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On a class of paracontact Riemannian manifold

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Abstract

We classify the paracontact Riemannian manifolds that their Riemannian curvature satisfies in the certain condition and we show that this classification holds for the special cases semi-symmetric and locally symmetric spaces. Finally we study paracontact Riemannian manifolds satisfying $R(X, \xi)\cdot S = 0$, where S is the Ricci tensor.

Keywords: Paracontact manifold; η-Einstein; semi-symmetric.
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1. Introduction

The notion of almost paracontact manifolds (respectively, almost paracontact Riemannian manifolds) as analogue of almost contact manifolds (respectively, almost contact Riemannian manifolds) was introduced by Sato in [7, 8]. Remarkable that an almost contact manifold is always odd dimensional but an almost paracontact manifold could be even dimensional as well. Then special classes of almost paracontact manifolds such as para Sasakian manifolds and semi-symmetric manifolds are studied by many authors (see [1, 2, 4]).

In [6], Perrone studied the contact Riemannian manifolds satisfying the conditions $R(X, \xi).R = 0$ for any X, and $l = -\phi^2$ where $l = R(\cdot, \xi)\xi$ and $R(X, \xi)$ acts on R as a derivation. In this paper we consider the paracontact Riemannian manifolds satisfying these conditions and obtain new results for this class of manifolds. The results of this paper are the paracontact analogues of the contact results proved in [6]. Since the semi-symmetric contact manifolds satisfying the condition $l = -\phi^2$ is a class of these manifolds, then we have the same results for this class. Also, since the notation of semi-symmetric spaces is a direct generalization of locally symmetric, then the results of this paper hold for the locally symmetric spaces.

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2. Paracontact metric structures

Let M be a $(2m+1)$-dimensional differentiable manifold and let ϕ be a $(1,1)$-tensor field, ξ a vector field and η a 1-form on M. Then (ϕ, ξ, η) is called an almost paracontact structure on M if

$$\phi^2 = I - \eta \otimes \xi, \tag{2.1}$$

$$\eta(\xi) = 1, \qquad \phi\xi = 0,$$

where I denotes the identity transformation. Moreover, the tensor ϕ induces an almost paracomplex structure on the distribution $D = ker\eta$, that is, the eigendistributions D^+, D^- corresponding to the eigenvalues 1, -1 of ϕ respectively, have equal dimension n ([9],[10]).

If a $2n+1$-dimensional almost paracontact manifold M with an almost paracontact structure (ϕ, ξ, η) admits a pseudo-Riemannian metric g of signature $(n+1, n)$ such that

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \tag{2.2}$$

or equivalently,

$$g(\phi X, Y) = -g(X, \phi Y), \qquad g(X, \xi) = \eta(X), \tag{2.3}$$

for any $X, Y \in TM$, then we say that M is an almost paracontact metric manifold. For a $(2n+1)$-dimensional manifold M with an almost paracontact metric structure (ϕ, ξ, η, g) one can also construct a local orthonormal basis. Let U be coordinate neighborhood on M and X_1 be a unit vector field on U orthogonal to ξ. Then ϕX_1 is a vector field orthogonal to both X_1 and ξ, and $|\phi X_1|^2 = -1$. Now choose a unit vector field X_2 orthogonal to ξ, X_1 and ϕX_1. Then ϕX_2 is also a vector field orthogonal to ξ, X_1, ϕX_1, and X_2 and $|\phi X_2|^2 = -1$. Proceeding in this way we obtain a local orthonormal basis $(Xi, \phi Xi, \xi)$, $(i = 1, \ldots, n)$ called a ϕ-basis. Indicating by $\pounds$ and R, the Lie differentiation operator and the curvature tensor of M^{2n+1} respectively, we define

$$h = \frac{1}{2}\pounds_\xi \phi, \quad l = R(., \xi)\xi, \quad \tau = \pounds_\xi g.$$

The $(1,1)$- type tensors h and l are symmetric and satisfy

$$h\xi = 0, \qquad l\xi = 0, \quad h\phi = -\phi h, \tag{2.4}$$

and, we can easily calculate the following formulas for a paracontact metric manifold M^{2n+1}:

$$\nabla_X \xi = -\phi X + \phi h X, \tag{2.5}$$

$$\nabla_\xi h = -\phi + \phi h^2 - \phi l, \tag{2.6}$$

$$l = -\phi l \phi + 2(h^2 - \phi^2), \tag{2.7}$$

$$\nabla_\xi \phi = 0, \tag{2.8}$$

where ∇ denotes the Riemannian connection of the Riemannian metric g.

