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Strong convergence theorem for a class of multiple-sets split variational inequality problems in Hilbert spaces

Chinedu Izuchukwu

School of Mathematics, Statistics and Computer Science, University of KwaZulu–Natal, Durban, South Africa

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Abstract

In this paper, we introduce a new iterative algorithm for approximating a common solution of certain class of multiple–sets split variational inequality problems. The sequence of the proposed iterative algorithm is proved to converge strongly in Hilbert spaces. As application, we obtain some strong convergence results for some classes of multiple–sets split convex minimization problems.

Keywords: split variational inequality problems; multiple–sets problems; convex minimization problems; strictly pseudo contractive mapping; inverse strongly monotone operators.

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1. Introduction

Let C be a nonempty, closed and convex subset of a real Hilbert space H. A mapping $T : C \to C$ is said to be

(i) *non–expansive*, if

$$\|Sx - Sy\| \leq \|x - y\|, \quad \forall x, y \in C,$$

(ii) *k–strictly pseudo contractive* in the sense of Browder and Petryshyn [8], if for $0 \leq k < 1$,

$$\|Sx - Sy\|^2 \leq \|x - y\|^2 + \mu\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Email address: izuchukwuc@ukzn.ac.za, izuchukwu_c@yahoo.com (Chinedu Izuchukwu)

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A point $x \in C$ is called a *fixed point* of S if $Sx = x$. We denote the set of fixed points of the mapping S by $F(S)$. It is well known that, if S is a k–strictly pseudo contractive mapping and $F(S) \neq \emptyset$, then $F(S)$ is closed and convex. For more information on strictly pseudo contrative mappings, see [1, 8, 24, 34] and references therein.
A mapping $M : C \to C$ is said to be
(i) *monotone*, if

$$\langle Mx - My, x - y\rangle \geq 0, \quad \forall x, y \in C,$$

(ii) α*–inverse strongly monotone (ism)*, if there exists a constant $\alpha > 0$ such that

$$\langle Mx - My, x - y\rangle \geq \alpha \|Mx - My\|^2, \quad \forall x, y \in C,$$

(iii) *firmly nonexpansive*, if

$$\langle Mx - My, x - y\rangle \geq \|Mx - My\|^2, \quad \forall x, y \in C,$$

(iv) *Lipschitz*, if there exists a constant $L > 0$ such that

$$||Mx - My|| \leq L||x - y||, \ \forall x, y \in C.$$

Remark 1.1. It is generally known that every α–ism mapping is $\frac{1}{\alpha}$–Lipschitz continuous (see [6]).

If M is a multivalued mapping, i.e. $M : H \to 2^H$, then M is called monotone, if

$$\langle x - y, u - v\rangle \geq 0 \ \ \forall x, y \in H, \ u \in M(x), \ v \in M(y)$$

and M is maximal monotone, if the graph $G(M)$ of M defined by

$$G(M) =: \{(x, y) \in H \times H : y \in M(x)\},$$

is not properly contained in the graph of any other monotone mapping. It is generally known that M is maximal if and only if for $(x, u) \in H \times H$, $\langle x - y, u - v\rangle \geq 0$ for all $(y, v) \in G(M)$ implies $u \in M(x)$.
A mapping $T : C \to C$ is said to be averaged non–expansive if $\forall x, y \in C, \quad T = (1 - \beta)I + \beta S$ holds for a non–expansive operator $S : C \to C$ and $\beta \in (0, 1)$. The term "averaged mapping" was first developed by Baillon et al [5]. Recall that a mapping T is firmly non–expansive if and only if T can be expressed as $T = \frac{1}{2}(I + S)$, where S is non–expansive (see [25]). Thus, we make the following remark which can be easily verified.

Remark 1.2. In a real Hilbert space, T is firmly non–expansive if and only if it is averaged with $\beta = \frac{1}{2}$.

The metric projection P_C is a map defined on H onto C which assigns to each $x \in H$, the unique point in C, denoted by $P_C x$ such that

$$||x - P_C x|| = \inf\{||x - y|| : \ y \in C\}.$$

It is well known that $P_C x$ is characterized by the inequality $\langle x - P_C x, z - P_C x\rangle \leq 0, \ \ \forall z \in C$ and P_C is a firmly non–expansive mapping. We also know that if f is β–inverse strongly monotone mapping with $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda f)$ is averaged non–expansive (see [15, Lemma 2.9]). Hence, from Remark 1.2, we obtain the following.

Remark 1.3. In a real Hilbert space, if f is β–inverse strongly monotone with $\lambda \in (0, 2\beta)$, then $P_C(I - \lambda f)$ is firmly non–expansive. Thus, $P_C(I - \lambda f)$ is non–expansive.

For more information on metric projections, see [19, 15] and the references therein. Recall that the normal cone of C at the point $z \in H$ is defined as

$$N_C z = \begin{cases} \{d \in H : \langle d, y - z\rangle \leq 0, \ \ \forall\, y \in C\}, & z \in C, \\ \emptyset, & \text{otherwise.} \end{cases}$$

In 1994, Censor and Elfving [13] introduced and studied the following Split Feasibility Problem (SFP): Find a point

$$x \in C \ \text{ such that } \ Ax \in Q, \tag{1.1}$$

where C and Q are nonempty closed and convex subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and A is an $m \times n$ real matrix. The SFP has many applications in a wide range of fields, which includes phase retrieval, medical image reconstruction, signal processing, radiation therapy treatment planning, among others (for example, see [9, 12, 13, 14] and the references therein).

