Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 207-218
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.307

Existence of solution and solving the integro-differential equations system by the multi-wavelet Petrov-Galerkin method

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(Communicated by M.B. Ghaemi)

Abstract

In this paper, we discuss about existence of solution for integro-differential system and then we solve it by using the Petrov-Galerkin method. In the Petrov-Galerkin method choosing the trial and test space is important, so we use Alpert multi-wavelet as basis functions for these spaces. Orthonormality is one of the properties of Alpert multi-wavelet which helps us to reduce computations in the process of discretizing and we drive a system of algebraic equations with small dimension which it leads to approximate solution with high accuracy. We compare the results with similar works and it guarantees the validity and applicability of this method.

Keywords: System of Integro-differential equations; Multi-wavelet; Petrov-Galerkin; Regular pairs; Trial space; Test space.
2010 MSC: Primary 45J05, 65L60, 65T60; Secondary 42C05, 34K28.

1. Introduction and preliminaries

Construction and applications of multi-wavelet have been explained in [2] such that these bases are orthonormal and also in [5, 9] the Petrov-Galerkin method has been used for solving integro-differential equations and in [4, 8] some integro-differential equations have been solved by using semiorthogonal spline wavelet. Some integro-differential system are solved in [1, 5] such that in [5] convergence of the Petrov-Galerkin method has been discussed with some restrictions on degrees of polynomial basis, but these restrictions removed in [9] and convergence is held for every degree of polynomial basis. Existence of solution for integro-differential equations or system of integro-differential equations has

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Received: December 2014 *Revised:* September 2015

not been discussed in the previous research articles, (see [11, 12, 13]). But we introduce a process for proof of existence of solution. In [12] the Tau method has introduced an approximate solution of integro-differential equations. In [6], six order compact finite difference method is given for solving second order integro-differential equation with different boundary condition, also it solved a system of integro-differential equation. This paper by proper choosing of test and trial space, the purposed method leads to diagonal matrices, so we obtain a final system with less dimension and computation. For validity and applicability the above proposed method we compare our results with[12, 13].

Consider the integro-differential equations system as follows:

$$\begin{aligned} \sum_{q=0}^{m}[\alpha_q(t)u_1^q(t)+\beta_q(t)u_2^q(t)]-\sum_{i=1}^{2}\int_0^1 k_i(t,s)u_i(s)ds=f_1(t), \\ \sum_{q=0}^{m}[\gamma_q(t)u_1^q(t)+\delta_q(t)u_2^q(t)]-\sum_{i=1}^{2}\int_0^1 k_{i+2}(t,s)u_i(s)ds=f_2(t), \\ u_1(0)=\alpha_1\ ,\ u_1(1)=\eta_1, \\ u_2(0)=\alpha_2\ ,\ u_2(1)=\eta_2, \end{aligned} \tag{1.1}$$

where q is differential order and $f_i(t)\in L^2[0,1], i=1,2$ and

$$k_i(t,s)\in L^2[0,1]^2, i=1,2,3,4,$$

are known functions and the $u_i(t), i=1,2$ are unknown functions which must be determined. For discussing about existence of solution of equations system (1.1) in conditions of the Petrov-Galerkin method we need operator form of equations system (1.1), so we suppose that $X=L^2[0,1]$, and $\kappa_i : X\to X$ for $i=1,2,3,4,$ then integral operators form are as follows

$$\kappa_i(u_j)=\int_o^1 k_i(t,s)u_j(s)ds\ ,\ i=1,2,3,4\ ,\ j=1,2 \tag{1.2}$$

where κ_i for $i=1,2,3,4,$ are linear compact operators. Also we assume that

$$D_i : X\to X\ ,\ i=1,2,3,4,$$

are linear compact operators such that

$$\begin{aligned} D_1(u_1)=\sum_{q=0}^{m}\alpha_q(t)u_1^q(t), \quad D_2(u_2)=\sum_{q=0}^{m}\beta_q(t)u_2^q(t), \\ D_3(u_1)=\sum_{q=0}^{m}\gamma_q(t)u_1^q(t), \quad D_4(u_2)=\sum_{q=0}^{m}\delta_q(t)u_2^q(t), \end{aligned}$$

without loos of generality, we can write equations system (1.1) in the following operator system:

$$\begin{aligned} u_1-(\kappa_1u_1+\kappa_2u_2-D_1u_1-D_2u_2)=f_1, \\ u_2-(\kappa_3u_1+\kappa_4u_2-D_3u_1-D_4u_2)=f_2. \end{aligned} \tag{1.3}$$