If the Reeb vector field ξ is Killing, that is, $\tau = 0$ or, equivalently $h = 0$, then the para- contact metric manifold M^{2n+1} is called a K-paracontact manifold. If a paracontact metric manifold M is normal, i.e., the tensor $N_\phi := [\phi, \phi] - 2d\eta \otimes \xi$ vanishes identically, then M is called a para Sasakian manifold. Equivalently, a paracontact metric manifold is para Sasakian if and only if $R_{XY}\xi = -(\eta(Y)X - \eta(X)Y)$. Any para Sasakian manifold is K-paracontact and in dimension 3 the converse also holds (see [5] for more details).

Lemma 2.1. *Let $(M^{2n+1}, \xi, \eta, \phi, g)$ be a paracontact metric manifold. Then the following equalities hold*

$$\begin{gathered}\tau(X,Y) = -2g(\phi X, hY), \quad \tau(\xi,.) = 0, \ \tau(X,Y) = \tau(Y,X), \\ \tau(\phi X, Y) = \tau(X, \phi Y), \ \tau(\phi X, \phi Y) = \tau(X,Y), \quad (\nabla_\xi \tau)(\xi,.) = 0, \\ (\nabla_\xi \tau)(X,Y) = -2g(\phi X, (\nabla_\xi h)Y), \ (\nabla_\xi \tau)(\phi X, \phi Y) = (\nabla_\xi \tau)(X,Y),\end{gathered}$$

for any $X, Y \in TM$.

Proof . We only prove $\tau(X,Y) = -2g(\phi X, hY)$. Using the definition of $\tau(X,Y)$ we have

$$\tau(X,Y) = \pounds_\xi g(X,Y) - g(\pounds_\xi X, Y) - g(X, \pounds_\xi Y).$$

Applying $\nabla g = 0$ and $\pounds_\xi = \nabla_\xi X - \nabla_X \xi$, the last equation reduces to

$$\tau(X,Y) = g(\nabla_X \xi, Y) + g(X, \nabla_Y \xi).$$

Setting (2.5) in the above relation we have

$$\tau(X,Y) = g(-\phi X + \phi h X, Y) + g(X, -\phi Y + \phi h Y).$$

From $h\phi = -\phi h$ we get the assertion. Similarity we can prove the another equalities. □

Proposition 2.2. *On a paracontact metric manifold, the following conditions are equivalent*

$$(i)\ \nabla_\xi h = 0, \qquad (ii)\ \nabla_\xi \tau = 0, \qquad (iii)\ \nabla_\xi l = 0.$$

Proof . $(i) \Rightarrow (ii)$ Since $\nabla_\xi h = 0$, then using Lemma 2.1 we deduce that $(\nabla_\xi \tau)(X,Y) = -2g(\phi X, (\nabla_\xi h)Y) = 0$.

$(ii) \Rightarrow (iii)$ Let $\nabla_\xi \tau = 0$. Then using Lemma 2.1 we obtain $2g(\phi X, (\nabla_\xi h)Y) = 0$. Replacing X by ϕX in this equation we obtain $g(\phi^2 X, (\nabla_\xi h)Y) = 0$, which gives us

$$g(X, (\nabla_\xi h)Y) - \eta(X) g(\xi, (\nabla_\xi h)Y) = 0.$$

But from (2.6) we derive that $g(\xi, (\nabla_\xi h)Y) = 0$. Setting this in the above equation we deduce that $g(X, (\nabla_\xi h)Y)$, which gives us $\nabla_\xi h = 0$. Further effecting ϕ to (2.6) and using $\eta l = 0$, we obtain

