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Fekete-Szegö problems for analytic functions in the space of logistic sigmoid functions based on quasi-subordination

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Abstract

In this paper, we define new subclasses $S_q^*(\alpha,\Phi)$, $M_q(\alpha,\Phi)$ and $L_q(\alpha,\Phi)$ of analytic functions in the space of logistic sigmoid functions based on quasi–subordination and determine the initial coefficient estimates $|a_2|$ and $|a_3|$ and also determine the relevant connection to the classical Fekete–Szegö inequalities. Further, we discuss the improved results for the associated classes involving subordination and majorization results briefly.

Keywords: univalent function; starlike function; quasi–subordination; logistic sigmoid function; Fekete–Szegö inequality.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}) \tag{1.1}$$

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which are analytic in the open disk $\mathbb{U} = \{z : |z| < 1\}$. Also, denote by $\mathcal{S}$ the class of all functions in $\mathcal{A}$ which are univalent and normalized by $f(0) = f'(0) - 1 = 0$ in $\mathbb{U}$. An analytic function f is subordinate to an analytic function g, written $f(z) \prec g(z)\ \ (z \in \mathbb{U})$, provided there is an analytic function w defined on $\mathbb{U}$ with $w(0) = 0$ and $|w(z)| < 1$ satisfying $f(z) = g(w(z))$. It follows from Schwarz Lemma that

$$f(z) \prec g(z) \quad \Longleftrightarrow (f(0) = g(0) \text{ and } \ f(\mathbb{U}) \subset g(\mathbb{U})) \ \ (z \in \mathbb{U}).$$

For details see [8], [11].

Further more, we recall the definition of quasi–subordination due to Robertson [18]. For two analytic functions $f, g \in \mathcal{A}$ the function f is quasi–subordinate to g, written as $f(z) \prec_q g(z),\ z \in \mathbb{U}$, if there exists an analytic function $\psi(z)$ with $|\psi(z)| \leq 1\ (z \in \mathbb{U})$ such that $\frac{f(z)}{\psi(z)}$ is analytic in $\mathbb{U}$ and

$$\frac{f(z)}{\psi(z)} \prec g(z) \quad (z \in \mathbb{U}), \tag{1.2}$$

where

$$\psi(z) = d_0 + d_1 z + d_2 z^2 + \cdots \quad (z \in \mathbb{U}), \tag{1.3}$$

then there exists a Schwarz function $\omega(z)$ such that

$$f(z) = \psi(z) g(\omega(z)).$$

It is observed that when $\psi(z) = 1$, then $f(z) = g(\omega(z))$, so that $f(z) \prec g(z)$ in $\mathbb{U}$. Also notice that if $\omega(z) = z$, then $f(z) = \psi(z)g(z)$ and it is said that f is majorized by g and written $f(z) \ll g(z)$ in $\mathbb{U}$. It is seen that quasi–subordination comprises subordination and majorization (see details in [1] and [6]).

Now we briefly recall about the definition and properties of sigmoid functions and its applications in the coefficient problems, studied recently by function theorists (see [3, 10, 12, 13, 14, 15]). Special functions deal with an information process that is inspired by the way nervous system such as brain processes information. It comprises of large number of highly interconnected processing elements (neurones) working together to solve a specific problem. The functions are out shinning by other fields like real analysis, algebra, topology, functional analysis, differential equations and so on because it mimicks the way human brain works. They can be programmed to solve a specific problem and it can also be trained by examples. Special functions can be categorized into three namely, threshold function, ramp function and the logistic sigmoid function. The most important one among all is the logistic sigmoid function because of its gradient descendent learning algorithm. It can be evaluated in different ways, most especially by truncated series expansion. The logistic sigmoid function of the form

$$h(z) = \frac{1}{1 + e^{-z}}, \tag{1.4}$$

is differentiable and has the following properties:

(i) it outputs real numbers between 0 and 1.
(ii) it maps a very large input domain to a small range of outputs.
(iii) it never loses information because it is a one–to–one function.
(iv) it increases monotonically.

It was shown that logistic Sigmoid function is very useful in geometric functions theory (for details see [3, 10, 12, 13, 14, 15]).

Lemma 1.1. (Fadipe–Joseph et al. [3]) Let $h(z)$ be a sigmoid function and

$$\Phi(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right)^m . \tag{1.5}$$

Then $\Phi(z) \in \mathcal{P}$, $|z| < 1$ where $\Phi(z)$ is a modified sigmoid function.

Lemma 1.2. (Fadipe–Joseph et al. [3]) Let

$$\Phi_{m,n}(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m} \left(\sum_{n=1}^{\infty} \frac{(-1)^n}{n!} z^n \right) . \tag{1.6}$$

Then $|\Phi_{m,n}(z)| < 2$.