2. Existence of solution

By considering the above mentioned operator system, we introduce operators κ and D in the following form
$\kappa = \begin{pmatrix} \kappa_1 & \kappa_2 \\ \kappa_3 & \kappa_4 \end{pmatrix}$, $D = \begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}$
with suppose $\hat{\kappa_i} = \kappa_i - D_i$, we can write $\hat{\kappa} = \kappa - D$, such that $\hat{\kappa}$ operator can be introduced by $\hat{\kappa} : L^2[0,1]^2 \to L^2[0,1]^2$, so from (1.3) and the above explanations we have,
$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - \hat{\kappa} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$.
By choosing $U = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $F = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ the operator form of Eq. (1.3) is given by Eq. (2.1),

$$(I - \hat{\kappa})\, U = F. \tag{2.1}$$

In the case of one dimensional in Eq. (2.1), if $\hat{\kappa}$ is an compact operator and it does not have 1 as an eigenvalue, so has a unique solution $u \in X = L^2[0,1]$, (see [3, 5]). But for Eq. (2.1) in the case of two-dimensional we must prove $\hat{\kappa}$ is a compact operator and $(I - \hat{\kappa})$ is invertible. So we suppose $\{U_n\}_{n\geq 1} = \left\{ \begin{matrix} u_{1,n} \\ u_{2,n} \end{matrix} \right\}_{n\geq 1}$, is a bounded sequence in $L^2[0,1]^2$ with a suitable norm such as

$$\|U_n\| = max\left(\|u_{1,n}\|, \|u_{2,n}\|\right), \quad \|x(t)\|_2 = \left(\int_0^1 x(t)^2 dt\right)^{\frac{1}{2}}, \tag{2.2}$$

so, $\{u_{i,n}\}_{n\geq 1}$ for $i = 1,2$ are bounded sequences, on the other hand because κ_i and D_i for $i = 1,2,3,4$ are assumed linear compact operator in $L^2[0,1]$ then $\hat{\kappa_i}s$ for i=1,2,3,4 are linear compact operator. So, every one of the following sequences has a convergence subsequence in $L^2[0,1]$,

$\{\hat{\kappa_1} u_{1,n}\}_{n\geq 1}, \{\hat{\kappa_2} u_{2,n}\}_{n\geq 1}, \{\hat{\kappa_3} u_{1,n}\}_{n\geq 1}, \{\hat{\kappa_4} u_{2,n}\}_{n\geq 1}$, $\{\hat{\kappa_1} u_{1,n} + \hat{\kappa_2} u_{2,n}\}$ and $\{\hat{\kappa_3} u_{1,n} + \hat{\kappa_4} u_{2,n}\}$, then $\{\hat{\kappa} U_n\} = \left\{ \begin{matrix} \hat{\kappa_1} u_{1,n} + \hat{\kappa_2} u_{2,n} \\ \hat{\kappa_3} u_{1,n} + \hat{\kappa_4} u_{2,n} \end{matrix} \right\}_{n\geq 1}$,

has a convergence subsequence in $L^2[0,1]^2$, so $\hat{\kappa}$ is a linear compact operator. Also $I - \hat{\kappa}$ operator introduce as,
$I - \hat{\kappa} : L^2[0,1]^2 \to L^2[0,1]^2$,
where $I_{2\times 2} - \hat{\kappa} = \begin{pmatrix} I_{1\times 1} - \hat{\kappa_1} & -\hat{\kappa_2} \\ -\hat{\kappa_3} & I_{1\times 1} - \hat{\kappa_4} \end{pmatrix}$ and $I_{2\times 2}$ is an identity two-dimensional operator and $I_{1\times 1}$ is an identity one-dimensional operator, so similar to one dimensional in [5] and concept of eigenvalues for $I - \hat{\kappa}$ we can rewrite $(I_{2\times 2} - \hat{\kappa})\, U = \lambda U$, so

$$\begin{pmatrix} 1 - \lambda_1 - \lambda & -\lambda_2 \\ -\lambda_3 & 1 - \lambda_4 - \lambda \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0,$$

where λ is an eigenvalue of $I - \hat{\kappa}$ and $\lambda_i s$ are eigenvalues of $\hat{\kappa_i} s$ for $i = 1,2,3,4$. So we have $\lambda^2 - (2 - \lambda_1 - \lambda_4)\lambda + (1 - \lambda_1)(1 - \lambda_4) - \lambda_2\lambda_3 = 0$ thus for obtaining $\lambda \neq 0$ for $(I - \hat{\kappa})$ and invertibility of the $I - \hat{\kappa}$ operator ,we assume there is a relation among eigenvalues of $\kappa_i s$ operators for $i = 1,2,3,4$ as follows:

$$(1 - \lambda_1)(1 - \lambda_4) - \lambda_2\lambda_3 \neq 0, \tag{2.3}$$

thus $(I - \hat{\kappa})$ is invertible, so Eq. (2.1) and equations system (1.1) have a unique solution as $u \in L^2[0,1]^2$.