$$h^2 - \phi^2 = l.$$

Differentiating the above equation with respect to ξ, we find that

$$\nabla_\xi l = \nabla_\xi h^2 = \nabla_\xi h.h + h.\nabla_\xi h = 0.$$

$(iii) \Rightarrow (i)$ Let $\nabla_\xi l = 0$, using (2.7) we get

$$\nabla_\xi h^2 = (\nabla_\xi h)h + h(\nabla_\xi h) = 0.$$

Applying ∇_ξ to the above equation we have

$$(\nabla_\xi \nabla_\xi h)h + 2(\nabla_\xi h)^2 = 0.$$

Setting $\nabla_\xi \nabla_\xi h = 0$ in the above equation we deduce that $\nabla_\xi h = 0$. □

Proposition 2.3. *Let $(M^{2n+1}, \phi, \xi, \eta)$ be a paracontact Riemannian manifold. Then $\nabla_\xi \tau = 0$ if and only if*

$$K(\xi, e') - K(\xi, e) = |e|^2 - |e'|^2 + |he'|^2 - |he|^2,$$

where $K(\xi, .)$ is the sectional curvature and e , e' tensor fields in D.

Proof . Relations (2.2), (2.4) and (2.6) yield

$$\begin{aligned} g(R(e,\xi)\xi, e) &= g((\nabla_\xi h)e + \phi e - \phi h^2 e, \phi e) \\ &= g((\nabla_\xi h)e, \phi e) + g(\phi e, \phi e) - g(h\phi e, h\phi e) \\ &= -\tfrac{1}{2}(\nabla_\xi \tau)(e,e) - |e|^2 + |he|^2. \end{aligned}$$

Therefore

$$K(\xi, e) = g(R(e,\xi)\xi, e) = -\frac{1}{2}(\nabla_\xi \tau)(e,e) - |e|^2 + |he|^2. \tag{2.9}$$

Similarly, we get

$$K(\xi, e') = -\frac{1}{2}(\nabla_\xi \tau)(e', e') - |e'|^2 + |he'|^2. \tag{2.10}$$

Now assume $\nabla_\xi \tau = 0$. From (2.9) and (2.10) we have

$$K(\xi, e') - K(\xi, e) = |e|^2 - |e'|^2 + |he'|^2 - |he|^2.$$

Conversely, setting $e' = \phi e$ and using (2.10) we obtain

$$K(\xi, \phi e) = \frac{1}{2}(\nabla_\xi \tau)(e,e) - |e|^2 + |he|^2. \tag{2.11}$$

Summing (2.9) and (2.11) we deduce

$$(\nabla_\xi \tau)(e,e) = K(\xi, \phi e) - K(\xi, e).$$

But according assumption we get $K(\xi, \phi e) - K(\xi, e) = 0$. Therefore $\nabla_\xi \tau = 0$. □

Lemma 2.4. *In a paracontact Riemannian manifold M, the following conditions are equivalent*

(*i*) $l = \kappa \phi^2$,
(*ii*) $\nabla_\xi l = 0, \qquad h^2 = (1+\kappa)\phi^2$,
(*iii*) $K(\xi, X) = \kappa, \quad \forall X \in D$.

Proof . (*i*) ⇒ (*ii*) Since $l = \kappa\phi^2$, then using (2.8) we have $\nabla_\xi l = \kappa \nabla_\xi \phi^2 = 0$. Also setting $l = \kappa\phi^2$ in (2.7) we deduce, $h^2 = (1+\kappa)\phi^2$.
(*ii*) ⇒ (*iii*) Let $\nabla_\xi l = 0$ then according to Proposition 2.2 we have $\nabla_\xi h = 0$. On the other hand, using (2.6) we get

$$\phi l = -\phi + \phi h^2.$$

Setting the last equation and $h^2 = (1+k)\phi^2$ in (2.7) we obtain $l = \kappa$. Therefore

$$K(X, \xi) = \frac{g(R(X,\xi)\xi, X)}{g(X,X)} = \frac{g(lX, X)}{g(X,X)} = \kappa.$$

