concept is somewhat similar to operator sequence spaces and has some connections with operator spaces and Banach lattices. Dales and Moslehian [9] investigated stability of Cauchy functional equation in multi-Banach spaces (see also [28], [35]).

In this paper, for a fixed positive integer $n \geq 2$, we introduce the following generalized quadratic functional equation:

$$\sum_{j=1}^{n} f\Big(-2x_j + \sum_{i=1, i\neq j}^{n} x_i\Big) = (n-6) f\Big(\sum_{i=1}^{n} x_i\Big) + 9\sum_{i=1}^{n} f(x_i). \tag{1.2}$$

Every solution of the functional equation (1.2) is said to be a *generalized quadratic mapping.*

We will adopt the idea of Cădariu and Radu [4], [6], [32], to prove the generalized Hyers–Ulam–Rassias stability of generalized quadratic functional equation on multi-Banach spaces.

2. Preliminaries

Assume that $(E, \|\cdot\|)$ is a complex linear space and let $m \in \mathbb{N}$. We denote by E^m the linear space $E \oplus E \oplus \ldots \oplus E$ consisting of m-tuples $(x_1, \ldots, x_m)$, where $x_1, \ldots, x_m \in E$. The linear operations on E^m are defined coordinatewise. When we write $(0, ..., 0, x_i, 0, ..., 0)$ for an element in E^m, understand that x_i appears in the i^{th} coordinate. The zero element of either E or E^m is denoted by 0. We denote by $\mathbb{N}_m$ the set $\{1, 2, \ldots, m\}$ and by σ_m the group of permutations on m symbols.

In this section, we recall the notion of a multi-normed space and some preliminaries concerning multi-normed spaces from [8].

Definition 2.1. [8] Let $(E, \|\cdot\|)$ be a complex normed space and let $m \in \mathbb{N}$. A multi-norm of level m on $\{E^s \ : \ s \in \mathbb{N}_m\}$ is a sequence

$$(\|\cdot\|_s) = (\|\cdot\|_s : s \in \mathbb{N}_m),$$

such that $\|\cdot\|_s$ is a norm on E^s for each $s \in \mathbb{N}_m$, such that $\|x\|_1 = \|x\|$ for each $x \in E$ and such that the following Axioms $(A1)-(A4)$ are satisfied for each $s \in \mathbb{N}_m$ with $s \geq 2$:

(A1) for each $\sigma \in \sigma_s$ and $x_1, \dots, x_s \in E$, we have

$$\|(x_{\sigma(1)}, \dots, x_{\sigma(s)})\|_s = \|(x_1, \dots, x_s)\|_s;$$

(A2) for each $\alpha_1, \dots, \alpha_s \in \mathbb{C}$ and $x_1, \dots, x_s \in E^s$, we have

$$\|(\alpha_1 x_1, \dots, \alpha_s x_s)\|_s \leq \big(\max_{i \in \mathbb{N}_s} |\alpha_i|\big) \|(x_1, \dots, x_s)\|_s;$$

(A3) for each $x_1, \dots, x_s \in E$, we have

$$\|(x_1, \dots, x_{s-1}, 0)\|_s = \|(x_1, \dots, x_{s-1})\|_{s-1};$$

(A4) for each $x_1, \dots, x_s \in E$, we have

$$\|(x_1, \dots, x_{s-1}, x_{s-1})\|_s = \|(x_1, \dots, x_{s-1})\|_{s-1}.$$

In this case, we say that $((E^s, \|\cdot\|_s) : s \in \mathbb{N}_m)$ is a *multi-normed space of level m.*

Definition 2.2. [8] A multi-norm on $\{E^s : s \in \mathbb{N}\}$ is a sequence

$$(\|\cdot\|_s) = (\|\cdot\|_s : s \in \mathbb{N}),$$

such that $(\|\cdot\|_s : s \in \mathbb{N}_m)$ is a multi-norm of level m for each $m \in \mathbb{N}$. In this case, we say that $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ is a multi-normed space.

Lemma 2.3. [8] Let $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ be a multi-normed space. The following properties are immediate consequences of the axioms for multi-normed spaces:

(i) for all $x \in E$ and $s \in \mathbb{N}$, we have

$$\|(x, \dots, x)\|_s = \|x\|,$$

(ii) for all $s \in \mathbb{N}$ and all $x_1, \dots, x_s \in E$, we have

$$\max_{i \in \mathbb{N}_s} \|x_i\| \leq \|(x_1, \dots, x_s)\|_s \leq \sum_{i=1}^{s} \|x_i\| \leq s \max_{i \in \mathbb{N}_s} \|x_i\|.$$

The following Lemma is a consequence of (ii):

Lemma 2.4. [8] Suppose that $(E, \|\cdot\|)$ is a Banach space. Then $(E^s, \|\cdot\|_s)$ is a Banach space for each $s \in \mathbb{N}$.

Definition 2.5. [8] Let $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ be a multi-normed space for which $(E, \|\cdot\|)$ is a Banach space. Then $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ is called a multi-Banach space.