Setting $m = 1$, Fadipe–Joseph et al. [3] remarked that

$$\Phi(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,$$

where $c_n = \frac{(-1)^{n+1}}{2n!}$. As such, $|c_n| \leq 2$, $n = 1, 2, 3, \ldots$ and the result is sharp for each n. It is given by

$$\Phi(z) = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^5}{240} - \frac{17z^7}{40320} + \cdots , \tag{1.7}$$

hence we get

$$\frac{\Phi(z) - 1}{\Phi(z) + 1} = \frac{1}{4}z - \frac{1}{16}z^2 - \frac{1}{192}z^3 - \frac{5}{768}z^4 - \frac{13}{15360}z^5 + \cdots . \tag{1.8}$$

Let $\mathcal{P}$ denote the class of function of p analytic in $\mathbb{U}$ such that $p(0) = 1$ and $\Re(p(z)) > 0$, where

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$

In [7], Ma and Minda introduced the following class

$$\mathcal{S}^*(\phi) = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \phi(z) \right\}, \tag{1.9}$$

where

$$\phi(z) = 1 + B_1 z + B_2 z^2 + B_3 z^3 + \cdots (B_1 > 0) \quad \phi \in \mathcal{P}. \tag{1.10}$$

with $\phi(\mathbb{U})$ is symmetric with respect to the real axis and starlike with respect to $\phi(0) = 1$ and $\phi'(0) > 0$. A function $f \in \mathcal{S}^*(\phi)$ is called Ma–Minda starlike with respect to ϕ. The class $\mathcal{C}(\phi)$ is the class of functions $f \in \mathcal{A}$ for which $1 + \frac{zf''(z)}{f'(z)} \prec \phi(z)$. The class $\mathcal{S}^*(\phi)$ and $\mathcal{C}(\phi)$ are well–known subclasses of starlike and convex functions. In the sequel, such a function has a series expansion of the form (1.10).

Recently, Mohd and Darus [9] introduced the classes $\mathcal{M}_q(\alpha, \phi)$ and $\mathcal{L}_q(\alpha, \phi)$ where $\alpha \geq 0$ satisfying the following analytic criteria

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)} \right) - 1 \prec_q \phi(z) - 1$$

and

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)}-1\prec_q \phi(z)-1.$$

A sharp bound of the functional $|a_3-\mu a_2^2|$ for univalent functions $f\in\mathcal{A}$ of the form (1.1) with real $\mu(0<\mu\le 1)$ was obtained by Fekete and Szegö [4] (se also [13, 16, 19, 20]).

We consider, due to Mohd and Darus [9] we define certain new subclasses of $\mathcal{S}$ denoted by $S_q^*(\alpha,\Phi)$, $M_q(\alpha,\Phi)$ and $L_q(\alpha,\Phi)$ based on quasi–subordination involving sigmoid functions.

Definition 1.3. Let the class $S_q^*(\alpha,\Phi)$ $(\alpha\ge 0)$, consist of functions $f\in\mathcal{A}$ satisfying the quasi–subordination

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}-1\prec_q \phi(z)-1. \tag{1.11}$$

Definition 1.4. Let the class $M_q(\alpha,\Phi)$ $(\alpha\ge 0)$, consist of functions $f\in\mathcal{A}$ satisfying the quasi–subordination

$$(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-1\prec_q \phi(z)-1. \tag{1.12}$$

Definition 1.5. Let the class $L_q(\alpha,\Phi)$ $(\alpha\ge 0)$, consist of functions $f\in\mathcal{A}$ satisfying the quasi–subordination

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)}-1\prec_q \phi(z)-1. \tag{1.13}$$

From (1.11), (1.12) and (1.13), we state the following:

Remark 1.6. If there exists an analytic function $\psi(z)$ with $|\psi(z)|\le 1$ $(z\in\mathbb{U})$, then equivalently we have

$$f\in S_q^*(\alpha,\Phi)\Leftrightarrow \frac{\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}-1}{\psi(z)}\prec \phi(z)-1, \tag{1.14}$$

$$f\in M_q(\alpha,\Phi)\Leftrightarrow \frac{(1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)-1}{\psi(z)}\prec \phi(z)-1 \tag{1.15}$$

and

$$L_q(\alpha,\Phi)\Leftrightarrow \frac{\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)}-1}{\psi(z)}\prec \phi(z)-1. \tag{1.16}$$

Remark 1.7. In the subordination conditions (1.14), (1.15) and (1.16), if $\psi(z)=1$, then

$$f\in S^*(\alpha,\Phi)\Leftrightarrow \frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}\prec \phi(z), \tag{1.17}$$

$$f\in M(\alpha,\Phi)\Leftrightarrow (1-\alpha)\frac{zf'(z)}{f(z)}+\alpha\left(1+\frac{zf''(z)}{f'(z)}\right)\prec \phi(z) \tag{1.18}$$

and

$$f\in L(\alpha,\Phi)\Leftrightarrow \left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1+\frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)}\prec \phi(z). \tag{1.19}$$

In this paper, motivated by the earlier works of [3, 9, 10, 13, 14, 16, 17], we determine the the classical Fekete–Szegö inequalities for functions f in the new subclasses $S_q^*(\alpha,\Phi)$, $M_q(\alpha,\Phi)$ and $L_q(\alpha,\Phi)$.