To approximate a solution of (1.1), Byrne [10] applied the forward–backward method, a type of projected gradient method, thus presenting the so–called CQ–iterative procedure which he defined as

$$x_{n+1} = P_C(I - \gamma A^*(I - P_Q)A)x_n, \ n \in \mathbb{N}, \tag{1.2}$$

where $\gamma \in (0, \frac{2}{\lambda})$ with λ being the spectral radius of the operator A^*A. Byrne [10] proved that the sequence generated by Algorithm 1.2 converges weakly to a solution of SFP (1.1).

The theory of Variational Inequality Problems (VIP) is well known, developed and appears to be one of the most important aspect in optimization and nonlinear analysis, since most mathematical problems can be modelled as a variational inequality problem. Let C be a nonempty, closed and convex subset of a real Hilbert space H and $f : H \to H$ be a nonlinear operator. The VIP defined for C and f is to find $x^* \in C$ such that

$$\text{x}^* \in (VI(C, f)) \ \text{ i.e.,} \ \ \langle f(x^*), \ x - x^*\rangle \geq 0, \ \ \forall x \in C. \tag{1.3}$$

Let f be an α–inverse strongly monotone operator on C and $N_C z$ be the normal cone of C at the point $z \in C$, we define the following set–valued operator $M : C \to 2^C$ by

$$Mz = fz + N_C z.$$

Then M is maximal monotone. Furthermore, $0 \in M(x^*) \iff x^* \in \text{VI}(C, f)$ (see [Theorem 3][27]). Moreover, $x^* \in \text{VI}(C, f)$ if and only if $x^* = P_C(I - \lambda f)(x^*), \ \ \forall \lambda > 0$ (see [16]).

In 2010, Censor *et. al.* [16] introduced a new class of problem (which is an important generalization of the SFP mentioned above) called the Split Variational Inequality Problem (SVIP) by combining the Variational Inequality Problem (VIP) and the SFP. They defined the SVIP as follows: Find $x^* \in C$ such that

$$\langle f(x^*), x - x^*\rangle \geq 0 \ \ \forall x \in C, \tag{1.4}$$

and such that $y^* = Ax^* \in Q$ solves

$$\langle g(y^*), y - y^*\rangle \geq 0 \ \ \forall y \in Q, \tag{1.5}$$

where C and Q are nonempty, closed and convex subsets of real Hilbert spaces H_1 and H_2 respectively, $A : H_1 \to H_2$ is a bounded linear operator, $f : H_1 \to H_1$ and $g : H_2 \to H_2$ are two given operators. If

(1.4) and (1.5) are considered separately, we have that (1.4) is a VIP with its solution set $VIP(C,f)$ and (1.5) is a VIP with its solution set $VIP(Q,g)$. To solve the SVIP (1.4)-(1.5), Censor *et. al.* [16] proposed the following algorithm and obtained a weak convergence result. For $x_1 \in H_1$, the sequence $\{x_n\}$ is generated by

$$x_{n+1} = P_C(I-\lambda f)(x_n + \gamma A^*(P_Q(I-\lambda g) - I)Ax_n),\ n \geq 1, \tag{1.6}$$

where $\gamma \in (0, \frac{1}{L})$ with L being the spectral radius of the operator A^*A.
In 2012, Censor *et. al.* [15] introduced the general Common Solutions to Variational Inequality Problem (CSVIP), which consist of finding common solutions to unrelated variational inequalities for finite number of sets. That is, find $x^* \in \cap_{i=1}^N C_i$ such that for each $i = 1, 2, \ldots, N$,

$$\langle A_i(x^*), x - x^* \rangle \geq 0,\ \text{for all } x \in C_i,\ i = 1, 2, \ldots, N, \tag{1.7}$$

where $A_i : H \to H$ is a nonlinear operator for each $i = 1, 2, \ldots, N$ and C_i is a nonempty, closed and convex subset of H. They obtained the solution of problem (1.7) by considering first, a case where $i = 1, 2$ and later obtained the result of the problem for $i = 1, 2, \ldots, N$. They proposed the following algorithm and proved the corresponding theorem

$$\begin{cases} x^0 \in H, \\ x^{k+1} = \prod_{i=1}^N (P_{C_i}(I - \lambda A_i))(x^k). \end{cases} \tag{1.8}$$

Theorem 1.4. Let H be a real Hilbert space and C_i be nonempty, closed and convex subsets of H for each $i = 1, 2, \ldots, N$. Let $A_i : H \to H$ be α_i–inverse strongly monotone operators with $\lambda \in (0, 2\alpha)$ and $\alpha := \min_i\{\alpha_i\}$. Assume that $\cap_{i=1}^N C_i \neq \emptyset$ and $\Gamma := \cap_{i=1}^N SOL(C_i, A_i) \neq \emptyset$. Then any sequence $\{x^k\}_{k=0}^\infty$ generated by Algorithm (1.8) converges weakly to a point $x^* \in \Gamma$ and furthermore,

$$x^* = \lim_{k\to\infty} P_\Gamma(x^k). \tag{1.9}$$

Very recently, Tian and Jiang [30] proposed a class of SVIP which is to find $x^* \in C$ such that

$$\langle f(x^*), x - x^* \rangle \geq 0\ \ \forall x \in C,\ \text{and such that } Ax^* \in F(S), \tag{1.10}$$

where C is a nonempty, closed and convex subset of H_1, $A : H_1 \to H_2$ is a bounded linear operator, $f : C \to H_1$ is a single valued operator and $S : H_2 \to H_2$ is a nonlinear mapping. To approximate solutions of (1.10), Tian and Jiang [30] proposed the following iterative algorithm by combining Algorithm (1.6) with the Korpelevich's extra–gradient method (see [22]) and Byrne's CQ algorithm: For arbitrary $x_1 \in C$, define the sequence $\{x_n\},\ \{y_n\}$ and $\{t_n\}$ by