(iii) ⇒ *(i)* Let $K(\xi, X) = \kappa$. Since $X \in D$, then we have, $g(\xi, X) = 0$. Therefore

$$\kappa = K(\xi, X) = \frac{g(R(X,\xi)\xi, X)}{g(X,X)},$$

which gives us

$$g(R(X,\xi)\xi - \kappa X, X) = 0.$$

From the above equation we get $R(X,\xi)\xi = \kappa X$. Since $X \in D$, then we deduce $X = \phi^2 X$. Therefore we obtain $lX = \kappa\phi^2 X$ for all $X \in D$. On the other hand we have $l\xi = 0 = \kappa\phi^2\xi$. Thus we deduce that $lX = \kappa\phi^2 X$ for all $X \in TM$. □

2.1. η-Einstein paracontact metric manifolds

Definition 2.5. A paracontact manifold M is said to be η-Einstein if its Ricci tensor S is of the form

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.12}$$

for any vector fields X and Y, where a, b are functions on M.

Proposition 2.6. *Let $(M^{2n+1}, \phi, \xi, \eta)$ be a paracontact metric manifold which satisfies the condition η-Einstein. Then*

$$a = \frac{r}{2n} + 1 - \frac{|\tau|^2}{8n}, \qquad b = -\frac{r}{2n} + (2n+1)(\frac{|\tau|^2}{8n} - 1), \tag{2.13}$$

where r is the scalar curvature.

Proof . Let $\{e_i, \phi e_i, \xi\}_{i=1}^n$ be a ϕ-basis of M. The Ricci tensor S and the scalar curvature r of M are defined by

$$S(X,Y) = \sum_{i=1}^n g(R(e_i,X)Y, e_i) - \sum_{i=1}^n g(R(\phi e_i, X)Y, \phi e_i) + g(R(\xi,X)Y,\xi),$$

where $X, Y \in \Gamma(TM)$, and

$$r = \sum_{i=1}^n S(e_i,e_i) - \sum_{i=1}^n S(\phi e_i, \phi e_i) + S(\xi,\xi), \tag{2.14}$$

respectively. Using (2.12) and $g(\phi e, \phi e) = -g(e,e)$ in the above equation we obtain

$$r = 2nag(e_i,e_i) + ag(\xi,\xi) + b\eta(\xi)\eta(\xi) = (2n+1)a + b. \tag{2.15}$$

Also from (2.12) we get

$$S(\xi,\xi) = a + b.$$

On the other hand, we have

$$S(\xi,\xi) = -2n + |h|^2 = -2n + \frac{1}{4}|\tau|^2 = 2n(\frac{|\tau|^2}{8n} - 1). \tag{2.16}$$

Two above equations show that

$$a + b = 2n(\frac{|\tau|^2}{8n} - 1). \tag{2.17}$$

Using (2.15) and (2.17) we have the assertion. □

Corollary 2.7. *Let (M, ϕ, ξ, η) be a paracontact metric manifold η-Einstein of dimension three. Then the curvature tensor has the form*

$$R(X,Y)Z = (3(-1+\frac{|\tau|^2}{8}) - \frac{r}{2})(g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) + (\frac{r}{2} + 2(1-\frac{|\tau|^2}{8}))(g(Y,Z)X - g(X,Z)Y).$$

Proof . On a 3-dimensional pseudo-Riemannian manifold, since the conformal curvature tensor vanishes identically, therefore the curvature tensor R takes the form

$$R(X,Y)Z = g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y - \frac{r}{2}((g(Y,Z)X - g(X,Z)Y)) \tag{2.18}$$

where $S(X,Y) = g(QX,Y)$. Using (2.12) we obtain

$$QX = aX + b\eta(X)\xi. \tag{2.19}$$

Setting (2.12) and (2.19) in the relation (2.18) we deduce

$$R(X,Y)Z = (2a - \frac{r}{2})[g(Y,Z)X - g(X,Z)Y] + b[g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y].$$