Now, we recall two important examples of multi-norms for arbitrary space $(E, \|\cdot\|)$. For other examples we refer to readers to [8].

Example 2.6. Let $(E, \|\cdot\|)$ be a normed space. For $m \in \mathbb{N}$, define $\|\cdot\|_m$ on E^m by

$$\|(x_1, \ldots, x_m)\|_m = \max_{i \in \mathbb{N}_m} \|x_i\| \qquad (x_1, \ldots, x_m \in E).$$

It is immediate that $((E^s, \|\cdot\|_s) : s \in \mathbb{N})$ is a multi-normed space. The sequence $(\|\cdot\|_m : m \in \mathbb{N})$ is called **minimum multi-norm**. The terminology 'minimum' is justified by Lemma 2.3.

Example 2.7. Let $(E, \|\cdot\|)$ be a normed space and $\{(\|\cdot\|_m^\alpha : m \in \mathbb{N}) : \alpha \in A\}$ be the (non-empty) family of all multi-norms on $\{E^s : s \in \mathbb{N}\}$. For $s \in \mathbb{N}$, define

$$|||(x_1, \ldots, x_s)|||_s = \sup_{\alpha \in A} \|(x_1, \ldots, x_s)\|_s^\alpha \quad (x_1, \ldots, x_s \in E).$$

Then $(|||.|||_m : m \in \mathbb{N})$ is a multi-norm on $\{E^s : s \in \mathbb{N}\}$, which is called **maximum multi-norm**.

We recall two fundamental results in fixed point theory.

Theorem 2.8. [4] Let (X, d) be a complete metric space and let $J : X \to X$ be strictly contractive, i.e.,

$$d(Jx, Jy) \leq Lf(x, y), \qquad \forall x, y \in X$$

for some Lipschitz constant $L < 1$. Then

(1) the mapping J has a unique fixed point $x^* = Jx^*$;

(2) the fixed point x^* is globally attractive, i.e.,

$$\lim_{n \to \infty} J^n x = x^*,$$

for any starting point $x \in X$;

(3) one has the following estimation inequalities:

$$\begin{aligned} d(J^n x, x^*) &\leq L^n d(x, x^*), \\ d(J^n x, x^*) &\leq \frac{1}{1-L} d(J^n x, J^{n+1} x), \\ d(x, x^*) &\leq \frac{1}{1-L} d(x, Jx), \end{aligned}$$

for all nonnegative integers n and all $x \in X$.

Definition 2.9. Let X be a set. A function $d : X \times X \to [0, \infty]$ is called a *generalized metric* on X if d satisfies

(1) $d(x, y) = 0$ if and only if $x = y$;

(2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(3) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Theorem 2.10. [10] Let (X, d) be a complete generalized metric space and let $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either

$$d(J^n x, J^{n+1}x) = \infty,$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1}x) < \infty, \qquad \forall n \geq n_0$;
(2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J;
(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0}x, y) < \infty\}$;
(4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

3. Main results

Throughout this paper, n will be a positive integer such that $n \geq 2$.

Lemma 3.1. *Let X and Y be linear spaces and suppose that a mapping $Q : X \to Y$ satisfies the functional equation (1.2) for all $x_1, \ldots, x_n \in X$. Then the mapping L is quadratic.*

Proof . Since n is a positive integer, putting $x_1 = \cdots = x_n = 0$ in (1.2), we get $Q(0) = 0$. Letting $x_m = 0$ in (1.2) for all $1 \leq m \leq n$ with $m \neq i, j$, we get

$$Q(x_i - 2x_j) + Q(x_j - 2x_i) + 4Q(x_i + x_j) = 9Q(x_i) + 9Q(x_j), \tag{3.1}$$

for all $x_i, x_j \in X$. Letting $x_j = 0$ in (3.1), we have

$$Q(-2x_i) = 4Q(x_i), \tag{3.2}$$

for all $x_i \in X$. Replacing x_i by $x_i + x_j$ in (3.1), we get

$$Q(x_i - x_j) + Q(-x_j - 2x_i) + 4Q(x_i + 2x_j) = 9Q(x_i + x_j) + 9Q(x_j), \tag{3.3}$$

for all $x_i, x_j \in X$. Replacing x_j by $-x_j$ in (3.3), we have

$$Q(x_i + x_j) + Q(x_j - 2x_i) + 4Q(x_i - 2x_j) = 9Q(x_i - x_j) + 9Q(-x_j), \tag{3.4}$$

for all $x_i, x_j \in X$. Letting $x_i = 0$ in (3.4), we get

$$2Q(x_j) + 4Q(-2x_j) = 18Q(-x_j), \tag{3.5}$$

for all $x_j \in X$. It follows from (3.2) and (3.5) that $Q(-x_j) = Q(x_j)$ for all $x_j \in X$, i.e. Q is even. Using evenness of Q and (3.4), we have

$$Q(x_i + x_j) + Q(2x_i - x_j) + 4Q(x_i - 2x_j) = 9Q(x_i - x_j) + 9Q(x_j), \tag{3.6}$$

