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# Some new results using Hadamard fractional integral

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**Abstract**

In this paper, we use Hadamard fractional integral to establish some new integral inequalities of Gruss type by using one or two parameters. Furthermore, other integral inequalities of reverse Minkowski's type are also obtained.

*Keywords:* Hadamard fractional integral; Fractional integral inequalities; Minkowski's inequality.
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## 1. Introduction

Fractional calculus is the field of mathematical analysis which deals with the investigation and applications of integrals and derivatives of arbitrary order.

In literature, there are several known forms of the fractional integrals which two have been studied extensively for their applications to many fields of sciences. The first is the Riemann-Liouville fractional integral defined by [9]

$$J_a^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \quad x > a. \tag{1.1}$$

For $a = 0$, we note

$$J_0^{\alpha} f(x) = J^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-t)^{\alpha-1} f(t)dt, \quad \alpha > 0, \quad x > 0. \tag{1.2}$$

The second is the Hadamard fractional integral introduced by J. Hadamard defined by [1]

$$_H D_1^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_1^x \ln(\frac{x}{t})^{\alpha-1} f(t) \frac{dt}{t}, \qquad \alpha > 0, \quad x > 1, \tag{1.3}$$

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where $\Gamma(\alpha) = \int_0^\infty e^{-u}u^{\alpha-1}du$.

From above definitions, we see the difference between Hadamard and Riemann-liouville fractional integrals as Kernel in the Hadamard integral has the from of $\ln(\frac{x}{t})$ instead of the form of $(x-t)$, which is involves in the Riemann- Liouville integral. The Hadamard derivative has the operator $(x\frac{d}{dx})^n$, while the Riemann-Liouville derivative has the operator $(\frac{d}{dx})^n$[10].

Many authors have been interested in the fractional calculus and their applications [2, 3, 5, 7, 4, 8, 10], however, the researches related to the Hadamard fractional integral are still less than that of Riemann-Liouville [11, 12].

The following theorems are some known results obtained in recent years:

In [7], Dahmani et al. studied the fractional integral inequalities, using the Riemann-Liouville fractional integral:
Let $p$ and $q$ be two positive functions on $[0,\infty[$ and let $f$ and $g$ be two integrable functions on $[0,\infty[$ satisfying the condition

$$\varphi \le f(x) \le \Phi,\ \psi \le g(x) \le \Psi,\ \varphi,\Psi,\psi,\Phi \in \mathbb{R},\ x \in [0,\infty]. \tag{1.4}$$

Then for all $t>0, \alpha>0, \beta>0$, we have

$$\begin{aligned} & |J^\alpha p(t)J^\beta qfg(t) + J^\beta q(t)J^\alpha pfg(t) - J^\alpha pf(t)J^\beta qg(t) - J^\beta qf(t)J^\alpha pg(t)| \\ \le\ & J^\alpha p(t)J^\beta q(t)((\Phi-\varphi)(\Psi-\psi)) \end{aligned}$$

and, let $f$ and $g$ be two lipschitzian functions on $[0,\infty[$ and Let $p$ and $q$ be two positive functions on $[0,\infty[$. Then for all $t>0, \alpha>0$, we have

$$\begin{aligned} & |J^\alpha q(t)J^\alpha pfg(t) + J^\alpha p(t)J^\alpha qfg(t) - J^\alpha pf(t)J^\alpha qg(t) - J^\alpha qf(t)J^\alpha pg(t)| \\ \le\ & L_1L_2(J^\alpha q(t)J^\alpha t^2p(t) + J^\alpha p(t)J^\alpha t^2q(t) - J^\alpha tp(t)J^\alpha tq(t)). \end{aligned}$$