$$\begin{cases} y_n = P_C(x_n - \gamma_n A^*(I-S)Ax_n), \\ t_n = P_C(y_n - \lambda_n f(y_n)), \\ x_{n+1} = P_C(y_n - \lambda_n f(t_n)), \end{cases} \tag{1.11}$$

for each $n \in \mathbb{N}$, where $\{\gamma_n\} \subset [a,b]$ for some $a, b \in (0, \frac{1}{||A||^2})$ and $\{\lambda_n\} \subset [c,d]$ for some $c, d \in (0, \frac{1}{k})$, $S : H_2 \to H_2$ is a non–expansive mapping and $f : C \to H_1$ is a monotone and k–Lipschitz continuous mapping. They proved that the sequence generated by Algorithm (1.11) converges weakly to a solution of (1.10). Furthermore, Tian and Jian [30] showed that Algorithm (1.11) can be used to solve the SVIP of Censor *et. al.* [16] by setting $S = P_Q(I - \lambda g)$ in Algorithm (1.11), since $P_Q(I - \lambda g)$ is a non–expansive mapping for $\lambda \in (0, 2\alpha)$. For more results on VIPs, see [2, 3, 4, 11, 15, 18, 21, 26] and the references therein.

Motivated by the works of Tian and Jiang [30], Censor *et. al.* [16] and Censor *et. al.* [15], we propose an extension of the class of SVIP studied by Tian and Jiang [30] to the following class of Multiple–Sets Split Variational Inequality Problem (MSSVIP): Find $x^* \in C := \cap_{i=1}^N C_i$ such that for each $i = 1, 2, \ldots, N$,

$$\langle f_i(x^*), x - x^* \rangle \geq 0 \ \forall x \in C_i, \text{ and such that } Ax^* \in F(S), \tag{1.12}$$

where $A : H_1 \to H_2$ is a bounded linear operator, $f_i : H_1 \to H_1$ is a single valued operator for each $i = 1, 2, \ldots, N$ and $S : H_2 \to H_2$ is a nonlinear mapping. Furthermore, we propose an iterative algorithm and using the algorithm, we state and prove some strong convergence results for the approximation of solutions of (1.12) and (1.4)–(1.5). Finally, we applied our results to study multiple–sets split convex minimization problems. Our results extend and improve the results of Censor *et. al.* [16], Censor *et. al.* [15], Tian and Jiang [30], and a host of other important results.

2. Preliminaries

We state some useful results which will be needed in the proof of our main theorem.

Lemma 2.1. (Chidume [17]) Let H be a Hilbert space, then for all $x, y \in H$ and $\alpha \in (0, 1)$, the following hold:

(i) $2\langle x, y \rangle = ||x||^2 + ||y||^2 - ||x - y||^2 = ||x + y||^2 - ||x||^2 - ||y||^2$,

(ii) $\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2$.

Lemma 2.2. (Xu [31]) Let H be a Hilbert space and $T : H \to H$ be a nonlinear mapping, then the following hold.

(i) f is non–expansive if and only if the complement $I - f$ is $\frac{1}{2}$–ism.

(ii) If f is ν–ism and $\gamma > 0$, then γf is $\frac{\nu}{\gamma}$–ism.

(iii) f is averaged if and only if the complement $I - f$ is ν–ism for some $\nu > \frac{1}{2}$. Indeed, for $\beta \in (0, 1)$, f is β–averaged if and only if $I - f$ is $\frac{1}{2\beta}$–ism.

(iv) If f_1 is β_1–averaged and f_2 is β_2-averaged, where $\beta_1, \beta_2 \in (0, 1)$, then the composite $f_1 f_2$ is β–averaged, where $\beta = \beta_1 + \beta_2 - \beta_1\beta_2$.

(v) If f_1 and f_2 are averaged and have a common fixed point, then $F(f_1 f_2) = F(f_1) \cap F(f_2)$.

Lemma 2.3. (Takahashi et. al. [28]) Let H_1 and H_2 be real Hilbert spaces. Let $A : H_1 \to H_2$ be a bounded linear operator with $A \neq 0$, and $S : H_2 \to H_2$ be a non–expansive mapping. Then $A^*(I - S)A$ is $\frac{1}{2||A||^2}$–ism.

Lemma 2.4. (Tian and Jiang [30]) Let H_1 and H_2 be real Hilbert spaces. Let C be a nonempty, closed and convex subset of H_1. Let $S : H_2 \to H_2$ be a non–expansive mapping and let $A : H_1 \to H_2$ be a bounded linear operator. Suppose that $C \cap A^{-1}F(S) \neq \emptyset$. Let $\gamma > 0$ and $x^* \in H_1$. Then the following are equivalent:

(i) $x^* = P_C(I - \gamma A^*(I-S)A)x^*$;

(ii) $0 \in A^*(I-S)Ax^* + N_C x^*$;

(iii) $x^* \in C \cap A^{-1}F(S)$.

Lemma 2.5. (Xu [32]) Let H be a real Hilbert space and $S : H \to H$ be a non–expansive mapping with $F(S) \neq \emptyset$. If $\{x_n\}$ is a sequence in H converging weakly to x^*, and if $\{(I-S)x_n\}$ converges strongly to y, then $(I-S)x^* = y$.