for all $x_i, x_j \in X$. Interchange x_i and x_j in (3.6), we get

$$Q(x_i + x_j) + Q(x_i - 2x_j) + 4Q(2x_i - x_j) = 9Q(x_i - x_j) + 9Q(x_i), \tag{3.7}$$

for all $x_i, x_j \in X$. Adding (3.6) and (3.7), we have

$$2Q(x_i + x_j) + 5Q(x_i - 2x_j) + 5Q(2x_i - x_j) = 18Q(x_i - x_j) + 9Q(x_i), +9Q(x_j) \tag{3.8}$$

for all $x_i, x_j \in X$. Using (3.1) and (3.8) we conclude that

$$Q(x_i + x_j) + Q(x_i - x_j) = 2Q(x_i) + 2Q(x_j),$$

for all $x_i, x_j \in X$. This means Q is quadratic. $\square$

Let X and Y be vector spaces. For a given mapping $f : X \to Y$, we define

$$Df(x_1, \dots, x_n) = \sum_{j=1}^{n} f\Big(-2x_j + \sum_{i=1, i\neq j}^{n} x_i\Big) - (n-6) f\Big(\sum_{i=1}^{n} x_i\Big) - 9\sum_{i=1}^{n} f(x_i),$$

for all $x_1, \dots, x_n \in X$.

Now, we prove the generalized Hyers–Ulam–Rassias stability of generalized quadratic mapping on multi-Banach spaces for the functional equation $Df(x_1, \dots, x_n) = 0$.

Theorem 3.2. *Let E be a linear space and $\{(F^l, \|\cdot\|_l) : l \in \mathbb{N}\}$ be a multi-Banach space. Suppose that $m \in \mathbb{N}$ and $f : E \to F$ is a mapping satisfying $f(0) = 0$ for which there exists a control function $\varphi : E^{nm} \to [0, \infty)$ such that*

$$\|\big(Df(X^{(1)}), \dots, Df(X^{(m)})\big)\|_m \le \varphi(X^{(1)}, \dots, X^{(m)}), \tag{3.9}$$

for all $X^{(1)} = (x_1^{(1)}, \dots, x_n^{(1)}), \dots, X^{(m)} = (x_1^{(m)}, \dots, x_n^{(m)}) \in E^n$. If there exists a Lipschitz constant $L < 1$ such that

$$\varphi\Big(X^{(1)}, \dots, X^{(m)}\Big) \le 4L\varphi\Big(\frac{X^{(1)}}{2}, \dots, \frac{X^{(m)}}{2}\Big),$$

for all $X^{(1)}, \dots, X^{(m)} \in E^n$, then there exists a unique quadratic mapping $Q : E \to F$ such that

$$\begin{aligned} \|f(x_1)-&Q(x_1), \dots, f(x_m) - Q(x_m)\|_m \\ &\le \frac{1}{4-4L}\Bigg[\frac{2}{9}\varphi\Big(X_{i,j}(0, 2x_1), X_{i,j}(0, 2x_2), \dots, X_{i,j}(0, 2x_m)\Big) \\ &\quad + \frac{1}{18}\varphi\Big(X_{i,j}(-2x_1, 2x_1), X_{i,j}(-2x_2, 2x_2), \dots, X_{i,j}(-2x_m, 2x_m)\Big) \\ &\quad + \varphi\Big(X_{i,j}(0, x_1), X_{i,j}(0, x_2), \dots, X_{i,j}(0, x_m)\Big)\Bigg], \end{aligned} \tag{3.10}$$

for all $x_1, \dots, x_m \in E$, where

$$X_{i,j}(x, y) = (0, \dots, 0, \underbrace{x}_{i\,th}, 0, \dots, 0, \underbrace{y}_{j\,th}, 0, \dots, 0),$$

for all $x, y \in E$.

Proof . For convenience, set

$$\begin{aligned} \varphi_{i,j}(x_1, x_2, \dots, x_m) = &\frac{2}{9}\varphi\Big(X_{i,j}(0, 2x_1), X_{i,j}(0, 2x_2), \dots, X_{i,j}(0, 2x_m)\Big) \\ &+ \frac{1}{18}\varphi\Big(X_{i,j}(-2x_1, 2x_1), X_{i,j}(-2x_2, 2x_2), \dots, X_{i,j}(-2x_m, 2x_m)\Big) \\ &+ \varphi\Big(X_{i,j}(0, x_1), X_{i,j}(0, x_2), \dots, X_{i,j}(0, x_m)\Big), \end{aligned}$$

where $x_1,\ldots,x_m \in E$ and $1 \le i < j \le n$. Consider the set $X := \{g : E \to F, \quad g(0) = 0\}$ and introduce the *generalized metric* on X:

$$d(g,h) = \inf\Big\{C \in \mathbb{R}_+ : \quad \|(g(x_1)-h(x_1),\ldots,g(x_m)-h(x_m))\|_m \\ \le C\varphi_{i,j}(x_1,x_2,\ldots,x_m), \quad \forall\, x_1,x_2,\ldots,x_m \in E\Big\}.$$