In [6], Dahmani proved some results associated with fractional integrals:
Let $p,q$ be two positive integrable functions on $[0,\infty[$ and Let $f,g$ be two integrable functions on $[0,\infty[$ satisfying the condition

$$|f(x)-f(y)| \le M|g(x)-g(y)|;\ M>0,\ x,y \in [0,\infty[. \tag{1.5}$$

Then for all $t>0, \alpha>0$, we have

$$\begin{aligned} & |J^\alpha q(t)J^\alpha pfg(t) + J^\alpha p(t)J^\alpha qfg(t) - J^\alpha pf(t)J^\alpha qg(t) - J^\alpha qf(t)J^\alpha pg(t)| \\ \le\ & M[J^\alpha p(t)J^\alpha qg^2(t) + J^\alpha q(t)J^\alpha pg^2(t) - 2J^\alpha pg(t)J^\alpha qg(t)]. \end{aligned}$$

In another paper, Dahmani [5] established the reverse Minkowski fractional inequalities as follows:
Let $\alpha>0$, $p\ge 1$ and let $f,g$ be two positive functions on $[0,\infty[$, such that for all $t>0$, $0 < J^\alpha f^p(t), J^\alpha g^p(t) < \infty$. If $0 < m \le f(\tau)/g(\tau) < \infty, \tau \in [0,t]$, then

$$[J^\alpha f^p(t)]^{\frac{1}{p}} + [J^\alpha g^p(t)]^{\frac{1}{p}} \le \frac{1+M(m+2)}{(m+1)(M+1)}[J^\alpha(f+g)^p(t)]^{\frac{1}{p}} \tag{1.6}$$

and

$$[J^\alpha f^p(t)]^{\frac{2}{p}} + [J^\alpha g^p(t)]^{\frac{2}{p}} \ge \left(\frac{(m+1)(M+1)}{M} - 2\right)[J^\alpha f^p(t)]^{\frac{1}{p}}[J^\alpha g^p(t)]^{\frac{1}{p}}. \tag{1.7}$$

The main objective of this paper is to establish some fractional integral inequalities by using Hadamard fractional integrals.

## 2. Main results

In this section, we present and prove the main results.

**Theorem 2.1.** *Let $p$ and $q$ be two positive functions on $[1,\infty[$ and let $f$ and $g$ be two functions defined on $[1,\infty[$ satisfying the condition (1.4) on $[1,\infty[$. Then for all $t>1, \alpha>0, \beta>0$, we have*

$$\begin{aligned} & |{}_HD_1^{-\beta}q(t){}_HD_1^{-\alpha}(pfg)(t) +{}_H D_1^{-\alpha}p(t){}_HD_1^{-\beta}(qfg)(t) \\ - & {}_HD_1^{-\alpha}(pf)(t){}_HD_1^{-\beta}(qg)(t) -{}_H D_1^{-\beta}(qf)(t){}_HD_1^{-\alpha}(pg)(t)| \\ \leq & {}_HD_1^{-\alpha}p(t){}_HD_1^{-\beta}q(t)\left((\Phi-\varphi)(\Psi-\psi)\right). \end{aligned}$$

**Proof .** Suppose that $f$ and $g$ be two functions defined on $[1,\infty[$ satisfying the condition (1.4). Then for all $t>1$, we have

$$|f(\tau)-f(\rho)| \leq \Phi-\varphi, \quad |g(\tau)-g(\rho)| \leq \Psi-\psi, \qquad \tau,\rho \in [1,\infty[, \tag{2.1}$$

which implies that

$$|\left(f(\tau)-f(\rho)\right)\left(g(\tau)-g(\rho)\right)| \leq (\Phi-\varphi)(\Psi-\psi). \tag{2.2}$$

Define

$$K(\tau,\rho) = f(\tau)g(\tau)+f(\rho)g(\rho)-f(\tau)g(\rho)-f(\rho)g(\tau), \tag{2.3}$$

then multiplying (2.3) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}p(\tau), \tau\in(1,t)$, and integrating the resulting identity with respect to $\tau$ from 1 to $t$ we obtain

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}p(\tau)K(\tau,\rho)\frac{d\tau}{\tau} \\ & = \frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}p(\tau)f(\tau)g(\tau)\frac{d\tau}{\tau} + \frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}p(\tau)f(\rho)g(\rho)\frac{d\tau}{\tau} \\ & -\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}p(\tau)f(\tau)g(\rho)\frac{d\tau}{\tau} - \frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}p(\tau)f(\rho)g(\tau)\frac{d\tau}{\tau}. \end{aligned} \tag{2.4}$$