3. Multi-wavelet Bases

We have introduced the multi-wavelet bases constructed by Alpert in [2] for $L^2[0,1]$. The interval $[0,1]$ is divided to 2^m subinterval where $f(x)$ has degree less than k in all subintervals. The set S_m^k is introduced as:

$$S_m^k=\left\{f|f=\begin{cases} Polynomial\ with\ degree<k, & \frac{n}{2^m}<t<\frac{n+1}{2^m}, 0\le n\le 2^m-1\\ 0, & \text{Otherwise}\end{cases}\right\}.$$

It is obvious that $dim(S_m^k)=2^m k$ and $S_0^k\subset S_1^k\subset\cdots$. Assume that R_m^k is the complement of S_m^k in S_{m+1}^k, then

$$S_{m+1}^k=S_m^k\oplus R_m^k,\quad S_m^k\perp R_m^k.$$

If we assume that $h_1,h_2,\dots,h_k:\mathbb{R}\to\mathbb{R}$ are the orthogonal basis functions for R_0^k, then

$$R_0^k\subset R_1^k\subset\cdots.$$

In [2] by orthonormalizing $h_1,h_2,\dots,h_k$, a basis for space R_m^k is obtained as:

$$\begin{aligned} R_m^k &= linear\ span\Big\{h_{j,m}^n|\ h_{j,m}^n(t)=2^{\frac{m}{2}}h_j(2^m t-n),\\ &\qquad j=1,2,\dots,k,\ n=0,1,\dots,2^m-1\Big\}.\end{aligned}$$

On the other hand,

$$S_m^k=S_{m-1}^k\oplus R_{m-1}^k=S_0^k\oplus_{p=0}^{m-1}R_p^k,$$

for example if $\{p_1,p_2,\dots,p_k\}$ are the Legendre orthonormal polynomials on $[0,1]$, then

$$\begin{aligned} S_m^k &= span\ B_k\\ &= span\Big\{\{p_1,\dots,p_k\}\\ &\quad \cup\{h_{j,p}^n|\ p=0,1,\dots,m-1,\ \ n=0,1,\dots,(2^p-1),\ \ j=1,2,\dots,k\}\Big\}\\ &= span\{b_j\}_{j=1}^{2^m k}.\end{aligned}$$

By choosing $S^k=\cup_{m=0}^{\infty}S_m^k$ then $\overline{S^k}=L^2[0,1]$ and B_k is known as the multi-wavelet basis of order k for $L^2[0,1]$ (see [2, 9]).

Now we choose trial and test spaces by the multi-wavelet bases and introduce B matrices: If

$$\begin{cases} X_n=S_0^2=span\{1,\sqrt{3}(2t-1)\},\\ Y_n=S_1^1=span\left\{1,\begin{cases}-1, & 0<t<\frac{1}{2}\\ 1, & \frac{1}{2}<t<1\end{cases}\right\},\end{cases}$$

then

$$B_{2\times2}=\begin{pmatrix}1 & 0\\ 0 & \frac{\sqrt{3}}{2}\end{pmatrix}.$$

If

$$\begin{cases} X_n=S_0^4=span\{1,\sqrt{3}(2t-1),\sqrt{5}(6t^2-6t+1),\sqrt{7}(20t^3-30t^2+12t-1)\},\\ Y_n=S_1^2=span\left\{1,\sqrt{3}(2t-1),\begin{cases}\sqrt{3}(-4t+1), & 0<t<\frac{1}{2}\\ \sqrt{3}(4t-3), & \frac{1}{2}<t<1\end{cases},\begin{cases}6t-1, & 0<t<\frac{1}{2}\\ 6t-5, & \frac{1}{2}<t<1\end{cases}\right\},\end{cases}$$

then

$$B_{4\times 4}=\begin{pmatrix} 1 & & & 0\\ & 1 & & \\ & & \frac{\sqrt{15}}{4} & \\ 0 & & & \frac{\sqrt{7}}{4}\end{pmatrix}.$$

If $X_n=S_0^6$ and $Y_n=S_1^3$ then

$$B_{6\times 6}=\begin{pmatrix} 1 & & & & & 0\\ & 1 & & & & \\ & & 1 & & & \\ & & & \frac{3\sqrt{7}}{8} & & \\ & & & & \frac{\sqrt{3}}{2} & \\ 0 & & & & & \frac{\sqrt{55}}{64}\end{pmatrix}.$$

$$\begin{aligned}
X_n=S_1^2 &= span\Big\{1,\sqrt{3}(2t-1),\begin{cases}\sqrt{3}(-4t+1), & 0<t<\frac{1}{2}\\ \sqrt{3}(4t-3), & \frac{1}{2}<t<1\end{cases},\begin{cases}6t-1, & 0<t<\frac{1}{2}\\ 6t-5, & \frac{1}{2}<t<1\end{cases}\Big\},\\
Y_n=S_2^1 &= span\Big\{1,\begin{cases}-1, & 0<t<\frac{1}{2}\\ 1, & \frac{1}{2}<t<1\end{cases},\begin{cases}-\sqrt{2}, & 0<t<\frac{1}{4}\\ \sqrt{2}, & \frac{1}{4}<t<\frac{1}{2}\end{cases},\begin{cases}-\sqrt{2}, & \frac{1}{2}<t<\frac{3}{4}\\ \sqrt{2}, & \frac{3}{4}<t<1\end{cases}\Big\},\\
B_{4\times 4} &= \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & \frac{\sqrt{3}}{2} & \frac{\sqrt{6}}{8} & \frac{\sqrt{6}}{8}\\ 0 & 0 & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4}\\ 0 & -\frac{1}{2} & \frac{3\sqrt{2}}{8} & \frac{3\sqrt{2}}{8}\end{pmatrix},
\end{aligned}$$

where $B^T.B=D$, which D is diagonal matrix with the positive elements. Thus $(B^T)^{-1}=B\ D^{-1}$ exists. To know how to prove this part which the matrices are $n\times n$ diagonal see [9].