It is easy to show that (X,d) is complete. Now we consider the linear mapping $J : X \to X$ such that $Jg(x) := \frac{1}{4}g(2x)$ for all $x \in E$. For any $g,h \in X$, we have

$$\begin{aligned}
& d(g,h) < C \\
& \quad \Longrightarrow \|(g(x_1)-h(x_1),\ldots,g(x_m)-h(x_m))\|_m \le \\
& \qquad\qquad C\varphi_{i,j}(x_1,x_2,\ldots,x_m), \quad (x_1,x_2,\ldots,x_m \in E) \\
& \quad \Longrightarrow \Big\|\Big(\frac{1}{4}g(2x_1)-\frac{1}{4}h(2x_1),\ldots,\frac{1}{4}g(2x_m)-\frac{1}{4}h(2x_m)\Big)\Big\|_m \le \\
& \qquad\qquad \frac{1}{4}C\varphi_{i,j}(2x_1,2x_2,\ldots,2x_m), \\
& \quad \Longrightarrow \Big\|\Big(\frac{1}{4}g(2x_1)-\frac{1}{4}h(2x_1),\ldots,\frac{1}{4}g(2x_m)-\frac{1}{4}h(2x_m)\Big)\Big\|_m \le \\
& \qquad\qquad LC\varphi_{i,j}(x_1,x_2,\ldots,x_m), \\
& \quad \Longrightarrow d(Jg,Jh) \le LC.
\end{aligned}$$

Therefore, we see that

$$d(Jg,Jh) \le L\,d(g,h), \qquad \forall g,h \in X.$$

This means J is a strictly contractive self-mapping of X, with the Lipschitz constant L.

For each $1 \le r \le n$ with $r \ne i,j$ and each $1 \le m \le s$, let $x_i^{(m)} = x_m, \quad x_j^{(m)} = y_m$ and $x_r^{(m)} = 0$ in (3.9), we get

$$\begin{aligned}
&\Big\|\Big(f(x_1-2y_1)+f(y_1-2x_1)+4f(x_1+y_1)-9f(x_1)-9f(y_1), \\
&\quad f(x_2-2y_2)+f(y_2-2x_2)+4f(x_2+y_2)-9f(x_2)-9f(y_2),\ldots, \\
&\quad f(x_m-2y_m)+f(y_m-2x_m)+4f(x_m+y_m)-9f(x_m)-9f(y_m)\Big)\Big\|_m \\
&\qquad \le \varphi\Big(X_{i,j}(x_1,y_1),X_{i,j}(x_2,y_2),\ldots,X_{i,j}(x_m,y_m)\Big),
\end{aligned} \tag{3.11}$$

for all $x_1,y_1,x_2,y_2,\ldots,x_m,y_m \in E$. Letting $x_1 = x_2 = \cdots = x_m = 0$ in (3.11), we get

$$\begin{aligned}
&\Big\|\Big(f(-2y_1)-4f(y_1),f(-2y_2)-4f(y_2),\ldots, \\
&\quad f(-2y_m)-4f(y_m)\Big)\Big\|_m \le \varphi\Big(X_{i,j}(0,y_1),X_{i,j}(0,y_2),\ldots,X_{i,j}(0,y_m)\Big),
\end{aligned} \tag{3.12}$$

for all $y_1,y_2,\ldots,y_m \in E$. Interchange $x_1,x_2,\ldots,x_m$ by $x_1+y_1,x_2+y_2,\ldots,x_m+y_m$, respectively, in (3.11), we get

$$\begin{aligned}
&\Big\|\Big(f(x_1-y_1)+f(-y_1-2x_1)+4f(x_1+2y_1)-9f(x_1+y_1)-9f(y_1), \\
&\quad f(x_2-y_2)+f(-y_2-2x_2)+4f(x_2+2y_2)-9f(x_2+y_2)-9f(y_2),\ldots, \\
&\quad f(x_m-y_m)+f(-y_m-2x_m)+4f(x_m+2y_m)-9f(x_m+y_m)-9f(y_m)\Big)\Big\|_m \\
&\qquad \le \varphi\Big(X_{i,j}(x_1+y_1,y_1),X_{i,j}(x_2+y_2,y_2),\ldots,X_{i,j}(x_m+y_m,y_m)\Big),
\end{aligned} \tag{3.13}$$

for all $x_1, y_1, x_2, y_2, \ldots, x_m, y_m \in E$. Replacing $y_1, y_2, \ldots, y_m$ by $-y_1, -y_2, \ldots, -y_m$, respectively, in (3.13), we have

$$\begin{aligned}
&\Big\|\Big(f(x_1+y_1)+f(y_1-2x_1)+4f(x_1-2y_1)-9f(x_1-y_1)-9f(-y_1),\\
&\quad f(x_2+y_2)+f(y_2-2x_2)+4f(x_2-2y_2)-9f(x_2-y_2)-9f(-y_2),\ldots,\\
&\quad f(x_m+y_m)+f(y_m-2x_m)+4f(x_m-2y_m)-9f(x_m-y_m)-9f(-y_m)\Big)\Big\|_m\\
&\qquad \le \varphi\Big(X_{i,j}(x_1-y_1,y_1), X_{i,j}(x_2-y_2,y_2),\ldots,X_{i,j}(x_m-y_m,y_m)\Big),
\end{aligned} \tag{3.14}$$