Consequently,

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}p(\tau)K(\tau,\rho)\frac{d\tau}{\tau} \\ & ={}_H D_1^{-\alpha}(pfg)(t)+f(\rho)g(\rho){}_HD_1^{-\alpha}p(t)-g(\rho){}_HD_1^{-\alpha}(pf)(t)-f(\rho){}_HD_1^{-\alpha}(pg)(t). \end{aligned} \tag{2.5}$$

Multiplying (2.5) by $\frac{(\ln(\frac{t}{\rho}))^{\beta-1}}{\rho\Gamma(\beta)}q(\rho), \rho\in(1,t), t>1$ and integrating the resulting identity with respect to $\rho$ from 1 to $t$ we get

$$\begin{aligned} & \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}\ln(\frac{t}{\rho})^{\beta-1}p(\tau)q(\rho)K(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho} \\ & ={}_H D_1^{-\beta}q(t){}_HD_1^{-\alpha}(pfg)(t) +{}_H D_1^{-\alpha}p(t){}_HD_1^{-\beta}(qfg)(t) \\ & -{}_HD_1^{-\alpha}(pf)(t){}_HD_1^{-\beta}(qg)(t) -{}_H D_1^{-\beta}(qf)(t){}_HD_1^{-\alpha}(pg)(t). \end{aligned} \tag{2.6}$$

On the other hand

$$\begin{aligned} & \frac{(\Phi-\varphi)(\Psi-\psi)}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}\ln(\frac{t}{\rho})^{\beta-1}p(\tau)q(\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho} \\ & ={}_H D_1^{-\alpha}p(t){}_HD_1^{-\beta}q(t)\left((\Phi-\varphi)(\Psi-\psi)\right). \end{aligned}$$

Hence

$$\left|\frac{(\Phi-\varphi)(\Psi-\psi)}{\Gamma(\alpha)\Gamma(\beta)}\int_1^t\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}\ln(\frac{t}{\rho})^{\beta-1}p(\tau)q(\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}\right|$$
$$\leq_H D_1^{-\alpha}p(t)_H D_1^{-\beta}q(t)\left((\Phi-\varphi)(\Psi-\psi)\right).$$

The result is proved. □

By taking $\alpha=\beta$ in Theorem 2.1 we obtain the following result.

**Corollary 2.2.** Let $p$ and $q$ be two positive functions on $[1,\infty[$ and let $f$ and $g$ be two functions defined on $[1,\infty[$ satisfying the condition (1.4) on $[1,\infty[$. Then for all $t>1,\alpha>0$, we have

$$\begin{aligned}&|_H D_1^{-\alpha}q(t)_H D_1^{-\alpha}(pfg)(t)+_H D_1^{-\alpha}p(t)_H D_1^{-\alpha}(qfg)(t)\\&-_H D_1^{-\alpha}(pf)(t)_H D_1^{-\alpha}(qg)(t)-_H D_1^{-\alpha}(qf)(t)_H D_1^{-\alpha}(pg)(t)|\\&\leq\left(_H D_1^{-\alpha}p(t)_H D_1^{-\alpha}q(t)\right)(\Phi-\varphi)(\Psi-\psi).\end{aligned}$$

**Theorem 2.3.** *Let $p,q$ be two positive functions on $[1,\infty[$ and let $f,g$ be two functions defined on $[1,\infty[$ satisfying the condition* (1.5) *on $[1,\infty[$. Then for all $t>1,\alpha>0$, we have*

$$\begin{aligned}&|_H D_1^{-\alpha}q(t)_H D_1^{-\alpha}(pfg)(t)+_H D_1^{-\alpha}p(t)_H D_1^{-\alpha}(qfg)(t)\\&-_H D_1^{-\alpha}(pf)(t)_H D_1^{-\alpha}(qg)(t)-_H D_1^{-\alpha}(qf)(t)_H D_1^{-\alpha}(pg)(t)|\\&\leq M\left[_H D_1^{-\alpha}p(t)_H D_1^{-\alpha}(qg^2)(t)+_H D_1^{-\alpha}q(t)_H D_1^{-\alpha}(pg^2)(t)-2_H D_1^{-\alpha}(pg)(t)_H D_1^{-\alpha}(qg)(t)\right].\end{aligned}\tag{2.7}$$

