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Generalized solution of Sine-Gordon equation

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Abstract

In this paper, we are interested to study the Sine-Gordon equation in generalized function theory, we give a result of existence and uniqueness of generalized solution with initial data are distributions (elements of the Colombeau algebra). Then we study the association concept with the classical solution.

Keywords: Generalized functions; hyperbolic system; Sine-Gordon equation.
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1. Introduction

In 1982, Colombeau introduced an algebra $\mathcal{G}$ of generalized functions to deal with the multiplication problem for distributions, see [4, 5]. This algebra $\mathcal{G}$ is a differential algebra which contains the space $\mathcal{D}'$ of distributions. Furthermore, nonlinear operations more general than the multiplication make sense in the algebra $\mathcal{G}$. Therefore the algebra $\mathcal{G}$ is a very convenient one to find and study solutions of nonlinear differential equations with singular data and coefficients. The paper is placed in the framework of algebras of generalized functions introduced by Colombeau in [4, 5]. Note also several examples have been studied by many authors in [12], [15, 16, 17] [13,16,17]. In particular, the authors [18] [18] processing the nonlinear wave with a data $u|\{t<0\} = 0$. In this paper, we study the Sine-Gordon equation which a nonlinear wave, but in this time with conditions initial are distribution.
The paper is organized as follows. In section 2, we recall the theory of Colombeau. Section 3, we proved the existence and uniqueness of solution in the algebra of Colombeau. The association with the classical solution is established in Section 4

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2. Notations

We shall fix the notation and introduce a number of known as well as new classes of generalized functions here. For more details, see [9].
Let Ω be an open subset of $\mathbb{R}^n$. The basic objects of the theory as we use it are families $(u_\varepsilon)_{\varepsilon\in(0,1]}$ of smooth functions $u_\varepsilon \in \mathcal{C}^\infty(\Omega)$ for $0 < \varepsilon \leq 1$. We single out the following subalgebras.

Moderate families, denoted by $\mathcal{E}_M(\Omega)$, are defined by the property :

$$\forall K \subseteq \Omega,\ \forall \alpha \in \mathbb{N}_0^n,\ \exists p \geq 0\ :\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}\left(\varepsilon^{-p}\right) \quad \text{as} \quad \varepsilon \to 0. \tag{2.1}$$

Null families, denoted by $\mathcal{N}(\Omega)$, are defined by the property :

$$\forall K \subseteq \Omega,\ \forall \alpha \in \mathbb{N}_0^n,\ \forall q \geq 0\ :\ \sup_{x\in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}\left(\varepsilon^{q}\right) \quad \text{as} \quad \varepsilon \to 0. \tag{2.2}$$

Thus moderate families satisfy a locally uniform polynomial estimate as $\varepsilon \to 0$,
together with all derivatives, while null functionals vanish faster than any power of ε in the same situation. The null families from a differential ideal in the collection of moderate families.

The *Colombeau algebra* is the factor algebra

$$\mathcal{G}(\Omega) = \mathcal{E}_M(\Omega)/\mathcal{N}(\Omega).$$

The algebra $\mathcal{G}(\Omega)$ just defined coincides with the *special Colombeau algebra* in [9], where the notation $\mathcal{G}^s(\Omega)$ has been employed. It was called the *simplified Colombeau algebra* in [1].
The Colombeau algebra on a closed half space $\mathbb{R}^n \times [0,\infty)$ is defined in a similary way. The restriction of an element $u \in \mathcal{G}\left(\mathbb{R}^n \times [0,\infty)\right)$ to the line $\{t = 0\}$ is defined on representatives by

$$u\mid_{\{t=0\}} = \text{class of } \Big(u_\varepsilon(.,0)\Big)_{\varepsilon\in(0,1]}.$$

Similary, restrictions of the elements of $\mathcal{G}(\Omega)$ to open subsets of Ω are defined on representatives. One can see that $\Omega \to \mathcal{G}(\Omega)$ is a sheaf of differential algebras on $\mathbb{R}^n$. The space of compactly supported distributions is imbedded in $\mathcal{G}(\Omega)$ by convolution :

$$i : \mathcal{E}'(\Omega) \to \mathcal{G}(\Omega), \quad i(w) = \text{class of } \Big(w * (\varphi_\varepsilon)\mid_\Omega\Big)_{\varepsilon\in(0,1]}, \tag{2.3}$$

where

$$\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi\Big(\frac{x}{\varepsilon}\Big) \tag{2.4}$$

is obtained by scaling a fixed test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$ of integral one with all moments vanishing. By the sheaf property, this can be extended in a unique way to an imbedding of the space of distributions $\mathcal{D}'(\omega)$.