for all $x_1, y_1, x_2, y_2, \ldots, x_m, y_m \in E$. Letting $x_1 = x_2 = \cdots = x_m = 0$ in (3.14), we get

$$\begin{aligned}
&\Big\|\Big(2f(y_1)+4f(-2y_1)-18f(-y_1),\\
&\quad 2f(y_2)+4f(-2y_2)-18f(-y_2),\ldots,\\
&\quad 2f(y_m)+4f(-2y_m)-18f(-y_m)\Big)\Big\|_m\\
&\qquad \le \varphi\Big(X_{i,j}(-y_1,y_1), X_{i,j}(-y_2,y_2),\ldots,X_{i,j}(-y_m,y_m)\Big),
\end{aligned} \tag{3.15}$$

for all $y_1, y_2, \ldots, y_m \in E$. It follows from (3.12) and (3.15) that

$$\begin{aligned}
&\big\|\big(f(y_1)-f(-y_1), f(y_2)-f(-y_2),\ldots,f(y_m)-f(-y_m)\big)\big\|_m\\
&\qquad \le \frac{2}{9}\varphi\Big(X_{i,j}(0,y_1), X_{i,j}(0,y_2),\ldots,X_{i,j}(0,y_m)\Big)\\
&\qquad\quad + \frac{1}{18}\varphi\Big(X_{i,j}(-y_1,y_1), X_{i,j}(-y_2,y_2),\ldots,X_{i,j}(-y_m,y_m)\Big),
\end{aligned} \tag{3.16}$$

for all $y_1, y_2, \ldots, y_m \in E$.

Now, using (3.12) and (3.16), we can conclude that

$$\begin{aligned}
&\big\|\big(f(2x_1)-4f(x_1), f(2x_2)-4f(x_2),\ldots,f(2x_m)-4f(x_m)\big)\big\|_m\\
&\qquad \le \frac{2}{9}\varphi\Big(X_{i,j}(0,2x_1), X_{i,j}(0,2x_2),\ldots,X_{i,j}(0,2x_m)\Big)\\
&\qquad\quad + \frac{1}{18}\varphi\Big(X_{i,j}(-2x_1,2x_1), X_{i,j}(-2x_2,2x_2),\ldots,X_{i,j}(-2x_m,2x_m)\Big)\\
&\qquad\quad + \varphi\Big(X_{i,j}(0,x_1), X_{i,j}(0,x_2),\ldots,X_{i,j}(0,x_m)\Big),
\end{aligned} \tag{3.17}$$

for all $x_1, x_2, \ldots, x_m \in E$. So

$$\Big\|\frac{1}{4}f(2x_1)-f(x_1), \frac{1}{4}f(2x_2)-f(x_2),\ldots,\frac{1}{4}f(2x_m)-f(x_m)\Big\|_m \le \frac{1}{4}\varphi_{i,j}(x_1,x_2,\ldots,x_m),$$

for all $x_1, x_2, \ldots, x_m \in E$. Hence $d(f, Jf) \le \frac{1}{4}$.

By Theorem 2.10, there exists a mapping $Q : E \to F$ such that
(1) Q is a fixed point of J, i.e.,

$$Q(x) = \frac{1}{4}Q(2x), \tag{3.18}$$

for all $x \in E$. The mapping Q is a unique fixed point of J in the set

$$Y = \{g \in X : d(f, g) < \infty\}.$$

This implies that Q is a unique mapping satisfying (3.18) such that there exists $C \in (0, \infty)$ satisfying

$$\|Q(x_1) - f(x_1), \ldots, Q(x_m) - f(x_m)\|_m \leq C\varphi_{i,j}(x_1, x_2, \ldots, x_m),$$

for all $x_1, x_2, \ldots, x_m \in E$.
(2) $d(J^k f, Q) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} \frac{1}{4^k} f\left(2^k x\right) = Q(x), \tag{3.19}$$

for all $x \in E$.
(3) $d(f, Q) \leq \dfrac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{1}{4 - 4L}.$$

This implies that the inequality (3.10) holds.