**Proof .** Let $f$ and $g$ be two functions satisfying the condition (1.5). Then for every $\tau,\rho\in(1,t);t>1$, we have

$$|f(\tau)-f(\rho)|\leq M|g(\tau)-g(\rho)|.\tag{2.8}$$

This implies that

$$|K(\tau,\rho)|\leq M(g(\tau)-g(\rho))^2,\qquad \tau,\rho\in(1,t).\tag{2.9}$$

Hence, it follows that

$$\begin{aligned}&\frac{1}{\Gamma(\alpha)}\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}p(\tau)|K(\tau,\rho)|\frac{d\tau}{\tau}\\&\quad\leq M\left(_H D_1^{-\alpha}(pg^2)(t)-2g(\rho)_H D_1^{-\alpha}(pg)(t)+g^2(\rho)_H D_1^{-\alpha}p(t)\right).\end{aligned}\tag{2.10}$$

Consequently,

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_1^t\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}\ln(\frac{t}{\rho})^{\alpha-1}p(\tau)q(\rho)|K(\tau,\rho)|\frac{d\tau}{\tau}\frac{d\rho}{\rho}\\&\leq M\left[_H D_1^{-\alpha}q(t)_H D_1^{-\alpha}(pg^2)(t)-2_H D_1^{-\alpha}(qg)(t)_H D_1^{-\alpha}(pg)(t)+_H D_1^{-\alpha}p(t)_H D_1^{-\alpha}(qg^2)(t)\right].\end{aligned}\tag{2.11}$$

Using (2.11) and the relation

$$\begin{aligned}&\frac{1}{\Gamma^2(\alpha)}\int_1^t\int_1^t \ln(\frac{t}{\tau})^{\alpha-1}\ln(\frac{t}{\rho})^{\alpha-1}p(\tau)q(\rho)K(\tau,\rho)\frac{d\tau}{\tau}\frac{d\rho}{\rho}\\&=\quad {}_H D_1^{-\alpha}q(t)_H D_1^{-\alpha}(pfg)(t)+_H D_1^{-\alpha}p(t)_H D_1^{-\alpha}(qfg)(t)\\&-_H D_1^{-\alpha}(pf)(t)_H D_1^{-\alpha}(qg)(t)-_H D_1^{-\alpha}(qf)(t)_H D_1^{-\alpha}(pg)(t),\end{aligned}\tag{2.12}$$

the result is follows. □

**Theorem 2.4.** *Let $p, q$ be two positive functions on $[1, \infty[$ and let $f, g$ be two functions defined on $[1, \infty[$ satisfying the condition* (1.5) *on* $[1, \infty[$. *Then the inequality*

$$\begin{aligned}
&|{}_H D_1^{-\beta} q(t)\, {}_H D_1^{-\alpha}(pfg)(t) + {}_H D_1^{-\alpha} p(t)\, {}_H D_1^{-\beta}(qfg)(t) \\
&- {}_H D_1^{-\alpha}(pf)(t)\, {}_H D_1^{-\beta}(qg)(t) - {}_H D_1^{-\beta}(qf)(t)\, {}_H D_1^{-\alpha}(pg)(t)| \qquad (2.13)\\
&\leq M \left[ {}_H D_1^{-\alpha} p(t)\, {}_H D_1^{-\beta}(qg^2)(t) + {}_H D_1^{-\beta} q(t)\, {}_H D_1^{-\alpha}(pg^2)(t) - 2\, {}_H D_1^{-\alpha}(pg)(t)\, {}_H D_1^{-\beta}(qg)(t) \right]
\end{aligned}$$