Thus according to Proposition 2.6 we have (2.18). □

Proposition 2.8. *Let (M, ϕ, ξ, η) be a 3-dimensional paracontact Riemannian manifold, then*

$$(\nabla_\xi \tau)(X,Y) = -S(X,Y) - S(\phi X, \phi Y) + \eta(Y)S(X,\xi) + \eta(X)S(Y,\xi) - \eta(X)\eta(Y)S(\xi,\xi), \qquad \forall X,Y \in \Gamma(TM).$$

Proof . Let $\{e, \phi e, \xi\}$ is an arbitrary ϕ-basis. Using (2.9) and (2.11) we have

$$K(\xi, \phi e) - K(\xi, e) = (\nabla_\xi \tau)(e,e).$$

From the above relation we get

$$S(\phi e, \phi e) + S(e,e) = -(\nabla_\xi \tau)(e,e). \tag{2.20}$$

Also we can deduce

$$S(\phi(e+e'), \phi(e+e')) + S(e+e', e+e') = -(\nabla_\xi \tau)(e+e', e+e'),$$

for e and e' in D. The above equations imply

$$S(\phi e, \phi e') + S(e,e') = -(\nabla_\xi \tau)(e,e'). \tag{2.21}$$

Since $X, Y \in TM$ therefore $\phi X, \phi Y \in D$. Putting $e = \phi X$, $e' = \phi Y$ and $\phi^2 = I - \eta \otimes \xi$ in (2.21) we get

$$S(\phi^2 X, \phi^2 Y) + S(\phi X, \phi Y) = -(\nabla_\xi \tau)(\phi X, \phi Y).$$

Setting $\phi^2 = I - \eta \otimes \xi$ in the above equation we get

$$S(X - \eta(X)\xi, Y - \eta(Y)\xi) + S(\phi X, \phi Y) = -(\nabla_\xi \tau)(X,Y).$$

This completes proof. □

Proposition 2.9. *In a 3-dimensional paracontact manifold M if we define the tensor S_1 as $S_1 = S - ag - b\eta \otimes \eta$, then the following identities hold:*

$$|S_1|^2 = 2|\sigma|^2 + \frac{1}{4}|\nabla_\xi \tau|^2, \tag{2.22}$$

$$< S_1, \nabla_\xi \tau > = < S, \nabla_\xi \tau > = -\frac{1}{2}|\nabla_\xi \tau|^2, \tag{2.23}$$

where $\sigma = S(\xi, .)_D$ and $a = \frac{r}{2} + 1 - \frac{|\tau|^2}{8}$, $b = -\frac{r}{2} - 3 + \frac{3}{8}|\tau|^2$.

Proof . Let $\{e, \phi e, \xi\}$ be a ϕ-basis. If we put $e' = \phi e$ in (2.21), then we get

$$S(e, \phi e) = -\frac{1}{2}(\nabla_\xi \tau)(e, \phi e). \tag{2.24}$$

Using (2.14) and (2.20) we have

$$r = S(e, e) - S(\phi e, \phi e) + S(\xi, \xi) = 2S(e, e) + S(\xi, \xi) + (\nabla_\xi \tau)(e, e).$$

Therefore we deduce

$$S(e, e) = \frac{r}{2} + 1 - \frac{|\tau|^2}{8} - \frac{1}{2}(\nabla_\xi \tau)(e, e). \tag{2.25}$$

Again (2.20) and the above equation give us

$$S(\phi e, \phi e) = -\frac{r}{2} - 1 + \frac{|\tau|^2}{8} - \frac{1}{2}(\nabla_\xi \tau)(e, e). \tag{2.26}$$

On the other hand, we can write

$$|S|^2 = S(e, e)^2 + S(\phi e, \phi e)^2 + S(\xi, \xi)^2 + 2S(e, \phi e)^2 + S(\xi, e)^2 + S(\xi, \phi e).$$