Also if $X_n=S_1^4$ and $Y_n=S_2^2$, then the matrix $B_{8\times 8}$ is the following form:

$$B_{8\times 8}=\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \frac{\sqrt{15}}{4} & 0 & \frac{\sqrt{\frac{15}{2}}}{16} & 0 & \frac{\sqrt{\frac{15}{2}}}{16} & 0\\
0 & 0 & 0 & \frac{\sqrt{7}}{4} & -\frac{5\sqrt{\frac{21}{2}}}{32} & \frac{\sqrt{\frac{7}{2}}}{32} & \frac{5\sqrt{\frac{21}{2}}}{32} & \frac{\sqrt{\frac{7}{2}}}{32}\\
0 & 0 & -\frac{2}{\sqrt{85}} & 0 & \frac{3\sqrt{\frac{5}{34}}}{2} & -\frac{7\sqrt{\frac{3}{170}}}{4} & \frac{3\sqrt{\frac{5}{34}}}{2} & \frac{7\sqrt{\frac{3}{170}}}{4}\\
0 & 0 & 0 & -\frac{2}{\sqrt{21}} & -\frac{5}{8\sqrt{14}} & \frac{\sqrt{\frac{21}{2}}}{8} & \frac{5}{8\sqrt{14}} & \frac{\sqrt{\frac{21}{2}}}{8}\\
0 & 0 & \frac{\sqrt{\frac{21}{85}}}{4} & 0 & -\frac{3\sqrt{\frac{105}{34}}}{16} & -\sqrt{\frac{14}{85}} & -\frac{3\sqrt{\frac{105}{34}}}{16} & \sqrt{\frac{14}{85}}\\
0 & 0 & 0 & \frac{5\sqrt{\frac{5}{21}}}{4} & \frac{23\sqrt{\frac{5}{14}}}{32} & \frac{\sqrt{\frac{105}{2}}}{32} & -\frac{23\sqrt{\frac{5}{14}}}{32} & \frac{\sqrt{\frac{105}{2}}}{32}
\end{pmatrix}.$$

4. Conditions of the Petrov-Galerkin Method

In the case of one dimensional integro-differential equations, conditions of the Petrov-Galerkin and their theorems have been proved in [9], we try to expand them for a system of integro-differential equations. So for $X = L^2[0,1]$ we consider an operator form of integro-differential equations as follows

$$(I - \hat{\kappa})\, u = f. \tag{4.1}$$

In this way, three conditions of the Petrov-Galerkin method are:

1. X is Banach space and $\hat{\kappa}$ is a compact linear operator it doesn't have 1 as an eigenvalue.
2. For $n \in N$, $X_n \subset X$, $Y_n \subset X^*$, $dimX_n = dimY_n < \infty$, i.e. X_n, Y_n are finite dimensional subsets of X, X^* sequentially.
3. H-Condition: For each $x \in X$ and $y \in X^*$, there exists $x_n \in X_n$ and $y_n \in Y_n$ such that $\|x_n - x\| \to 0$ and $\|y_n - y\| \to 0$ as $n \to +\infty$.

The Petrov-Galerkin method for equation (4.1) is a numerical method for finding $u_n \in X_n$ such that:

$$< u_n - \hat{\kappa}_n, y_n >=< f, y_n >, \qquad \forall y_n \in Y_n. \tag{4.2}$$

Definition 4.1. *For linear operator $P_n : X \to X_n$, given $x \in X$, an element $P_n x \in X_n$ is called a generalized best approximation from X_n to x with respect to Y_n if it satisfies the equation*

$$< x - P_n x, y_n >= 0, \;\; \forall y_n \in Y_n,$$

similarly, given $y \in X^*$, an element $P_n' y \in Y_n$ is called a generalized best approximation from Y_n to y with respect to X_n if it satisfies the equation

$$< x_n, y - P_n' y >= 0, \;\; \forall x_n \in X_n.$$

Above three conditions with Definition 4.1 illustrate that in the Petrov-Galerkin method, we are looking for the generalized best approximation $x \in X$ with respect to Y_n in [5]. Also Eq. (4.2) is equivalent to

$$u_n - P_n \hat{\kappa} u_n = f. \tag{4.3}$$

This equation indicates that the Petrov-Galerkin method is a projection method. By assuming $X = L^2[0,1]^2$ in the case of two-dimensional and Eq. (2.1), and by considering definition of operator $\hat{\kappa}$, we show three conditions of the Petrov-Galerkin are held. Condition 1 is held because we proved $\hat{\kappa}$ is linear and compact operator. For invertibility we construct a condition on eigenvalues of $\hat{\kappa}$ as follows,

$$(1 - \lambda_1^p)\,(1 - \lambda_4^q) \neq \lambda_2^r \lambda_3^f, \tag{4.4}$$

where λ_i's are eigenvalues of $\hat{\kappa_i}$ operators for $i = 1, 2, 3, 4$ and p, q, r and f are number of eigenvalue of $\hat{\kappa_1}, \hat{\kappa_2}, \hat{\kappa_3}$ and $\hat{\kappa_4}$ respectively.