One of the main features of the Colombeau construction is the fact that this imbedding renders $\mathcal{C}^\infty(\Omega)$ a faithful subalgebra. In fact, given $f \in \mathcal{C}^\infty(\Omega)$, one can define a corresponding element of $\mathcal{G}(\omega)$ by the constant imbedding $\sigma(f) = \text{class of } [(\varepsilon, x) \to f(x)]$. Then the important equality $i(f) = \sigma(f)$ holds in $\mathcal{G}(\Omega)$.

If $u \in \mathcal{G}(\Omega)$ and f is a smooth function which is of at most polynomial growth at infinity, together with all its derivatives, the superposition $f(u)$ is a well-defined element of $\mathcal{G}(\Omega)$.

We need a couple of further notions from the theory of Colombeau generalized functions. An element u of $\mathcal{G}(\Omega)$ is called of *local* L^p-*type* $(1 \leq p \leq \infty)$, if it has a representative with the property

$$\limsup_{\varepsilon\to 0} \|u_\varepsilon\|_{L^p(K)} < \infty$$

for every $K \subset \Omega$.

Regularity theory is based on the subalgebra $\mathcal{G}^\infty(\Omega)$ of *regular generalized functions* in $\mathcal{G}(\Omega)$. It is defined by those elements which have a representative satisfying

$$\forall K \subset \Omega \; \exists p \geq 0 \; \forall \alpha \in \mathbb{N}_0^n \; : \; \sup_{x \in K} |\partial^\alpha u_\varepsilon(x)| = \mathcal{O}\Big(\epsilon^{-p}\Big) \quad \text{as } \varepsilon \to 0. \tag{2.5}$$

Observe the change of quantifiers with respect to formula (2.1); locally, all derivatives of a regular generalized function have the same order of growth in $\varepsilon > 0$. One has that (see [13]).

$$\mathcal{G}^\infty(\Omega) \cap \mathcal{D}'(\Omega) = \mathcal{C}^\infty(\Omega).$$

For the purpose of describing the regularity of Colombeau generalized functions, $\mathcal{G}^\infty(\Omega)$ plays the same role as $\mathcal{C}^\infty(\Omega)$ does in the setting of distributions.

A net $\Big(r_\varepsilon\Big)_{\varepsilon \in (0,1]}$ of complex numbers is called a *slow scale net* if

$$|r_\varepsilon|^p = \mathcal{O}\Big(\varepsilon^{-1}\Big) \quad \text{as} \quad \varepsilon \to 0$$

for every $p \geq 0$. We refer to [6] for a detailed discussion of slow scale nets.
Finally, an element $u \in \mathcal{G}(\Omega)$ is called of *total slow scale type*, if for some representative, $\|\partial^\alpha u_\varepsilon\|_{L^\infty(K)}$ forms a slow scale net for every $K \subset \Omega$ and $\alpha \in \mathbb{N}_0^n$.

We end this section by recalling the *association relation* on the Colombeau algebra $\mathcal{G}(\Omega)$. It identifies elements of $\mathcal{G}(\Omega)$ if they coincide in the weak limit. That is, u, $v \in \mathcal{G}(\Omega)$ are called associated,

$$u \approx v, \quad \text{if} \quad \lim_{\varepsilon \to 0} \int \Big(u_\varepsilon(x) - v_\varepsilon(x)\Big)\psi(x)dx = 0$$

for all test functions $\psi \in \mathcal{D}(\Omega)$. We shall also say that u is associated with a distribution w if $u_\varepsilon \to w$ in the sense of distributions as $\varepsilon \to 0$

3. Existence/uniqueness of generalized solutions

This section is devoted to solving the Sine-Gordon equation in the Colombeau algebra $\mathcal{G}\Big(\mathbb{R} \times [0, \infty)\Big)$. Recall first that if u is a classical solution of the following problem

$$\begin{cases} (\partial_t^2 - \partial_x^2)\, u = 2\sin(u) & \forall x \in \mathbb{R},\; t \in \mathbb{R}_+, \\ u(x,0) = a(x),\; \partial_t u(x,0) = b(x) & x \in \mathbb{R}. \end{cases} \tag{3.1}$$

Then it solves the integral equation

$$u(x,t) = \frac{1}{2}\Big(a(x-t) + a(x+t)\Big) + \frac{1}{2}\int_{x-t}^{x+t} b(x)dx + \int_0^t \int_{x-t+s}^{x+t-s} \sin\left(u(z,s)\right) dz\, ds. \tag{3.2}$$

Let $K_0 = [-\kappa, \kappa]$ b a compact interval. For $0 \leq t \leq s \leq \kappa$, the interval I_t and the trapezoidal region K_s are defined by

$$I_t = \Big\{x \in \mathbb{R} \,|\, |x| \leq \kappa - t\Big\}$$

$$K_s = \Big\{(x,t) \in \mathbb{R} \times [0,\infty) \,|\, 0 \leq t \leq s,\; x \in I_t\Big\}.$$