Setting (2.24), (2.25) and (2.26) in the above equation we have

$$|S|^2 = 4(\frac{|\tau|^2}{8} - 1)^2 + \frac{1}{2}(r + 2 - \frac{|\tau|^2}{4})^2 + \frac{1}{4}|\nabla_\xi \tau|^2 + 2|\sigma|^2. \tag{2.27}$$

Using the formula of S_1 and (2.12) it is easy to see that

$$|S_1|^2 = |S|^2 - 4(\frac{|\tau|^2}{8} - 1)^2 - \frac{1}{2}(r + 2 - \frac{|\tau|^2}{4})^2. \tag{2.28}$$

Replacing (2.27) in The last equation we get (2.22). For the proof of (2.23), we consider

$$< S_1, \nabla_\xi \tau > = < S, \nabla_\xi \tau > - a < g, \nabla_\xi \tau > - b < \eta \otimes \eta, \nabla_\xi \tau > .$$

Using (2.3) and $g(\phi e, \phi e) = -g(e, e)$, the last equation reduces to

$$< S_1, \nabla_\xi \tau > = < S, \nabla_\xi \tau > .$$

Applying Lemma (2.1), (2.20) and (2.24) in the above relation we get

$$\begin{aligned} < S, \nabla_\xi \tau > &= < S, \nabla_\xi \tau > (e, e) + < S, \nabla_\xi \tau > (\phi e, e) \\ &\quad + < S, \nabla_\xi \tau > (e, \phi e) + < S, \nabla_\xi \tau > (\phi e, \phi e) \\ &= \nabla_\xi \tau(e, e)\{S(e, e) + S(\phi e, \phi e)\} + 2S(e, \phi e)\nabla_\xi \tau(e, \phi e) \\ &= -\frac{1}{2} < \nabla_\xi \tau, \nabla_\xi \tau > . \end{aligned}$$

This completes the proof. □

Two above propositions give us the following theorem:

Theorem 2.10. Let M be a 3-dimensional paracontact Riemannian manifold. Then $\sigma = S(\xi, \cdot)_D = 0$ if and only if the Ricci tensor is given by

$$S = -\frac{1}{2}\nabla_\xi \tau + (\frac{r}{2} + 1 - \frac{|\tau|^2}{8})g + (-\frac{r}{2} - 3 + \frac{3}{8}|\tau|^2)\eta \otimes \eta.$$

From Theorem 2.10 we can conclude the following proposition.

Proposition 2.11. *Let M be a 3-dimensional paracontact Riemannian manifold. If $l = 0$. Then $\sigma = 0$ if and only if the Ricci tensor is given by*

$$S = \frac{r}{2}(g - \eta \otimes \eta). \tag{2.29}$$

Proof . Let $\{e, \phi e, \xi\}$ be an arbitrary ϕ-basis, and $\sigma = 0$. Thus according to Theorem 2.10 we have

$$S = -\frac{1}{2}\nabla_\xi \tau + (\frac{r}{2} + 1 - \frac{|\tau|^2}{8})g + (-\frac{r}{2} - 3 + \frac{3}{8}|\tau|^2)\eta \otimes \eta. \tag{2.30}$$

since $l = 0$, therefore

$$trl = S(\xi, \xi) = 0.$$

On the other hand, setting $n = 1$ in (2.16) we get

$$S(\xi, \xi) = 2(\frac{|\tau|^2}{8} - 1).$$

Two above equations gives us $|\tau|^2 = 8$. Also from $l = 0$ we have $\nabla_\xi l = 0$. Thus according to Proposition 2.10 we deduce $\nabla_\xi \tau = 0$. Replacing $|\tau|^2 = 8$ and $\nabla_\xi \tau = 0$ in (2.30) we obtain (2.29). □

Lemma 2.12. *Let (M^3, ξ, η, g) be a paracontact metric manifold with $l = 0$. If $R(X, \xi)S = 0$, then $\sigma = S(\xi, .)_D = 0$.*