Lemma 2.6. (Xu [33]) Let $\{a_n\}$ be a sequence of nonnegative real numbers such that

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n\delta_n, \quad n \geq 0,$$

where $\{\gamma_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that
(i) $\sum_{n=0}^{\infty}\gamma_n = \infty$,
(ii) $\limsup\limits_{n\to\infty} \delta_n \leq 0$ or $\sum_{n=0}^{\infty}|\delta_n\gamma_n| < \infty$.
Then $\lim\limits_{n\to\infty} a_n = 0$.

Lemma 2.7. (Zhou [34]) Let H be a real Hilbert space and $S : H \to H$ be k–strictly pseudo contractive mapping with $k \in [0,1)$. Let $T_\beta := \beta I + (1-\beta)S$, where $\beta \in [\mu, 1)$. Then

(i) $F(S) = F(T_\beta)$,

(ii) T_β is a non–expansive mapping.

Lemma 2.8. (Maingé [23]) Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity, in the sense that there exists a subsequence $\{\Gamma_{n_j}\}_{j\geq 0}$ of $\{\Gamma_n\}$such that

$$\Gamma_{n_j} < \Gamma_{n_j+1} \quad \forall j \geq 0.$$

Also consider the sequence of integers $\{\tau(n)\}_{n\geq n_0}$ defined by

$$\tau(n) = \max\{k \leq n \mid \Gamma_k < \Gamma_{k+1}\}.$$

Then $\{\Gamma_n\}_{n\geq n_0}$ is a nondecreasing sequence such that $\tau(n) \to \infty$, as $n \to 0$, and for all $n \geq n_0$, the following two estimates hold:

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \Gamma_n \leq \Gamma_{\tau(n)+1}.$$

3. Main results

Theorem 3.1. *Let H_1 and H_2 be real Hilbert spaces, and for each $i = 1, 2, \ldots, N$, let C_i be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $f_i : H_1 \to H_1$ be an α_i-inverse strongly monotone mapping and $S : H_2 \to H_2$ be k–strictly pseudo contractive mapping. Assume that $\Gamma = \{z \in \cap_{i=1}^N VI(C_i, f_i) : \ Az \in F(S)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I-T_\beta)Au_n), \\ x_{n+1} = P_{C_N}(I-\lambda f_N) \circ P_{C_{N-1}}(I-\lambda f_{N-1}) \circ \ldots \circ P_{C_1}(I-\lambda f_1)y_n, \ n \geq 1, \end{cases} \tag{3.1}$$

where $T_\beta := \beta I + (1-\beta)S$ with $\beta \in [k,1)$, $C := \cap_{i=1}^N C_i \neq \emptyset$, $\{\gamma_n\} \subset [a,b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right)$, $\lambda \in (0, 2\alpha)$, $\alpha := \min\{\alpha_i, \ i = 1, 2, \ldots, N\}$ and $\{\beta_n\} \subset (0,1)$ such that $\lim\limits_{n\to\infty}\beta_n = 0$ and $\sum_{n=1}^{\infty}\beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma u$.

Proof . From Lemma 2.7, Lemma 2.2 (ii), (iii), (iv) and Lemma 2.3, we obtain that $P_C(I-\gamma_n A^*(I-T_\gamma)A)$ is $\frac{1+\gamma_n\|A\|^2}{2}$–averaged. That is, $P_C(I-\gamma_n A^*(I-T_\gamma)A) = (1-\alpha_n)I+\alpha_n T_n$, where $\alpha_n = \frac{1+\gamma_n\|A\|^2}{2}$ and T_n is a non–expansive mapping for each $n \geq 1$. Thus, we rewrite y_n as

$$y_n = (1-\alpha_n)u_n + \alpha_n T_n u_n. \tag{3.2}$$

Let $p \in \Gamma$ and $\Phi^N = P_{C_N}(I-\lambda f_N) \circ P_{C_{N-1}}(I-\lambda f_{N-1}) \circ \ldots \circ P_{C_1}(I-\lambda f_1)$, where $\Phi^0 = I$, then from (3.1), (3.2) and Lemma 2.1, we have

$$\begin{aligned}
\|x_{n+1}-p\|^2 &= \|P_{C_N}(I-\lambda f_N)\left(\Phi^{N-1}y_n\right)-p\|^2 \\
&\leq \|\Phi^{N-1}y_n - p\|^2 \\
&\ \vdots \\
&\leq \|y_n-p\|^2 \\
&= \|(1-\alpha_n)(u_n-p)+\alpha_n(T_nu_n-p)\|^2 \\
&= (1-\alpha_n)\|u_n-p\|^2+\alpha_n\|T_nu_n-p\|^2 \\
&\quad -\alpha_n(1-\alpha_n)\|u_n-T_nu_n\|^2 \\
&\leq \|u_n-p\|^2-\alpha_n(1-\alpha_n)\|u_n-T_nu_n\|^2 \\
&\leq ||(1-\beta_n)(x_n-p)+\beta_n(u-p)||^2 \\
&\leq (1-\beta_n)\|x_n-p\|^2+\beta_n\|u-p\|^2 \\
&\leq \max\{\|x_n-p\|^2, \|u-p\|^2\} \\
&\ \vdots \\
&\leq \max\{\|x_1-p\|^2, \|u-p\|^2\}.
\end{aligned} \tag{3.3}$$

Therefore $\{||x_n-p||^2\}$ is bounded. Consequently, $\{x_n\}$, $\{y_n\}$ $\{u_n\}$ and $\{Au_n\}$ are all bounded. From (3.1), we obtain

$$\lim_{n\to\infty} ||u_n-x_n||^2 = \lim_{n\to\infty} \beta_n||u-x_n||^2 = 0. \tag{3.4}$$