For condition 2 we choose: $X^2 = L^2[0,1]^2$, $X_n^2 = \begin{pmatrix} S_m^k \\ S_m^k \end{pmatrix}$, and also $X^{*2} = L^2[0,1]^2$ (see [7]), and $Y_n^2 = \begin{pmatrix} S_{m'}^{k'} \\ S_{m'}^{k'} \end{pmatrix} \subset X^{*2}_{\;n}$, where $S_m^k, S_{m'}^{k'}$ have been introduced in [9]. Of course X_n in [9] is one dimensional but in this paper X_n^2 is two dimensional, also according to S_m^k in [9] dim $X_n^2 = 2(2^m k) =$dim $Y_n^2 \prec \infty$, so condition 2 is satisfied. By considering $X^2 = L^2[0,1]^2$ and $X^{*2} = L^2[0,1]^2$ condition 3 holds.

5. Solution of the Integro-Differential Equations System

Again we consider to the equations system (1.1):

we suppose $u_1(t), u_2(t)$ are linear combination of a basis in $X = L^2[0,1]$, so $U_n = \begin{pmatrix} u_{1,n} \\ u_{2,n} \end{pmatrix}$ is a linear combination of a basis in $X_n^2 \subseteq L^2[0,1]^2$. Regarding to Definition (1) we can write,

$P_n : X^2 = L^2[0,1]^2 = \begin{pmatrix} \overline{\cup_{m=0}^{\infty} S_m^k} \\ \\ \overline{\cup_{m=0}^{\infty} S_m^k} \end{pmatrix} \to X_n^2 = \begin{pmatrix} S_m^k \\ S_m^k \end{pmatrix},$

$P_n(u(t)) = U_n(t) = \begin{pmatrix} \sum_{i=1}^n c_{1i} b_i(t) \\ \sum_{j=1}^n c_{2j} b_j(t) \end{pmatrix}, \ \forall u(t) \in X.$

In other words $\{b_i(t), b_j(t)\}$ is a base for subspace X_n^2 from $L^2[0,1]^2$. P_n is an operator which explained completely in [9]. By substituting $u_1(t), u_2(t)$ in system (1.1) we have

$\sum_{q=0}^m \sum_{i=1}^n \alpha_q(t) c_{1i} b_i^q(t) + \beta_q(t) c_{2i} b_i^q(t) - \int_0^1 \left[\sum_{i,j=1}^n k_{1ij} b_i(t) b_j(s)\right] \left[\sum_{k=1}^n c_{1k} b_k(s)\right] ds$
$- \int_0^1 \left[\sum_{i,j=1}^n k_{2ij} b_i(t) b_j(s)\right] \left[\sum_{k=1}^n c_{2k} b_k(s)\right] ds = \sum_{i=1}^n f_{1i}(t) b_i(t),$
$\sum_{q=0}^m \sum_{i=1}^n \gamma_q(t) c_{1i} b_i^q(t) + \delta_q(t) c_{2i} b_i^q(t) - \int_0^1 \left[\sum_{i,j=1}^n k_{3ij} b_i(t) b_j(s)\right] \left[\sum_{k=1}^n c_{1k} b_k(s)\right] ds$
$- \int_0^1 \left[\sum_{i,j=1}^n k_{4ij} b_i(t) b_j(s)\right] \left[\sum_{k=1}^n c_{2k} b_k(s)\right] ds = \sum_{i=1}^n f_{2i}(t) b_i(t).$

We assume that

$$g_1(t) = \sum_{q=0}^m \alpha_q(t) b_i^q(t), \quad g_2(t) = \sum_{q=0}^m \beta_q(t) b_i^q(t), \tag{5.1}$$
$$g_3(t) = \sum_{q=0}^m \gamma_q(t) b_i^q(t), \quad g_4(t) = \sum_{q=0}^m \delta_q(t) b_i^q(t),$$

with substituting (5.1) in last system we have

$$\sum_{i=1}^n [c_{1i} g_{1i}(t) + c_{2i} g_{2i}(t)] - \left[\sum_{i,j=1}^n k_{1ij} b_i(t) c_{1j} + \sum_{i,j=1}^n k_{2ij} b_i(t) c_{2j}\right] = \sum_{i=1}^n f_{1i} b_i(t),$$
$$\sum_{i=1}^n [c_{1i} g_{3i}(t) + c_{2i} g_{4i}(t)] - \left[\sum_{i,j=1}^n k_{3ij} b_i(t) c_{1j} + \sum_{i,j=1}^n k_{4ij} b_i(t) c_{2j}\right] = \sum_{i=1}^n f_{2i} b_i(t).$$