Proof . Let $\{e, \phi e, \xi\}$ be a ϕ-basis, and $\sigma_x \neq 0$ in some point x. Also we suppose that $e \in T_xM$ be a vector field with $|e| = 1$ such that $\sigma_x(e) = S(\xi, e) = 0$ and $|\sigma| = \sigma_x(\phi(e)) = S(\xi, \phi e) \neq 0$. Setting $X = Z = e$ and $y = \xi$ in Corollary 2.7 we get

$$R(e, \xi)e = S(\xi, e)e - S(e, e)\xi + \eta(e)Q(e) - g(e, e)Q(\xi) - \frac{r}{2}(\eta(e)e - g(e, e)\xi).$$

Since $g(e, e) = 1$ and $\eta(e) = 0$, from the last relation we deduce

$$R(e, \xi)e = -S(e, e)\xi - Q(\xi) + \frac{r}{2}\xi.$$

On the other hand, according assumption since $R(X, \xi)S = 0$, thus we get

$$S(R(e, \xi)e, \xi) = 0.$$

From the last two equations we obtain

$$S(Q(\xi), \xi) = S(\xi, \xi)\{\frac{r}{2} - S(e, e)\}.$$

But, since $trl = S(\xi, \xi) = 0$ thus

$$S(Q(\xi), \xi) = 0. \tag{2.31}$$

Moreover, we have (sraeily)

$$Q\xi = (\alpha + \beta)\xi + \sigma(e)e + \sigma(\phi e)\phi e.$$

Therefore

$$S(Q\xi, \xi) = (\alpha + \beta)S(\xi, \xi) + S(\sigma(e)e + \sigma(\phi e)\phi e, \xi). \tag{2.32}$$

Using (2.31), (2.32) and $S(e, \xi) = 0 = S(\xi, \xi)$ we obtain

$$0 = S(|\sigma|\phi e, \xi) = |\sigma|^2.$$

Therefore $\sigma = 0$. we have the assertion. □

Corollary 2.13. Let (M^3, ξ, η, g) be a paracontact metric manifold satisfies in the condition $l = 0$ and $R(X, \xi)S = 0$. Then the Ricci tensor has the form

$$S = \frac{r}{2}(g - \eta \otimes \eta). \tag{2.33}$$

3. Semi-symmetric paracontact Riemannian manifold

A Riemannian manifold M is locally symmetric if its curvature tensor R satisfies $\nabla R = 0$, where ∇ is Levi-Civita connection of the Riemannian metric, and a Riemannian manifold M is said to be semi-symmetric if its curvature tensor R satisfies

$$R(X, Y)R = 0, \quad \forall X, Y \in \Gamma(TM)$$

where $R(X, Y)$ acts on R as a derivation.

Proposition 3.1. *Let (M^{2n+1}, ξ, η, g) be a paracontact metric manifold with $n > 1$ satisfies in the condition $R(X, \xi)R = 0$. If $0 \neq l = \kappa\phi^2$, then M is para Sasakian manifold of constant curvature -1.*

Proof . From $R(X, Y)R = 0$ we have

$$\begin{aligned} &R(X, \xi)R(Y, Z)V - R(R(X, \xi)Y, Z)V - R(Y, R(X, \xi)Z)V \\ &\quad -R(Y, Z)R(X, \xi)V = 0, \qquad \forall Y, Z, V \in \Gamma(TM). \end{aligned} \tag{3.1}$$