We now divide our proof into two cases:
Case 1: Suppose that $\{\|x_n-p\|^2\}$ is monotone decreasing, then $\{\|x_n-p\|^2\}$ is convergent. Thus,

$$\lim_{n\to\infty}\left(||x_n-p||^2-||x_{n+1}-p||^2\right) = 0. \tag{3.5}$$

It follows from (3.3) that

$$\begin{aligned}
\alpha_n(1-\alpha_n)||u_n-T_nu_n||^2 &\leq ||u_n-p||^2-||x_{n+1}-p||^2 \\
&\leq (1-\beta_n)||x_n-p||^2+\beta_n||u-p||^2 \\
&\quad -||x_{n+1}-p||^2 \to 0, \text{ as } n\to\infty.
\end{aligned} \tag{3.6}$$

Since $\alpha_n = \frac{1+\gamma_n\|A\|^2}{2}$, then by the condition on γ_n, we obtain

$$\lim_{n\to\infty} ||u_n-T_nu_n||^2 = 0. \tag{3.7}$$

Furthermore, (3.2) and (3.7) yields

$$\lim_{n\to\infty} ||y_n-u_n||^2 = \lim_{n\to\infty} \alpha_n||T_nu_n-u_n||^2 = 0. \tag{3.8}$$

Also, we obtain from (3.4) and (3.8) that

$$\lim_{n\to\infty} ||y_n - x_n||^2 = 0. \tag{3.9}$$

Since $P_{C_N}(I - \lambda f_N)$ is firmly nonexpansive (see Remark 1.3), we obtain

$$\begin{aligned} ||x_{n+1} - p||^2 &= ||P_{C_N}(I - \lambda f_N)\Phi^{N-1}y_n - p||^2 \\ &\leq \langle x_{n+1} - p, \Phi^{N-1}y_n - p\rangle \\ &= \frac{1}{2}\left[||x_{n+1} - p||^2 + ||\Phi^{N-1}y_n - p||^2 - ||x_{n+1} - \Phi^{N-1}y_n||^2\right], \end{aligned}$$

which implies from (3.5) and (3.9) that

$$\begin{aligned} ||x_{n+1} - \Phi^{N-1}y_n||^2 &\leq ||\Phi^{N-1}y_n - p||^2 - ||x_{n+1} - p||^2 \\ &\vdots \\ &\leq ||y_n - p||^2 - ||x_{n+1} - p||^2 \\ &\leq ||y_n - x_n||^2 + 2||y_n - x_n||||x_n - p|| \\ &\quad +||x_n - p||^2 - ||x_{n+1} - p||^2 \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.10}$$

By similar argument as above, we obtain

$$\begin{aligned} ||\Phi^{N-1}y_n - \Phi^{N-2}y_n||^2 &\leq ||\Phi^{N-2}y_n - p||^2 - ||\Phi^{N-1}y_n - p||^2 \\ &\vdots \\ &\leq ||y_n - p||^2 - ||\Phi^{N-1}y_n - p||^2 \\ &\leq ||y_n - p||^2 - ||x_{n+1} - p||^2 \\ &\leq ||y_n - x_n||^2 + 2||y_n - x_n||||x_n - p|| \\ &\quad +||x_n - p||^2 - ||x_{n+1} - p||^2 \to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.11}$$

Continuing in the same manner, we have that

$$\lim_{n\to\infty} ||\Phi^{N-2}y_n - \Phi^{N-3}y_n|| = \cdots = \lim_{n\to\infty} ||\Phi^2 y_n - \Phi^1 y_n|| = \lim_{n\to\infty} ||\Phi^1 y_n - y_n|| = 0. \tag{3.12}$$

From (3.10), (3.11) and (3.12), we conclude that

$$\lim_{n\to\infty} ||\Phi^i y_n - \Phi^{i-1}y_n|| = 0, \ i = 1, 2, \ldots, N. \tag{3.13}$$

By Remark 1.1, we have that f_i is Lipschitz continuous for each $i = 1, 2, \ldots N$. Thus,

$$\lim_{n\to\infty} ||f_i\Phi^i y_n - f_i\Phi^{i-1}y_n|| = 0, \ i = 1, 2, \ldots, N. \tag{3.14}$$

Also,

$$||x_{n+1} - y_n|| \leq ||\Phi^N y_n - \Phi^{N-1}y_n|| + ||\Phi^{N-1}y_n - \Phi^{N-2}y_n|| + \cdots + ||\Phi^1 y_n - y_n||,$$

which implies from (3.13) that

$$\lim_{n\to\infty} ||x_{n+1} - y_n|| = 0. \tag{3.15}$$

From (3.8) and (3.15), we have

$$\lim_{n\to\infty} ||x_{n+1} - u_n||^2 = 0. \tag{3.16}$$

Now, since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ that converges weakly to x^*. It then follows from (3.4) that there exists a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ that converges weakly to x^*. Without loss of generality, the subsequence $\{\gamma_{n_k}\}$ of $\{\gamma_n\}$ converges to a point $\bar{\gamma} \in \left(0, \frac{1}{\|A\|^2}\right)$. By Lemma 2.3, $A^*(I-T_\beta)A$ is inverse strongly monotone, thus $\{A^*(I-T_\beta)Au_{n_k}\}$ is bounded. It then follows from the nonexpansivity of P_C that

$$\|P_C(I-\gamma_{n_k}A^*(I-T_\beta)A)u_{n_k}-P_C(I-\bar{\gamma}A^*(I-T_\beta)A)u_{n_k}\| \leq |\gamma_{n_k}-\bar{\gamma}|\|A^*(I-T_\beta)Au_{n_k}\| \to 0, \text{ as } k \to \infty.$$