Replacing $X^{(1)} = \cdots = X^{(m)} = (x_1, x_2, \ldots, x_n) := X$ in (3.9), using the properties of norm in multi-normed spaces and (3.19), we have

$$\begin{aligned}
&\|DQ(x_1, \ldots, x_n), \ldots, DQ(x_1, \ldots, x_n)\|_s \\
&= \lim_{k\to\infty} \frac{1}{4^k} \|Df(2^k x_1, \ldots, 2^k x_n), \ldots, Df(2^k x_1, \ldots, 2^k x_n)\|_m \\
&= \lim_{k\to\infty} \frac{1}{4^k} \|Df(2^k x_1, \ldots, 2^k x_n)\|_m \\
&\leq \lim_{k\to\infty} \frac{1}{4^k} \varphi(2^k X, \ldots, 2^k X) \\
&\leq \lim_{k\to\infty} L^k \varphi(X, \ldots, X) = 0,
\end{aligned}$$

for all $x_1, x_2, \ldots, x_n \in E$. So

$$\sum_{j=1}^{n} Q\Big(- 2x_j + \sum_{i=1, i\neq j}^{n} x_i\Big) = (n-6)Q\Big(\sum_{i=1}^{n} x_i\Big) + 9\sum_{i=1}^{n} Q(x_i).$$

By Lemma 3.1, the mapping $Q : E \to F$ is Quadratic, i.e., $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ for all $x, y \in E$. $\square$

Corollary 3.3. *Let E be a linear space and $\{(F^l, \|\cdot\|_l) : l \in \mathbb{N}\}$ be a multi-Banach space. Suppose that $m \in \mathbb{N}$ and $0 < p < 2$ and $f : E \to F$ is a mapping with $f(0) = 0$ satisfying*

$$\|Df(X^{(1)}), \ldots, Df(X^{(m)})\|_m \leq \epsilon \sum_{k=1}^{m} \sum_{t=1}^{n} \|x_t^{(k)}\|^p,$$

for all $X^{(1)}, \ldots, X^{(m)} \in E^n$. Then there exists a unique quadratic mapping $Q : E \to F$ such that

$$\|f(x_1) - Q(x_1), \ldots, f(x_m) - Q(x_m)\|_m \leq \left(\frac{2^p}{3(2^2 - 2^p)} + \frac{1}{2^2 - 2^p}\right) \epsilon \sum_{k=1}^{m} \|x_k\|^p,$$

for all $x_1, \ldots, x_m \in E$.

Proof . The proof follows from Theorem 3.2 by taking

$$\varphi(X^{(1)},\dots,X^{(m)})=\epsilon\sum_{m=1}^{s}\sum_{t=1}^{n}\|x_t^{(m)}\|^p,$$

for all $X^{(1)},\dots,X^{(m)}\in E^n$. We can choose $L=\dfrac{1}{2^{2-p}}$ to get the desired result. □

Theorem 3.4. *Let E be a linear space and $\{(F^l,\|\cdot\|_l)\ :l\in\mathbb{N}\}$ be a multi-Banach space. Suppose that $m\in\mathbb{N}$ and $f:E\to F$ is a mapping satisfying $f(0)=0$ for which there exists a control function $\varphi:E^{nm}\to[0,\infty)$ satisfying (3.9) for all $X^{(1)},\dots,X^{(m)}\in E^n$. If there exists a Lipschitz constant $L<1$ such that*

$$\varphi\Big(X^{(1)},\dots,X^{(m)}\Big)\le\frac{1}{4}L\varphi\Big(2X^{(1)},\dots,2X^{(m)}\Big),$$

for all $X^{(1)},\dots,X^{(m)}\in E^n$, then there exists a unique quadratic mapping $Q:E\to F$ such that

$$\begin{aligned}\|f(x_1)-&Q(x_1),\dots,f(x_m)-Q(x_m)\|_m\\ &\le\frac{L}{4-4L}\bigg[\frac{2}{9}\varphi\Big(X_{i,j}(0,2x_1),X_{i,j}(0,2x_2),\dots,X_{i,j}(0,2x_m)\Big)\\ &\quad+\frac{1}{18}\varphi\Big(X_{i,j}(-2x_1,2x_1),X_{i,j}(-2x_2,2x_2),\dots,X_{i,j}(-2x_m,2x_m)\Big)\\ &\quad+\varphi\Big(X_{i,j}(0,x_1),X_{i,j}(0,x_2),\dots,X_{i,j}(0,x_m)\Big)\bigg],\end{aligned}\tag{3.20}$$

for all $x_1,\dots,x_m\in E$, where

$$X_{i,j}(x,y)=(0,\dots,0,\underbrace{x}_{i\,th},0,\dots,0,\underbrace{y}_{j\,th},0,\dots,0),$$

for all $x,y\in E$.

Proof . Similar to the proof of Theorem 3.2, we consider the linear mapping $J:X\to X$ such that $Jg(x):=4g\big(\frac{1}{2}x\big)$ for all $x\in E$. We can conclude that J is a strictly contractive self-mapping of X, with the Lipschitz constant L.

It follows from (3.17) that

$$\Big\|f(x_1)-4f(\frac{1}{2}x_1),f(x_2)-4f(\frac{1}{2}x_2),\dots,f(x_m)-4f(\frac{1}{2}x_m)\Big\|_m\le\frac{L}{4}\varphi_{i,j}(x_1,x_2,\dots,x_m),$$

for all $x_1,x_2,\dots,x_m\in E$. Hence $d(f,Jf)\le\dfrac{L}{4}$.