Now we project $g_{ij}(t)$, $i = 1,2,3,4$ on X_n , so

$$\sum_{i,j=1}^n [c_{1i} g_{1ij} b_i(t) + c_{2i} g_{2ij} b_i(t)] - \left[\sum_{i,j=1}^n (k_{1ij} c_{1j} + k_{2ij} c_{2j}) b_i(t)\right] = \sum_{i=1}^n f_{1i} b_i(t),$$
$$\sum_{i,j=1}^n [c_{1i} g_{3ij} b_i(t) + c_{2i} g_{4ij} b_i(t)] - \left[\sum_{i,j=1}^n (k_{3ij} c_{1j} + k_{4ij} c_{2j}) b_i(t)\right] = \sum_{i=1}^n f_{2i} b_i(t). \tag{5.2}$$

According to the petrov-Galerkin method we multiply both sides of Eq. (5.2) in bases elements of Y_n (Y_n is a dual space of X_n) and from orthonormality of test and trial spaces X_n, Y_n we have

$$\sum_{i,j=1}^{n} c_{1i}g_{1ij} < b_j(t), b_p^*(t) > + \sum_{i,j=1}^{n} c_{2i}g_{2ij} < b_j(t), b_p^*(t) > -$$
$$\left[\sum_{i,j=1}^{n} (k_{1ij}c_{1j} + k_{2ij}c_{2j}) < b_j(t), b_p^*(t) >\right] = \sum_{i=1}^{n} f_{1i} < b_j(t), b_p^*(t) >,$$
$$\sum_{i,j=1}^{n} c_{1i}g_{3ij} < b_j(t), b_p^*(t) > + \sum_{i,j=1}^{n} c_{2i}g_{4ij} < b_j(t), b_p^*(t) > -$$
$$\left[\sum_{i,j=1}^{n} (k_{3ij}c_{1j} + k_{4ij}c_{2j}) < b_j(t), b_p^*(t) >\right] = \sum_{i=1}^{n} f_{2i} < b_j(t), b_p^*(t) >,$$

where $< b_j(t), b_p^*(t) >$, are elements of the B matrix, so the inner product equal to 1 when $j = p$ and it's 0 when $j \neq p$. So, we can write,

$$\sum_{i,j=1}^{n} c_{1i}g_{1ip} + \sum_{i,j=1}^{n} c_{2i}g_{2ip} - \left[\sum_{i,j=1}^{n} (k_{1pj}c_{1j} + k_{2pj}c_{2j})\right] = f_{1p},$$
$$\sum_{i,j=1}^{n} c_{1i}g_{3ip} + \sum_{i,j=1}^{n} c_{2i}g_{4ip} - \left[\sum_{i,j=1}^{n} (k_{3pj}c_{1j} + k_{4pj}c_{2j})\right] = f_{2p}.$$

By using boundary conditions we have following system and $P = 1, 2, ..., n$, instead of two other equations we apply boundary conditions, so

$$\sum_{i=1}^{n} c_{1i}(g_{1pi} - k_{1ip}) + \sum_{i=1}^{n} c_{2i}(g_{2pi} - k_{2ip}) = f_{1p}, \tag{5.3}$$
$$\sum_{i=1}^{n} b_i(0)c_{1i} = \alpha_1, \qquad \sum_{i=1}^{n} b_i(1)c_{1i} = \eta_1,$$
$$\sum_{i=1}^{n} c_{1i}(g_{3pi} - k_{3ip}) + \sum_{i=1}^{n} c_{2i}(g_{4pi} - k_{4ip}) = f_{2p},$$
$$\sum_{i=1}^{n} b_i(0)c_{2i} = \alpha_2, \qquad \sum_{i=1}^{n} b_i(1)c_{2i} = \eta_2.$$

where in the system of (5.3) we have

$$k_{1ij} = \int_0^1\int_0^1 k_1(t,s)b_i(t)b_j(s), \quad k_{2ij} = \int_0^1\int_0^1 k_2(t,s)b_i(t)b_j(s),$$
$$k_{3ij} = \int_0^1\int_0^1 k_3(t,s)b_i(t)b_j(s), \quad k_{4ij} = \int_0^1\int_0^1 k_4(t,s)b_i(t)b_j(s),$$
$$f_{1p} = \int_0^1 f_1(t)b_p(t)dt, \qquad f_{2p} = \int_0^1 f_2(t)b_p(t)dt.$$

Matric form of system (5.3) is the following form

$$B_{n\times n}\begin{pmatrix} g_1^t - k_1 & g_2^t - k_2 \\ g_3^t - k_3 & g_4^t - k_4 \end{pmatrix} C_{n\times 1} = F.$$

6. Application of proposed method

Here before solving systems we are going to show that condition (6) is held. To this end we have chosen some kernels. By solving them it's clear that condition (6) is held. (Other kernels are the same.)