That is,

$$\lim_{k\to\infty} \|y_{n_k} - P_C(I-\bar{\gamma}A^*(I-T_\beta)A)u_{n_k}\| = 0,$$

which implies from (3.8) that

$$\lim_{k\to\infty} ||u_{n_k} - P_C(I-\bar{\gamma}A^*(I-T_\beta)A)u_{n_k}|| = 0. \tag{3.17}$$

It then follows from Lemma 2.5 that $x^* \in F(P_C(I-\bar{\gamma}A^*(I-T_\beta)A))$. Thus, from Lemma 2.4, we obtain that

$$x^* \in C \cap A^{-1}F(T_\beta).$$

Thus, from Lemma 2.7, we obtain that

$$Ax^* \in F(T_\beta) = F(S).$$

We now show that $x^* \in \cap_{i=1}^N VI(C_i, f_i)$. For each $i = 1, 2, \dots, N$, let $N_{C_i}z$ be the normal cone of C_i at a point $z \in C_i$, we define the operator $M_i : C_i \to 2^{H_1}$, for each $i = 1, 2, \dots, N$ by

$$M_i z = f_i z + N_{C_i} z.$$

Then, M_i is maximal monotone for each $i = 1, 2, \dots, N$. Let $(z, w) \in G(M_i)$, then $w - f_i z \in N_{C_i} z$. For $\Phi^i y_{n_k} \in C_i$, we have

$$\langle z - \Phi^i y_{n_k}, w - f_i z\rangle \geq 0, \ i = 1, 2, \dots, N. \tag{3.18}$$

From $\Phi^i y_{n_k} = P_{C_i}(I-\lambda f_i)\Phi^{i-1}y_{n_k}$, we have

$$\langle z - \Phi^i y_{n_k}, \Phi^i y_{n_k} - (\Phi^{i-1}y_{n_k} - \lambda f_i \Phi^{i-1}y_{n_k})\rangle \geq 0, \quad (i = 1, 2, \dots, N),$$

which implies $\langle z - \Phi^i y_{n_k}, \frac{\Phi^i y_{n_k} - \Phi^{i-1}y_{n_k}}{\lambda} + f_i\Phi^{i-1}y_{n_k}\rangle \geq 0$, for each $i = 1, 2, \dots, N$. From (3.18), we get

$$\begin{aligned}
\langle z - \Phi^i y_{n_k}, w\rangle &\geq \langle z - \Phi^i y_{n_k}, f_i z\rangle \\
&\geq \langle z - \Phi^i y_{n_k}, f_i z\rangle - \langle z - \Phi^i y_{n_k}, \frac{\Phi^i y_{n_k} - \Phi^{i-1}y_{n_k}}{\lambda} + f_i\Phi^{i-1}y_{n_k}\rangle \\
&= \langle z - \Phi^i y_{n_k}, f_i z - f_i\Phi^{i-1}y_{n_k} - \frac{\Phi^i y_{n_k} - \Phi^{i-1}y_{n_k}}{\lambda}\rangle \\
&= \langle z - \Phi^i y_{n_k}, f_i z - f_i\Phi^i y_{n_k}\rangle + \langle z - \Phi^i y_{n_k}, f_i\Phi^i y_{n_k} - f_i\Phi^{i-1}y_{n_k}\rangle \qquad (3.19)\\
&\quad - \langle z - \Phi^i y_{n_k}, \frac{\Phi^i y_{n_k} - \Phi^{i-1}y_{n_k}}{\lambda}\rangle \\
&\geq \langle z - \Phi^i y_{n_k}, f_i\Phi^i y_{n_k} - f_i\Phi^{i-1}y_{n_k}\rangle - \langle z - \Phi^i y_{n_k}, \frac{\Phi^i y_{n_k} - \Phi^{i-1}y_{n_k}}{\lambda}\rangle. \qquad (3.20)
\end{aligned}$$

Using (3.13) and (3.14) together with the fact that $\{x_{n_k+1}\} = \{\Phi^i y_{n_k}\}$ converges weakly to x^*, we obtain from (3.19) that $\langle z - x^*, w\rangle \geq 0$. Also, M_i is maximal monotone for each $i = 1, 2, \ldots, N$, this gives us that $x^* \in M_i^{-1}(0)$, which implies that $0 \in M_i(x^*)$ for each $i = 1, 2, \ldots, N$. Hence, $x^* \in \cap_{i=1}^N VI(C_i, f_i)$. Therefore, $x^* \in \Gamma$.
Next, we claim that $\limsup_{n\to\infty}\langle u - z, x_n - z\rangle \leq 0$, where $z \in P_\Gamma u$.
Now, since $\{x_{n_k}\}$ converges weakly to x^*, we obtain by the property of P_Γ that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z, x_n - z\rangle &= \lim_{k\to\infty}\langle u - z, x_{n_k} - z\rangle \\ &= \langle u - z, x^* - z\rangle \\ &\leq 0. \end{aligned} \tag{3.21}$$

We now show that $\{x_n\}$ converges strongly to z. From (3.3), we obtain

$$\begin{aligned}\|x_{n+1} - z\|^2 &\leq \|u_n - z\|^2 \\ &= \|(1-\beta_n)(x_n - z) + \beta_n(u - z)\|^2 \\ &= (1-\beta)^2||x_n - z||^2 + \beta_n^2||u - z||^2 + 2\beta_n(1-\beta_n)\langle x_n - z, u - z\rangle \\ &\leq (1-\beta_n)||x_n - z||^2 + \beta_n\left[\beta_n||u - z||^2 + 2(1-\beta_n)\langle u - z, x_n - z\rangle\right]. \end{aligned} \tag{3.22}$$