By Theorem 2.10, there exists a mapping $Q:E\to F$ such that
(1) Q is a fixed point of J, i.e.,

$$Q(x)=4Q(\frac{x}{2}),\tag{3.21}$$

for all $x\in E$. The mapping Q is a unique fixed point of J in the set

$$Y=\{g\in X:d(f,g)<\infty\}.$$

This implies that Q is a unique mapping satisfying (3.21) such that there exists $C \in (0, \infty)$ satisfying

$$\|f(x_1) - Q(x_1), \ldots, f(x_m) - Q(x_m)\|_m \leq C\varphi_{i,j}(x_1, x_2, \ldots, x_m),$$

for all $x_1, x_2, \ldots, x_m \in E$.
(2) $d(J^k f, Q) \to 0$ as $k \to \infty$. This implies the equality

$$\lim_{k\to\infty} 4^k f\left(\frac{x}{2^k}\right) = Q(x),$$

for all $x \in E$.
(3) $d(f, Q) \leq \dfrac{1}{1-L} d(f, Jf)$, which implies the inequality

$$d(f, Q) \leq \frac{L}{4-4L}.$$

This implies that the inequality (3.20) holds.

The rest of the proof is similar to the proof of Theorem 3.2. □

Corollary 3.5. *Let E be a linear space and $\{(F^l, \|\cdot\|_l) : l \in \mathbb{N}\}$ be a multi-Banach space. Suppose that $m \in \mathbb{N}$ and $p > 2$ and $f : E \to F$ is a mapping with $f(0) = 0$ satisfying*

$$\|Df(X^{(1)}), \ldots, Df(X^{(m)})\|_m \leq \epsilon \sum_{k=1}^{m} \sum_{t=1}^{n} \|x_t^{(k)}\|^p,$$

for all $X^{(1)}, \ldots, X^{(m)} \in E^n$. Then there exists a unique quadratic mapping $Q : E \to F$ such that

$$\|f(x_1) - Q(x_1), \ldots, f(x_m) - Q(x_m)\|_m \leq \left(\frac{2^p}{3(2^p - 2^2)} + \frac{1}{2^p - 2^2}\right) \epsilon \sum_{k=1}^{m} \|x_k\|^p,$$

for all $x_1, \ldots, x_m \in E$.

Proof . The proof follows from Theorem 3.4 by taking

$$\varphi(X^{(1)}, \ldots, X^{(m)}) = \epsilon \sum_{m=1}^{s} \sum_{t=1}^{n} \|x_t^{(m)}\|^p,$$

for all $X^{(1)}, \ldots, X^{(m)} \in E^n$. We can choose $L = \dfrac{1}{2^{p-2}}$ to get the desired result. □

Acknowledgements

The authors would like to thank Marand Branch, Islamic Azad University for the financial support of this research, which is based on a research project contract.

References

[1] T. Aoki, *On the stability of the linear transformation in Banach spaces*, J. Math. Soc. Japan 2 (1950) 64–66.

[2] C. Borelli and G. L. Forti, *On a general Hyers–Ulam stability result,* Internat. J. Math. Math. Sci. 18 (1995) 229–236.

[3] D.G. Bourgin, *Classes of transformations and bordering transformations,* Bull. Amer. Math. Soc. 57 (1951) 223–237.

[4] L. Cădariu and V. Radu, *Fixed points and the stability of Jensen's functional equation*, J. Inequal. Pure Appl. Math., **4**, no. 1, Art. 4 (2003).

[5] L. Cădariu and V. Radu, *Fixed points and the stability of quadratic functional equations*, An. Univ. Timişoara, Ser. Mat. Inform. 41 (2003) 25–48.

[6] L. Cădariu and V. Radu, *On the stability of the Cauchy functional equation: a fixed point approach*, Grazer Math. Ber. 346 (2004) 43–52.

[7] S. Czerwik, *The stability of the quadratic functional equation, in: Th. M. Rassias, J. Tabor (Eds.), Stability of Mappings of Hyers–Ulam Type,* Hadronic Press, Florida, 1994, 81-91.

[8] H.G. Dales and M.E. Polyakov, *Multi-normed spaces and multi-Banach algebras*, arxiv:1112.5148v2 (2012).

[9] H.G. Dales and M.S. Moslehian, *Stability of mappings on multi-normed spaces*, Glasg. Math. J. 49 (2007) 321–332.

[10] J. Diaz and B. Margolis, *A fixed point theorem of the alternative for contractions on a generalized complete metric space*, Bull. Amer. Math. Soc. 74 (1968) 305–309.

[11] M. Eshaghi Gordji and H. Khodaei, *Solution and stability of generalized mixed type cubic, quadratic and additive functional equation in quasi–Banach spaces*, Nonlinear Anal.-TMA, 71 (2009) 5629–5643.

[12] M. Eshaghi Gordji and M. Bavand Savadkouhi, *Stability of a mixed type cubic–quartic functional equation in non-Archimedean spaces*, Appl. Math. Lett. 23 (2010) 1198–1202.