*is valid for all $t > 1, \alpha > 0, \beta > 0$.*

**Proof .** Using the relation (2.10), we obtain

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_1^t \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} \ln(\frac{t}{\rho})^{\beta-1} p(\tau) q(\rho) |K(\tau, \rho)| \frac{d\tau}{\tau} \frac{d\rho}{\rho} \\
&\leq \frac{M}{\Gamma(\beta)} \int_1^t \left( \ln(\frac{t}{\rho})^{\beta-1} q(\rho) \left[ {}_H D_1^{-\alpha} pg^2(t) - 2g(\rho)\, {}_H D_1^{-\alpha} pg(t) + g^2(\rho)\, {}_H D_1^{-\alpha} p(t) \right] \right) \frac{d\rho}{\rho}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
&\frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_1^t \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} \ln(\frac{t}{\rho})^{\beta-1} p(\tau) q(\rho) |K(\tau, \rho)| \frac{d\tau}{\tau} \frac{d\rho}{\rho} \qquad (2.14)\\
&\leq M \left[ {}_H D_1^{-\alpha} p(t)\, {}_H D_1^{-\beta}(qg^2)(t) + {}_H D_1^{-\beta} q(t)\, {}_H D_1^{-\alpha}(pg^2)(t) - 2\, {}_H D_1^{-\alpha}(pg)(t)\, {}_H D_1^{-\beta}(qg)(t) \right].
\end{aligned}$$

Theorem 2.4 is thus proved. □

**Remark 2.5.** Applying Theorem 2.4 for $\alpha = \beta$, we obtain Theorem 2.3.

**Theorem 2.6.** *Let $f$ and $g$ be two Lipschitzian functions on $[1, \infty[$ and Let $p$ and $q$ be two positive functions on $[1, \infty[$. Then for all $t > 1, \alpha > 0$, the inequality*

$$\begin{aligned}
&|{}_H D_1^{-\alpha} q(t)\, {}_H D_1^{-\alpha}(pfg)(t) + {}_H D_1^{-\alpha} p(t)\, {}_H D_1^{-\alpha}(qfg)(t) \qquad (2.15)\\
&- {}_H D_1^{-\alpha}(pf)(t)\, {}_H D_1^{-\alpha}(qg)(t) - {}_H D_1^{-\alpha}(qf)(t)\, {}_H D_1^{-\alpha}(pg)(t)| \\
&\leq L_1 L_2 \left( {}_H D_1^{-\alpha} q(t)\, {}_H D_1^{-\alpha}(\tau^2 p)(t) + {}_H D_1^{-\alpha} p(t)\, {}_H D_1^{-\alpha}(\tau^2 q)(t) - 2\, {}_H D_1^{-\alpha}(\tau p)(t)\, {}_H D_1^{-\alpha}(\tau q)(t) \right)
\end{aligned}$$

*is valid.*

**Proof .** Let $f$ and $g$ be two lipschitzian functions on $[1, \infty[$. Then for all $t > 1$, we have

$$|f(\tau) - f(\rho)| \leq L_1 |\tau - \rho|, |g(\tau) - g(\rho)| \leq L_2 |\tau - \rho|; \tau, \rho \in (1, t). \qquad (2.16)$$

Hence

$$|K(\tau, \rho)| \leq L_1 L_2 (\tau - \rho)^2. \qquad (2.17)$$

Then, using the same argument as before, we can write

$$\begin{aligned}
&|\frac{1}{\Gamma^2(\alpha)} \int_1^t \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} \ln(\frac{t}{\rho})^{\alpha-1} p(\tau) q(\rho) |K(\tau, \rho)| \frac{d\tau}{\tau} \frac{d\rho}{\rho}| \\
&\leq L_1 L_2 \left( {}_H D_1^{-\alpha} q(t)\, {}_H D_1^{-\alpha}(\tau^2 p)(t) + {}_H D_1^{-\alpha} p(t)\, {}_H D_1^{-\alpha}(\tau^2 q)(t) - 2\, {}_H D_1^{-\alpha}(\tau p)(t)\, {}_H D_1^{-\alpha}(\tau q)(t) \right).
\end{aligned}$$