By (3.21) and Lemma 2.6, we conclude that $\{x_n\}$ converges strongly to z.
Case 2. Assume that $\{||x_n - p||^2\}$ is not monotone decreasing. Set $\Gamma_n = ||x_n - p||^2$ and let $\tau : \mathbb{N} \to \mathbb{N}$ be a mapping defined for all $n \geq n_0$ (for some large n_0) by

$$\tau(n) := \max\{k \in \mathbb{N} : k \leq n, \Gamma_k \leq \Gamma_{k+1}\}.$$

Then, by Lemma 2.8, we have that $\{\tau(n)\}$ is a nondecreasing sequence such that $\tau(n) \to \infty$, as $n \to \infty$ and

$$\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}, \quad \forall n \geq n_0.$$

From (3.6), we have

$$\begin{aligned}\alpha_{\tau(n)}(1-\alpha_{\tau(n)})||u_{\tau(n)} - T_{\tau(n)}u_{\tau(n)}||^2 &\leq ||x_{\tau(n)} - p||^2 - ||x_{\tau(n)+1} - p||^2 + \beta_{\tau(n)}||u - p||^2 \\ &\quad - \beta_{\tau(n)}||x_{\tau(n)} - p||^2 \\ &\leq \beta_{\tau(n)}(||u - p||^2 - ||x_{\tau(n)} - p||^2) \to 0, \text{ as } n \to \infty. \end{aligned}$$

By condition on $\{\alpha_{\tau(n)}\}$, we obtain

$$\lim_{n\to\infty}||u_{\tau(n)} - T_{\tau(n)}u_{\tau(n)}||^2 = 0. \tag{3.23}$$

Following the same line of argument as in Case 1, we can show that

$$\lim_{n\to\infty}||\Phi^i y_{\tau(n)} - \Phi^{i-1}y_{\tau(n)}|| = 0, \ i = 1, 2, \ldots, N$$

and that $\{x_{\tau(n)}\}$ converges weakly to $z \in \Gamma$. Now for all $n \geq n_0$, we have from (3.22) that

$$\begin{aligned}0 &\leq ||x_{\tau(n)+1} - z||^2 - [||x_{\tau(n)} - z||^2 \\ &\leq (1-\beta_{\tau(n)})||x_{\tau(n)} - z||^2 + \beta_{\tau(n)}[\beta_{\tau(n)}||u - z||^2 + 2(1-\beta_{\tau(n)})\langle x_{\tau(n)} - z, u - z\rangle] - ||x_{\tau(n)} - z||^2,\end{aligned}$$

which implies

$$||x_{\tau(n)} - z||^2 \leq \beta_{\tau(n)}||u - z||^2 + 2(1 - \beta_{\tau(n)})\langle x_{\tau(n)} - z, u - z\rangle \to 0, \text{ as } n \to \infty.$$

Hence

$$\lim_{n\to\infty} ||x_{\tau(n)} - z||^2 = 0.$$

Therefore,

$$\lim_{n\to\infty} \Gamma_{\tau(n)} = \lim_{n\to\infty} \Gamma_{\tau(n)+1} = 0.$$

Moreover, for $n \geq n_0$, it is clear that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ if $n \neq \tau(n)$ (that is $\tau(n) < n$) because $\Gamma_j > \Gamma_{j+1}$ for $\tau(n) + 1 \leq j \leq n$. Consequently for all $n \geq n_0$,

$$0 \leq \Gamma_n \leq \max\{\Gamma_{\tau(n)}, \Gamma_{\tau(n)+1}\} = \Gamma_{\tau(n)+1}.$$

Thus, $\lim_{n\to\infty} \Gamma_n = 0$. That is $\{x_n\}$ converges strongly to z. □

If S is non–expansive and $N = 1$ in Theorem 3.1, then we obtain the following result.

Corollary 3.2. Let H_1 and H_2 be real Hilbert spaces and C be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $f : H_1 \to H_1$ be an α-inverse strongly monotone mapping and $S : H_2 \to H_2$ be non–expansive mapping. Assume that $\Gamma = \{z \in VI(C, f) : \ Az \in F(S)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by

$$\begin{cases} u_n = (1 - \beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - S)Au_n), \\ x_{n+1} = P_C(I - \lambda f)y_n, \ n \geq 1, \end{cases} \tag{3.24}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right)$, $\lambda \in (0, 2\alpha)$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma u$.

If $H = H_1 = H_2$ and $S = A = I$(where I is the identity mapping on H) in Theorem 3.1, we obtain the following result.

Corollary 3.3. Let H be a real Hilbert space and for each $i = 1, 2, \ldots, N$, and C_i be a nonempty closed and convex subset of H. Let $f_i : H \to H$ be an α_i–inverse strongly monotone mapping. Assume that $\Gamma = \{z \in \cap_{i=1}^N VI(C_i, f_i)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H$ by

$$\begin{cases} y_n = P_C\left((1 - \beta_n)x_n + \beta_n u\right), \\ x_{n+1} = P_{C_N}(I - \lambda f_N) \ \circ \ P_{C_{N-1}}(I - \lambda f_{N-1}) \ \circ \ \ldots \ \circ \ P_{C_1}(I - \lambda f_1)y_n, \ n \geq 1, \end{cases} \tag{3.25}$$

where $C := \cap_{i=1}^N C_i \neq \emptyset$, $\lambda \in (0, 2\alpha)$, $\alpha := \min\{\alpha_i, \ i = 1, 2, \ldots, N\}$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma u$.

In the following Theorem, we study the class of SVIP introduced by Censor *et. al.* [16].