Using (3.2), the following estimates are easily deduced ($0 \le t \le T \le \kappa$)

$$\|u\|_{L^\infty(K_T)} \le \|a\|_{L^\infty(I_0)} + T\,\|b\|_{L^\infty(I_0)} + 2T \int_0^T \|\sin(u)\|_{L^\infty(K_s)}\,ds, \tag{3.3}$$

$$\|u(.,t)\|_{L^\infty(I_t)} \le \|a\|_{L^\infty(I_0)} + T\,\|b\|_{L^\infty(I_0)} + 2T \int_0^T \|\sin(u(.,t))\|_{L^\infty(I_s)}\,ds. \tag{3.4}$$

Proposition 3.1. *Let a, $b \in \mathcal{G}(\mathbb{R})$, then the problem (3.2) has a unique solution $u \in \mathcal{G}\Big(\mathbb{R}\times[0,\infty)\Big)$.*

Proof . To prove the existence of a solution, take representatives a_ε, b_ε and let $u_\varepsilon \in \mathcal{C}^\infty\Big(\mathbb{R}\times[0,\infty)\Big)$ be the unique solution to the Sine-Gordon equation with regularized data

$$\begin{cases} (\partial_t^2 - \partial_x^2)\,u_\varepsilon = 2\sin(u_\varepsilon) & \forall x \in \mathbb{R},\ t \in \mathbb{R}_+, \\ u_\varepsilon(x,0) = a_\varepsilon(x),\ \partial_t u_\varepsilon(x,0) = b_\varepsilon(x) & x \in \mathbb{R}. \end{cases} \tag{3.5}$$

The classical solution u_ε to (3.5) is constructed by rewriting (3.5) as an integral equation and invoking a fixed point argument (this involves applying estimate (3.3) successively to all derivatives). If we show that the net $(u_\varepsilon)_{\varepsilon\in(0,1]}$ belongs to $\mathcal{E}_M\Big(\mathbb{R}\times[0,\infty)\Big)$, its equivalence class in $\mathcal{G}\Big(\mathbb{R}\times[0,\infty)\Big)$ will be a solution. To show that the zero-th derivative of u_ε satisfies the estimate (2.3), we take a region K_T with its horizontal slices I_t and invoke inequality (3.3) to see that

$$\|u_\epsilon\|_{L^\infty(K_T)} \le \|a_\epsilon\|_{L^\infty(I_0)} + T\,\|b_\epsilon\|_{L^\infty(I_0)} + 2T \int_0^T \|\sin(u_\epsilon)\|_{L^\infty(K_s)}\,ds. \tag{3.6}$$

Using that each of the terms involving a_ε, b_ε is of order $\mathcal{O}(\varepsilon^{-p})$ for some p, we infer from Gronwall's inequality for the function $s \to \|u_\varepsilon\|_{K_s}$. Thus u_ε is moderates on the region K_T, that is, it satisfies the estimate (3.3) there. To get the estimates for the higher order derivatives, one just differentiates the equation and employs the same arguments inductively, using that the lower order terms are already known to be moderate from the previous steps.

To prove uniqueness, we consider representatives u_ε, $v_\varepsilon \in \mathcal{E}\Big(\mathbb{R}\times[0,\infty)\Big)$ of two solutions u and v. Their difference satisfies

$$\begin{gathered} \Big(\partial_t^2 - \partial_x^2\Big)(u_\varepsilon - v_\varepsilon) = 2\sin(u_\varepsilon) - 2\sin(v_\varepsilon) + \eta_\varepsilon \\ (u_\varepsilon - v_\varepsilon)(x,0) = a_\varepsilon(x) + \eta_{0\varepsilon}, \quad \partial_t(u_\varepsilon - v_\varepsilon)(x,0) = b_\varepsilon(x) + \eta_{1\varepsilon} \end{gathered}$$

for certain null elements η_ε, $\eta_{0\varepsilon}$, $\eta_{1\varepsilon}$. Thus $u_\varepsilon - v_\varepsilon$ satisfies an estimate of the form (3.6), but with the null elements η_ε, $\eta_{0\varepsilon}$, $\eta_{1\varepsilon}$ replacing a_ε, b_ε there. This implies as above that the $L^\infty-$norm of $u_\varepsilon - v_\varepsilon$ on K_T is of order $\mathcal{O}(\varepsilon^q)$ for every $q \ge 0$. By [9], the null estimate (2.4) on $u_\varepsilon - v_\varepsilon$ suffices to have null estimates on all derivatives. Thus $u = v$ in $\mathcal{G}\Big(\mathbb{R}\times[0,\infty)\Big)$. □