Now using (2.17) and the properties of modulus, we deduce

$$\begin{aligned}
&|\frac{1}{\Gamma^2(\alpha)} \int_1^t \int_1^t \ln(\frac{t}{\tau})^{\alpha-1} \ln(\frac{t}{\rho})^{\alpha-1} p(\tau) q(\rho) |K(\tau, \rho)| \frac{d\tau}{\tau} \frac{d\rho}{\rho}| \qquad (2.18)\\
&\leq L_1 L_2 \left( {}_H D_1^{-\alpha} q(t)\, {}_H D_1^{-\alpha}(\tau^2 p)(t) + {}_H D_1^{-\alpha} p(t)\, {}_H D_1^{-\alpha}(\tau^2 q)(t) - 2\, {}_H D_1^{-\alpha}(\tau p)(t)\, {}_H D_1^{-\alpha}(\tau q)(t) \right).
\end{aligned}$$

Theorem 2.6 is thus proved. □

**Theorem 2.7.** *Let $f$ and $g$ be two lipschitzian functions on $[1,\infty[$ and let $p$ and $q$ be two positive functions on $[1,\infty[$. Then for all $t>1, \alpha>0, \beta>0$, we have*

$$\begin{aligned}
&|{}_HD_1^{-\beta}q(t){}_HD_1^{-\alpha}(pfg)(t) +_H D_1^{-\alpha}p(t){}_HD_1^{-\beta}(qfg)(t) \\
-&\quad {}_HD_1^{-\alpha}(pf)(t){}_HD_1^{-\beta}(qg)(t) -_H D_1^{-\beta}(qf)(t){}_HD_1^{-\alpha}(pg)(t)| \\
&\leq L_1L_2\left({}_HD_1^{-\beta}q(t){}_HD_1^{-\alpha}(\tau^2p)(t) +_H D_1^{-\alpha}p(t){}_HD_1^{-\beta}(\tau^2q)(t) - 2{}_HD_1^{-\alpha}(\tau p)(t){}_HD_1^{-\beta}(\tau q)(t)\right).
\end{aligned} \tag{2.19}$$

**Proof .** To obtain the right hand side of (2.19), we multiply (2.17)by

$$\frac{\left(\ln(\frac{t}{\tau})^{\alpha-1}\right)\left(\ln(\frac{t}{\rho})^{\beta-1}\right)}{\tau\rho\Gamma(\tau)\Gamma(\rho)}p(\tau)q(\rho), \qquad \tau,\rho\in(1,t), \tag{2.20}$$

then we integrate with respect to $\tau$ and $\rho$ over $(1,t)^2$, the inequality (2.19) follows. □

**Remark 2.8.** Applying Theorem 2.7 for $\alpha=\beta$, we obtain Theorem 2.6.

**Theorem 2.9.** *Let $\alpha>0$, $p\geq 1$, and let $f$ and $g$ be two positive functions on $[1,\infty[$ such that $t>1$, $0<_H D_1^{-\alpha}(f^p)$, ${}_HD_1^{-\alpha}(g^p)<\infty$. If $0<m\leq f(\tau)/g(\tau)\leq M<\infty$, $\tau\in[1,t]$, then we have*

$$[{}_HD_1^{-\alpha}(f^p)(t)]^{\frac{1}{p}} + [{}_HD_1^{-\alpha}(g^p)(t)]^{\frac{1}{p}} \leq \frac{1+M(m+2)}{(m+1)(M+1)}[{}_HD_1^{-\alpha}((f+g)^p)(t)]^{\frac{1}{p}}. \tag{2.21}$$