Theorem 3.4. *Let H_1 and H_2 be real Hilbert spaces and C, Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $f : H_1 \to H_1$ be α–inverse strongly monotone mapping and $g : H_2 \to H_2$ be β–inverse strongly monotone mapping. Assume that $\Gamma = \{z \in VI(C, f) : Az \in VI(Q, g)\} \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by*

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - P_Q(I-\lambda g))Au_n), \\ x_{n+1} = P_C(I-\lambda f)y_n, \ n \geq 1, \end{cases} \tag{3.26}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right)$, $0 < \lambda < 2\alpha, 2\beta$ and $\{\beta_n\} \subset (0,1)$ such that $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z \in \Gamma$, where $z = P_\Gamma u$.

Proof . We know that, for any $\lambda > 0$, $F(P_Q(I - \lambda g)) = VI(Q, g)$ and for $\lambda \in (0, 2\beta)$, $P_Q(I-\lambda g)$ is non–expansive. Thus, setting $S = P_Q(I - \lambda g)$ in Corollary 3.2, we obtain the desired result. □

4. Application to multiple–sets split convex minimization problems

Let $F : C \to \mathbb{R}$ be a convex and differentiable function. We know that if ∇F is $\frac{1}{\alpha}$–Lipschitz continuous, then it is α–inverse strongly monotone, where ∇F is the gradient of F (see Remark 1.1). Moreover,

$$x^* = \arg\min_{x\in C} F(x) \Leftrightarrow x^* \in VI(C, \nabla F).$$

Now, consider the following class of Multiple-Sets Split Convex Minimization Problem (MSSCMP): Find

$$x^* \in \cap_{i=1}^N C_i \text{ such that } x^* = \arg\min_{x\in C_i} F_i(x), \ i = 1, 2, \dots, N \text{ and such that } Ax^* \in F(S), \tag{4.1}$$

where $A : H_1 \to H_2$ is a bounded linear operator, F_i is as defined above, $S : H_2 \to H_2$ is a k–strictly pseudo contractive mapping. Suppose the solution set of problem (4.1) is Ω, then setting $f_i = \nabla F_i$ for each $i = 1, 2, \dots, N$ in Theorem 3.1, we obtain the following result.

Theorem 4.1. Let H_1 and H_2 be real Hilbert spaces, and for each $i = 1, 2, \dots, N$, let C_i be a nonempty closed and convex subset of H_1. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $F_i : H_1 \to \mathbb{R}$ be a convex and differentiable function such that ∇F_i is $\frac{1}{\alpha_i}$–Lipschitz continuous. Let $S : H_2 \to H_2$ be k-strictly pseudo contractive mapping. Suppose $\Omega \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - S)Au_n), \\ x_{n+1} = P_{C_N}(I-\lambda\nabla F_N) \circ P_{C_{N-1}}(I-\lambda\nabla F_{N-1}) \circ \dots \circ P_{C_1}(I-\lambda\nabla F_1)y_n, \ n \geq 1, \end{cases} \tag{4.2}$$

where $C := \cap_{i=1}^N C_i \neq \emptyset$, $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right)$, $\lambda \in (0, 2\alpha)$, $\alpha := \min\{\alpha_i, \ i = 1, 2, \dots, N\}$ and $\{\beta_n\} \subset (0,1)$ such that $\lim\limits_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega u$.

Next, we consider the following class of Split Convex Minimization Problem (SCMP): Find

$$x^* \in C \text{ such that } x^* = \arg\min_{x \in C} F(x) \tag{4.3}$$

and such that $y^* = Ax^* \in Q$, solves

$$y^* = \arg\min_{y \in Q} G(y). \tag{4.4}$$

Suppose the solution set of problem (4.3)–(4.3) is Ω, then setting $f = \nabla F$ and $g = \nabla G$ in Theorem 3.4, we obtain the following result.

Theorem 4.2. Let H_1 and H_2 be real Hilbert spaces, and C, Q be nonempty closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$. Let $F : H_1 \to H_1$ be convex and differentiable function such that ∇F is $\frac{1}{\alpha}$–Lipschitz continuous and $G : Q \to H_2$ be convex and differentiable function such that ∇G is $\frac{1}{\beta}$–Lipschitz continuous. Assume that $\Omega \neq \emptyset$ and the sequence $\{x_n\}$ be generated for arbitrary $x_1, u \in H_1$ by

$$\begin{cases} u_n = (1-\beta_n)x_n + \beta_n u, \\ y_n = P_C(u_n - \gamma_n A^*(I - P_Q(I - \lambda \nabla G))Au_n), \\ x_{n+1} = P_C(I - \lambda \nabla F)y_n, \ n \geq 1, \end{cases} \tag{4.5}$$

where $\{\gamma_n\} \subset [a, b]$ for some $a, b \in \left(0, \frac{1}{||A||^2}\right)$, $0 < \lambda < 2\alpha, 2\beta$ and $\{\beta_n\} \subset (0, 1)$ such that $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=1}^{\infty} \beta_n = \infty$. Then, the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega u$.

Remark 4.3. Our results extend and improve the results of Tian and Jiang [30], Censor *et. al.* [16] and Censor *et. al.* [15] in the following ways:

(i) The results obtained in this paper extend the results of Tian and Jiang [30] from split problems to multiple–sets split problems.

(ii) Our results extend the result of Censor *et. al.* [15] from finite family of VIP to finite family of SVIP.

(iii) As seen in Theorem 3.4, the main results of this paper generalizes the main results in [16].

(iv) The authors in [30], [15] and [16] obtained weak convergence results, while in this paper, we obtained strong convergence results.

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