Example 6.1. In this example we solve

$$u_1''(t) - u_2'(t) - \int_0^1 e^{t+s} u_1(s)ds - \int_0^1 (t+s)u_2(s)ds = 2 + e^t - 2e^{t+1} - 2(\frac{1}{3} + \frac{t}{2}),$$

$$u_1' - u_2'' - \int_0^1 e^{ts} u_1(s)ds - \int_0^1 (t+s)^2 u_2(s)ds = 1 + \frac{e^t - 1}{t^2} + \frac{2 - 3e^t}{t} - 2(\frac{1}{4} + \frac{2t}{3} + \frac{t^2}{2}),$$

$$u_1(0) = 0 \ , \quad u_1(1) = 1,$$

$$u_2(0) = 0 \ , \ u_2(1) = 2,$$

with exact solution of $u_1(t) = t$, $u_2(t) = 2t$. At first we show that condition (6) is held for these kernels.

$$k_1(u(s)) = \int_0^1 e^{t+s} u(s)ds, \qquad k_2(u(s)) = \int_0^1 (t+s)u(s)ds,$$

$$k_3(u(s)) = \int_0^1 e^{ts} u(s)ds, \qquad k_4(u(s)) = \int_0^1 (t+s)^2 u(s)ds,$$

eigenvalue for k_i th kernel is $k_i u = \lambda_i u$, $1 \leq i \leq 4$, so for k_1 kernel we have $k_1 u = \lambda_1 u$, also

$$\int_0^1 e^{t+s} u(s)ds = \lambda_1 u(t), \tag{6.1}$$

$$e^t \int_0^1 e^s u(s)ds = \lambda_1 u(t),$$

by choosing $a = \int_0^1 e^s u(s)ds$, we have

$$u(t) = \frac{a}{\lambda_1} e^t, \tag{6.2}$$

and by substituting (6.2) in (6.1) eigenvalue for k_1 is like this $\lambda_1^1 = \frac{e^2-1}{2}$. Also for kernel k_2 we can write
$k_2 u = \lambda_2 u$, so

$$\int_0^1 (t+s)u(s)ds = \lambda_2 u(t), \tag{6.3}$$

we suppose $\int_0^1 u(s)ds = a_1, \int_0^1 su(s)ds = a_2$, so we have

$$a_1 t + a_2 = \lambda_2 u(t), \tag{6.4}$$

and by substituting (6.4) in (6.3), we obtain
$\frac{1}{\lambda_2}\left((\frac{t}{2} + \frac{1}{3})a_1 + \left(t + \frac{1}{2}\right)a_2\right) = ta_1 + a_2,$

by reordering following relation and by knowing that $\{1, t\}$ are independent and a_is are not zero we will reach the following system

$$\begin{pmatrix} \left(\frac{1}{2}-\lambda_2\right) & 1 \\ \frac{1}{3} & \left(\frac{1}{2}-\lambda_2\right) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix},$$

so eigenvalues for kernel κ_2 are

$$\begin{cases} \lambda_2^1 = \frac{1}{2} - \frac{\sqrt{3}}{3}, \\ \lambda_2^2 = \frac{1}{2} + \frac{\sqrt{3}}{3}. \end{cases}$$

Also for kernel k_3 we can write $k_3 u(s) = \lambda_3 u(t)$. So,

$$\int_0^1 e^{ts} u(s) ds = \lambda_3 u(t) \tag{6.5}$$

and according to following expansion

$$e^{st} = 1 + (st) + \frac{(st)^2}{2} + ...,$$

and by choosing $a_i = \frac{t^{i-1}}{i\text{-}1} \int_0^1 s^{i-1} u(s) ds, i = 1, 2, 3,$ in Eq. (6.5) we obtain

$$u(t) = \frac{a_1 + t a_2 + t^2 a_3}{\lambda_3} \tag{6.6}$$

by substituting (6.6) in (6.5) we have

$$\left(a_1 + \frac{a_2}{2} + \frac{a_3}{3} - a_1\lambda_3\right) + t\left(\frac{a_1}{2} + \frac{a_2}{3} + \frac{a_3}{4} - \lambda_3 a_2\right) + t^2\left(\frac{a_1}{6} + \frac{a_2}{8} + \frac{a_3}{10} - \lambda_3 a_3\right) = 0.$$

Because $\{1, t, t^2\}$ are linearly independent so the last system will be

$$\begin{pmatrix} (1-\lambda_3) & \frac{1}{2} & \frac{1}{3} \\ \frac{1}{2} & \left(\frac{1}{3}-\lambda_3\right) & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{8} & \left(\frac{1}{10}-\lambda_3\right) \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

So eigenvalues for k_3 will be produced as:

$$\begin{cases} \lambda_3^1 = 0.1.33628 - i1.4803 \times 10^{-17}, \\ \lambda_3^2 = 0.0952323 + i2.36848 \times 10^{-16}, \\ \lambda_3^3 = 0.00181901 - i2.36848 \times 10^{-16}. \end{cases}$$