**Proof .** Using the condition $f(\tau)/g(\tau)\leq M$ for all $t>1$, $\tau\in[1,t]$, we can get

$$(M+1)^pf^p(\tau)\leq M^p(f+g)^p(\tau). \tag{2.22}$$

Multiplying both sides of (2.22) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\Gamma(\alpha)}, \tau\in(1,t)$ $t>1$ and integrating the resulting inequalities with respect to $\tau$ from 1 to $t$, we obtain

$$(M+1)^p_HD_1^{-\alpha}(f^p)(t)\leq M^P_HD_1^{-\alpha}((f+g)^p)(t).$$

Hence, we can write

$$[{}_HD_1^{-\alpha}(f^p)(t)]^{\frac{1}{p}}\leq\frac{M}{M+1}[{}_HD_1^{-\alpha}((f+g)^p)(t)]^{\frac{1}{p}}. \tag{2.23}$$

On the other hand, using the condition $m\leq f(\tau)/g(\tau)$, we can get

$$(m+1)^pg^p(\tau)\leq(f+g)^p(\tau). \tag{2.24}$$

Multiplying both sides of (2.24) by $\frac{(\ln(\frac{t}{\tau}))^{\alpha-1}}{\tau\gamma(\alpha)}$, $\tau\in(1,t)$, $t>1$ and integrating the resulting inequalities with respect to $\tau$ from 1 to $t$, we obtain

$$(m+1)^p_HD_1^{-\alpha}(g^p)(t)\leq_H D_1^{-\alpha}((f+g)^p)(t).$$

Hence, we can write

$$[{}_HD_1^{-\alpha}(g^p)(t)]^{\frac{1}{p}}\leq\frac{1}{m+1}[{}_HD_1^{-\alpha}((f+g)^p)(t)]^{\frac{1}{p}}. \tag{2.25}$$

Adding the inequality (2.23) and (2.25), we obtain the inequality (2.21). □

**Theorem 2.10.** *Let $\alpha > 0$, $p \geq 1$, and let $f$ and $g$ be two positive functions on $[1, \infty[$ such that $t > 1$, $0 <_H D_1^{-\alpha}(f^p)$, ${}_H D_1^{-\alpha}(g^p) < \infty$. If $0 < m \leq f(\tau)/g(\tau) \leq M < \infty$, $\tau \in [1, t]$, then we have*

$$[{}_H D_1^{-\alpha}(f^p)(t)]^{\frac{2}{p}} + [{}_H D_1^{-\alpha}(g^p)(t)]^{\frac{2}{p}} \geq \left(\frac{(m+1)(M+1)}{M} - 2\right)[{}_H D_1^{-\alpha}(f^p)(t)]^{\frac{1}{p}}[{}_H D_1^{-\alpha}(g^p)(t)]^{\frac{1}{p}}. \quad (2.26)$$

**Proof .** Multiplying the inequality (2.23) and (2.25), we obtain

$$\frac{(m+1)(M+1)}{M}[{}_H D_1^{-\alpha}(f^p)(t)]^{\frac{1}{p}}[{}_H D_1^{-\alpha}(g^p)(t)]^{\frac{1}{p}} \leq \left([{}_H D_1^{-\alpha}((f+g)^p)(t)]^{\frac{1}{p}}\right)^2. \quad (2.27)$$

Applying Minkowski's inequality to the right side of (2.27), we get

$$\begin{aligned} &\left([{}_H D_1^{-\alpha}((f+g)^p)(t)]^{\frac{1}{p}}\right)^2 \leq \left([{}_H D_1^{-\alpha}(f^p)(t)]^{\frac{1}{p}} + [{}_H D_1^{-\alpha}(g^p)(t)]^{\frac{1}{p}}\right)^2 \\ = \;& [{}_H D_1^{-\alpha}(f^p)(t)]^{\frac{2}{p}} + [{}_H D_1^{-\alpha}(g^p)(t)]^{\frac{2}{p}} + 2[{}_H D_1^{-\alpha}(f^p)(t)]^{\frac{1}{p}}[{}_H D_1^{-\alpha}(g^p)(t)]^{\frac{1}{p}}. \quad (2.28) \end{aligned}$$

Combining (2.27) and (2.28), we obtain 2.26. □

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