Replacing $Z = V = \xi$ in the above equation, we deduce

$$R(X, \xi)lY - lR(X, \xi)Y - R(Y, lX)\xi - R(Y, \xi)lX = 0. \tag{3.2}$$

On the other hand, from (2.1) we get

$$lX = \kappa\phi^2 X = \kappa(X - \eta(X)\xi). \tag{3.3}$$

Setting (3.3) in (3.2) gives us

$$R(X, Y)\xi + R(\xi, Y)X = \kappa\{\eta(Y)X - 2\eta(X)Y + g(X, Y)\xi\}.$$

Using the first Bianchi identity and then altering X and Y, we have

$$2R(Y, X)\xi + R(\xi, Y)X = \kappa\{\eta(X)Y - 2\eta(Y)X + g(X, Y)\xi\}.$$

Substracting two last equation we derive

$$R(X, Y)\xi = \kappa(\eta(Y)X - \eta(X)Y).$$

Applying $V = \xi$ in (3.1) and using the above relation we obtain

$$R(Y, Z)X = \kappa(g(X, Z)Y - g(X, Y)Z).$$

Since $\kappa \neq 0$ therefore we can deduce $\kappa = const = -1$. i.e M is para Sasakian. □

Proposition 3.2. *Let (M^3, ξ, η, g) be a paracontact metric manifold with $R(X, \xi)R = 0$. If $l = \kappa\phi^2$, then M is either flat or of constant sectional curvature -1.*

Proof . Let $l = 0$, thus setting $Y = \xi$ in (2.18) we get

$$Q(X) = -\frac{r}{2}\eta(X)\xi + \eta(X)Q(\xi) + \frac{r}{2}X. \tag{3.4}$$

Replacing (2.33) and (3.4) in (2.18) we obtain

$$R(X, \xi)Z = -g(X, Y)Q(\xi) + \eta(X)\eta(Z)Q(\xi) = g(\phi X, \phi Z). \tag{3.5}$$

Similarly we can obtain

$$R(\xi, Z)X = -g(\phi X, \phi Z). \tag{3.6}$$

On the other hand, from the first identity of Bianchi, we have

$$R(X, \xi)Z + R(\xi, Z)X + R(Z, X)\xi = 0. \tag{3.7}$$

From (3.5), (3.6) and (3.7) we get

$$R(Z, X)\xi = 0.$$

Therefore locally M^3 is the product of a flat 2-dimensional manifold and 1-dimensional manifold of negative constant curvature equal -4.
Now, if $l \neq 0$ then similar to the proof of Theorem 3.1 we can deduce M has constant curvature -1. □

Definition 3.3. A para Sasakian manifold is said to be Einstein-para Sasakian if the Ricci tensor S is of the form $S(X, Y) = \lambda g(X, Y)$, where λ is a constant.

Theorem 3.4. *Let M be a $(2n + 1)$-dimension paracontact Riemanniann manifold satisfying*

$$R(X, \xi).S = 0 \quad (n \geq 2).$$

If

$$R(X, Y)\xi = \kappa\{\eta(Y)X - \eta(X)Y\}, \tag{3.8}$$

then either M is locally as the product of a flat $(n + 1)$-dimensional manifold and n-dimensional manifold of negative constant curvature -4 or M is a Einstein manifold.

Proof . If $k = 0$, then we have $R(X, Y)\xi$. Thus from Theorem 3.3 of [10] the first part of the theorem is proved. Now let $k \neq 0$. The condition $R(X, \xi).S = 0$ gives us

$$S(R(X, \xi)Y, Z) = -S(Y, R(X, \xi)Z), \quad \forall X, Y, Z \in \Gamma(TM).$$

Replacing $Z = \xi$ in the above relation we get

$$S(R(X, \xi)Y, \xi) = -S(Y, lX). \tag{3.9}$$

On the other hand, Since M is para Sasakian manifold thus we have

$$R(X, Y)\xi = -\{\eta(Y)X - \eta(X)Y\}, \tag{3.10}$$

Applying (3.10) we deduce

$$S(X,\xi) = \sum_{i=1}^{2n+1} g(R(e_i, X)\xi, e_i) = -2n\eta(X). \tag{3.11}$$

From (3.9) and (3.11) we have

$$S(lX, Y) = -2n\eta(R(X,\xi)Y).$$

that is,

$$S(lX, Y) = 2ng(lX, Y).$$

Setting $Y = \xi$ in (3.10) we get

$$lX = -X + \eta(X)\xi.$$

Two above equations give us

$$S(X, Y) = -2ng(X, Y).$$

Therefore $S = -2ng$. □

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