Also for kernel k_4 we can write $k_4 u(s) = \lambda_4 u(t)$. So,

$\int_0^1 (t+s)^2 u(s) ds = \lambda_4 u(t),$

$\int_0^1 u(s) ds + t \int_0^1 s u(s) ds + t^2 \int_0^1 \frac{s^2}{2} u(s) ds = \lambda_4 u(t),$

by choosing $a_i = \frac{t^{i-1}}{i\text{-}1} \int_0^1 s^{i-1} u(s) ds, \quad i = 1, 2, 3,$ we can write $u(t) = \frac{a_3 + t a_2 + t^2 a_1}{\lambda_4}$, so

$t^2\left(\frac{1}{3}a_1 + \frac{1}{2}a_2 + a_3 - \lambda_4 a_1\right) + t\left(\frac{1}{2}a_1 + \frac{2}{3}a_2 + a_3 - \lambda_4 a_2\right) + \left(\frac{1}{3}a_1 + \frac{1}{4}a_2 + \frac{1}{3}a_3 - \lambda_4 a_3\right) = 0.$

According to previous, eigenvalues for k_4 are

$$\begin{cases} \lambda_4^1 = 1.433664 + i9.86865 \times 10^{-18}, \\ \lambda_4^2 = 1.106375 + i1.75898 \times 10^{-16}, \\ \lambda_4^3 = 0.00607153 - i1.57898 \times 10^{-16}. \end{cases}$$

Table1. Numerical results for Example 6.1

error of $u_1(t)$ $X_n = S_0^4$ $Y_n = S_1^2$	error of $u_2(t)$ $X_n = S_0^4$ $Y_n = S_1^2$
0	1.26276×10^{-17}
0	1.66533×10^{-16}
0	0
4.44089×10^{-16}	0
0	0
0	0
0	0
0	0
0	0
0	2.22045×10^{-16}
0	0

condition (6) is held for kernels $k_i, 1 \leq i \leq 4$. By using system (5.3) for solving Example 6.1, the absolute errors in some different points are shown in Table 1.

Example 6.2. In this example we solve and compare our method with system of integro-differential equations which was solved in [12, 13]. Results are shown in Table 2. (also this example is solved by the Tau method in [12] and Yousufouglu solved the same example with another method in [13]. Differences are in Table 3. Of course condition (6) is held as before.

$$u_1^{''}(t) + u_2^{'}(t) - \int_0^1 -2tsu_1(s)ds - \int_0^1 6tsu_2(s)ds = 3t^2 + \frac{3t}{10} + 8,$$

$$u_1^{'}(t) + u_2^{''}(t) - 3\int_0^1 (-2t - s^2)u_1(s)ds - 6\int_0^1 (2t + s^2)u_2(s)ds = 21t + \frac{4}{5},$$

$$u_1(0) = 1\ ,\quad u_1(1) = 4, \qquad\qquad u_2(0) = -1\ ,\ u_2(1) = 2,$$

exact solutions are $u_1(t) = 3t^2 + 1$, $u_2(t) = t^3 + 2t - 1$. The absolute errors in some different points are shown in Table 3.

Table 2. Numerical results for Example 6.2 by proposed method

error of $u_1(t)$ $X_n = S_0^4$, $Y_n = S_1^2$ $N = 4$	error of $u_2(t)$ $X_n = S_0^4$, $Y_n = S_1^2$ $N = 4$
1.11022×10^{-16}	1.11022×10^{-16}
4.44089×10^{-16}	2.22045×10^{-16}
4.44089×10^{-16}	2.22045×10^{-16}
2.22045×10^{-16}	0
4.44089×10^{-16}	0
2.22045×10^{-16}	0
4.44089×10^{-16}	0
0	1.11022×10^{-16}
4.44089×10^{-16}	0
0	2.22045×10^{-16}
0	0

Table 3. Numerical results for Example 6.2 by [12, 13]

E$u_1(t)$ in [12] $N=10$	E$u_2(t)$ in [12] $N=10$	E$u_1(t)$ in [13]	E$u_2(t)$ in [13]
1.38237×10^{-13}	8.41984×10^{-14}	0	1.0000014305
1.12493×10^{-13}	6.90537×10^{-14}	0.0768043995	0.9970476627
9.05143×10^{-14}	5.68951×10^{-14}	0.2955164909	0.9553389549
7.75923×10^{-14}	4.71163×10^{-14}	0.6031990051	0.7975912094
6.75932×10^{-14}	3.91115×10^{-14}	0.8844869137	0.4665645361
4.65236×10^{-14}	3.22747×10^{-14}	0.9999994636	0.0000001788
		0.884486439	0.4665646553
		0.6031984091	0.7975909710
		0.2955157161	0.9553378820
		0.768044963	0.9970461726
		0	1.0000000000

7. Conclusion

In this paper, we proved existence of solution for integro-differential system and also we used Alpert multi-wavelet bases to solve system of integro-differential equations. Suitable choices of test and trial space is very important. These orthogonal bases reduce the computations. So final systems have small dimension and enough accuracy. Of course there is not restriction on degrees of chosen polynomial basis.

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