[13] M. Eshaghi Gordji, A. Najati and A. Ebadian, *Stability and superstability of Jordan homomorphisms and Jordan derivations on Banach algebras and C * -algebras: a fixed point approach*, Acta Math. Sci. Ser. B Engl. Ed. 31 (2011) 1911–1922.

[14] M. Eshaghi Gordji and S. Abbaszadeh, *Stability of Cauchy-Jensen inequalities in fuzzy Banach spaces*, Appl. Comput. Math. 11 (2012) 27–36.

[15] M. Eshaghi Gordji, H. Khodaei and Th.M. Rassias, *Fixed points and stability for quadratic mappings in β -normed left Banach modules on Banach algebras*, Results Math. 61 (2012) 393–400.

[16] P. Găvruta, *A generalization of the Hyers–Ulam–Rassias stability of approximately additive mappings*, J. Math. Anal. Appl. 184 (1994) 431–436.

[17] D.H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. 27 (1941) 222–224.

[18] S.-M. Jung and J.M. Rassias, *A fixed point approach to the stability of a functional equation of the spiral of Theodorus,* Fixed Point Theory and Applications, 2008 (2008), Art. ID: 945010.

[19] L. Maligranda, *A result of Tosio Aoki about a generalization of Hyers–Ulam stability of additive functions - a question of priority,* Aeq. Math. 75 (2008) 289–296.

[20] F. Moradlou, *Additive functional inequalities and derivations on Hilbert C^*-modules,* Glasg. Math. J. 55 (2013) 341–348.

[21] F. Moradlou, *Approximate Euler - Lagrange - Jensen type additive mapping in multi-Banach spaces, A Fixed Point Approach,* Commun. Korean Math. Soc. 28 (2013) 319–333.

[22] F. Moradlou, H. Vaezi and C. Park, *Fixed points and stability of an additive functional equation of n-Apollonius type in C^*-algebras,* Abstract and Applied Analysis, vol. 2008, Article ID 672618, 13 pages, 2008. doi:10.1155/2008/672618.

[23] F. Moradlou, A. Najati and H. Vaezi, *Stability of homomorphisms and derivations on C^*-ternary rings associated to an Euler–Lagrange type additive mapping,* Results Math. 55 (2009) 469–486.

[24] F. Moradlou, H. Vaezi and G.Z. Eskandani, *Hyers–Ulam–Rassias stability of a quadratic and additive functional equation in quasi-Banach spaces*, Mediterr. J. Math. 6 (2009) 233–248.

[25] F. Moradlou and M. Eshaghi Gordji, *Approximate Jordan derivations on Hilbert C^*- modules*, Fixed Point Theory 14 (2013) 413–426.

[26] F. Moradlou and Th.M. Rassias, *Generalized Hyers-Ulam-Rassias stability for a general additive functional equation in quasi-β-normed spaces*, Bull. Korean Math. Soc. 50 (2013) 2061–2070.

[27] M.S. Moslehian, *Approximate C^*-ternary ring homomorphisms,* Bull. Braz. Math. Soc. 38 (2007) 611–622.

[28] M.S. Moslehian, K. Nikodem and D. Popa, *Asymptotic aspect of the quadratic functional equation in multi-normed spaces*, J. Math. Anal. Appl. 355 (2009) 717–724.

[29] C. Park, *On the stability of the linear mapping in Banach modules*, J. Math. Anal. Appl. 275 (2002) 711–720.

[30] C. Park, *Fixed points and Hyers–Ulam–Rassias stability of Cauchy-Jensen functional equations in Banach algebras*, Fixed Point Theory and Applications, 2007 (2007), Art. ID 50175.
[31] C. Park and Th.M. Rassias, *Hyers–Ulam stability of a generalized Apollonius type quadratic mapping,* J. Math. Anal. Appl. 322 (2006) 371–381.
[32] V. Radu, *The fixed point alternative and stability of functional equations*, Fixed Point Theory, Cluj-Napoca, IV(1) (2003) 91–96.
[33] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, J. Funct. Anal. 46 (1982) 126–130.
[34] J.M. Rassias, *On approximation of approximately linear mappings by linear mappings*, Bull. Sci. Math. 108 (1984) 445–446.
[35] T. Zhou Xu, J.M. Rassias and W. Xin Xu, *Generalized Ulam-Hyers stability of a general mixed AQCQ-functional equation in multi-Banach spaces: a fixed point approach*, Eur. J. Pure Appl. Math. 3 (2010) 1032–1047.
[36] Th.M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. 72 (1978) 297–300.
[37] Th.M. Rassias, *The problem of S. M. Ulam for approximately multiplicative mappings*, J. Math. Anal. Appl. 246 (2000) 352–378.
[38] Th.M. Rassias, *On the stability of functional equations in Banach spaces*, J. Math. Anal. Appl. 251 (2000) 264–284.
[39] F. Skof, *Local properties and approximations of operators,* Rend. Sem. Mat. Fis. Milano 53 (1983) 113–129.
[40] S.M. Ulam, *A Collection of the Mathematical Problems*, Interscience Publ. New York, 1960.