Remark 3.2. In case the data are continuous or smooth functions, the relation of the generalized solution to the classical solution is as follows:
Assume first that a, b belong to $\mathcal{C}^\infty(\mathbb{R})$, let $w \in \mathcal{C}^\infty\Big(\mathbb{R}\times[0,\infty)\Big)$ be the classical solution. Then w coincides with the generalized solution $u \in \mathcal{G}\Big(\mathbb{R}\times[0,\infty)\Big)$, that is, $u = i(w)$ in $\mathcal{G}\Big(\mathbb{R}\times[0,\infty)\Big)$. This follows from the fact that the imbedding i coincides with the constant imbedding σ on $\mathcal{C}^\infty(\mathbb{R})$, so

$u_\varepsilon \equiv w$ is a representative of the generalized solution. Second, assume the data a, b are continuous functions and let $w \in \mathcal{C}^\infty\Big(\mathbb{R} \times [0,\infty)\Big)$ be the corresponding continuous (weak) solution. Then the generalized solution $u \in \mathcal{G}\Big(\mathbb{R} \times [0,\infty)\Big)$ is associated with w. This follows from the classical result of continuous dependence of the continuous solution on the initial data. Third, when the data are distributions, there may be no meaning for a distributional solution, in general. Yet the solution in $\mathcal{G}\Big(\mathbb{R} \times [0,\infty)\Big)$ may still be associated with a distribution. Some incidents of such a situation will be described in the next section.

4. Association with classical solution

Let v the solution to

$$\begin{cases} (\partial_t^2 - \partial_x^2)\, v = 0 & \forall x \in \mathbb{R}\ ,\ t \in \mathbb{R}_+, \\ v(x,0) = a(x)\ ,\ \partial_t v(x,0) = b(x) & x \in \mathbb{R} \end{cases}$$

and w the solution to

$$\begin{cases} (\partial_t^2 - \partial_x^2)\, w = 2\sin(w+m) & \forall x \in \mathbb{R}\ ,\ t \in \mathbb{R}_+, \\ w(x,0) = 0\ ,\ \partial_t w(x,0) = 0 & x \in \mathbb{R} \end{cases}$$

with

$$m(x,t) = \frac{1}{2}\beta_0\Big(H\left(t - |x|\right) - H\left(-t - |x|\right)\Big),$$

where H is the Heaviside function.

Proposition 4.1. *The generalized solution u_ε of (3.5) in $\mathcal{G}(\mathbb{R} \times \mathbb{R}_+)$ is associated with $v + w$.*

Proof . Let v_ε by the classical solution to

$$\begin{cases} (\partial_t^2 - \partial_x^2)\, v_\epsilon = 0 & \forall x \in \mathbb{R}\ ,\ t \in \mathbb{R}_+, \\ v_\epsilon(x,0) = a_\epsilon(x)\ ,\ \partial_t v_\epsilon(x,0) = b_\epsilon(x) & x \in \mathbb{R}. \end{cases}$$

We have

$$\begin{cases} (\partial_t^2 - \partial_x^2)\left(u_\epsilon - v_\epsilon - w\right) = 2\sin\left(u_\epsilon\right) - 2\sin(w+m) & \forall x \in \mathbb{R}\ ,\ t \in \mathbb{R}_+, \\ \left(u_\epsilon - v_\epsilon - w\right)(x,0) = 0\ ,\ \partial_t\left(u_\epsilon - v_\epsilon - w\right)(x,0) = 0 & x \in \mathbb{R}. \end{cases}$$

By using (3.4) with L^1-norm we obtain

$$\begin{aligned} \|u_\epsilon - v_\epsilon - w\|_{L^1(K_T)} &\leq 2T\int_0^T \|\sin\left(u_\epsilon\right) - \sin(w+m)\|_{L^1(K_s)}\, ds \\ &\leq 2T\int_0^T \|\sin(u_\epsilon) - \sin(v_\epsilon + w)\|_{L^1(K_s)}\, ds \\ &\qquad + 2T^2\left\|\sin\left(v_\epsilon + w\right) - \sin(w+m)\right\|_{L^1(K_T)} \\ &\leq 2T\int_0^T \|u_\epsilon - v_\epsilon - w\|_{L^1(K_s)}\, ds \\ &\qquad + 2T^2\left\|\sin\left(v_\epsilon + w\right) - \sin(w+m)\right\|_{L^1(K_T)}. \end{aligned} \tag{4.1}$$

Since $\sin(v_\epsilon + w) - \sin(w + m)$ converges to zero almost everywhere and remains bounded, Lebesgue's theorem shows that its L^1-norm on K_T converges to zero. By Gronwall's lemma, it follows that the L^1-norm of $u_\epsilon - v_\epsilon - w$ converges to zero on any K_T as well. Hence u_ϵ converges to $v + w$ weakly,which translates into the claimed association result. □

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