2. Main Results

Motivated by the earlier work of [3, 9, 10, 13, 14, 16, 17], and making use of the the following Lemmas 2.1, 2.2 and 2.3, we determine the the classical Fekete–Szegö inequalities for functions f in the above defined subclasses $S_q^*(\alpha, \Phi)$, $M_q(\alpha, \Phi)$ and $L_q(\alpha, \Phi)$.

Lemma 2.1. (Keogh and Merkes [5]) Let w be the analytic function in $\mathbb{U}$ with $w(0) = 0$, $|w(z)| < 1$ and

$$\omega(z) = \omega_1 z + \omega_2 z^2 + \omega_3 z^3 + \cdots . \tag{2.1}$$

Then $|w_2 - \mu w_1^2| \leq max\{1, |\mu|\}$, where $\mu \in \mathbb{C}$. The result is sharp for the functions $w(z) = z^2$ or $w(z) = z$.

Lemma 2.2. (Duren [2]) Let w be the analytic function in $\mathbb{U}$ with $w(0) = 0$, $|w(z)| < 1$ and be given by (2.1). Then

$$|w_n| \leq \begin{cases} 1 & \text{for} \quad n = 1, \\ 1 - |w_1|^2 & \text{for} \quad n \geq 2, \end{cases}$$

The result is sharp for the functions $w(z) = z^n$ or $w(z) = z$.

Lemma 2.3. (Keogh and Merkes [5]) Let $\psi(z)$ be the analytic function in $\mathbb{U}$ with $|\psi(z)| < 1$ given by (1.3). Then $|d_0| \leq 1$ and $|d_n| \leq 1 - |d_0|^2 \leq 1$ for $n > 0$.

To start with, let $f \in \mathcal{A}$ be of the form (1.1) where $\alpha \geq 0$ then we have the following:

$$\frac{zf'(z)}{f(z)} + \alpha \frac{z^2 f''(z)}{f(z)} - 1 = (1 + 2\alpha)a_2 z + [2(1 + 3\alpha)a_3 - (1 + 2\alpha)a_2^2]z^2 + \cdots , \tag{2.2}$$

$$(1 - \alpha)\frac{zf'(z)}{f(z)} + \alpha \left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 = (1 + \alpha)a_2 z + [2(1 + 2\alpha)a_3 - (1 + 3\alpha)a_2^2]z^2 + \cdots \tag{2.3}$$

and

$$\begin{aligned} &\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} \left(1 + \frac{zf''(z)}{f'(z)}\right)^{(1-\alpha)} - 1 \\ &\qquad = (2 - \alpha)a_2 z + \frac{1}{2}[((2 - \alpha)^2 - 3(4 - 3\alpha))a_2^2 + 4(3 - 2\alpha)a_3]z^2 + \cdots . \end{aligned} \tag{2.4}$$

Since $\Phi(z)$ is a modified logistic sigmoid function given by (1.7) and from (1.8) we have

$$\frac{\Phi(\omega(z)) - 1}{\Phi(\omega(z)) + 1} = \frac{\omega_1}{4} z + \left(\frac{\omega_2}{4} - \frac{\omega_1^2}{192}\right) z^2 + \cdots ,$$

hence

$$\phi\left(\frac{\Phi(\omega(z)) - 1}{\Phi(\omega(z)) + 1}\right) = 1 + \frac{B_1 \omega_1}{4} z + \left(\frac{(B_2 - B_1)\omega_1^2}{16} + \frac{B_1 \omega_2}{4}\right) z^2 + \cdots .$$

For $\psi(z) \in \mathbb{U}$ of the form (1.3), then let

$$\begin{aligned}\psi(z)&\left(\phi\left[\frac{\Phi(\omega(z))-1}{\Phi(\omega(z))+1}\right]-1\right)\\ &=\left(d_0+d_1z+d_2z^2+\cdots\right)\left[\frac{B_1\omega_1}{4}z+\left(\frac{(B_2-B_1)\omega_1^2}{16}+\frac{B_1\omega_2}{4}\right)z^2+\cdots\right]\\ &=\frac{d_0B_1\omega_1}{4}z+\left(\frac{\omega_2B_1d_0}{4}+\frac{(B_2-B_1)\omega_1^2d_0}{16}+\frac{\omega_1B_1d_1}{4}\right)z^2+\cdots, \end{aligned}\tag{2.5}$$

unless otherwise stated.

Theorem 2.4. *Let $f \in \mathcal{A}$ of the form (1.1) belong to the class $S_q^*(\alpha,\Phi)$. Then*

$$|a_2| \le \frac{B_1}{4(1+2\alpha)}\tag{2.6}$$

and for some $\mu \in \mathbb{C}$

$$|a_3-\mu a_2^2| \le \frac{B_1}{8(1+3\alpha)}\max\left\{1,\left|\frac{1}{4}\left(\frac{B_1}{1+2\alpha}+\frac{B_2-B_1}{B_1}-\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2}\right)\right|\right\}.\tag{2.7}$$

Proof . Since $f \in S_q^*(\alpha,\Phi)$ we have

$$\frac{zf'(z)}{f(z)}+\alpha\frac{z^2f''(z)}{f(z)}-1=\psi(z)\left(\phi\left(\frac{\Phi(\omega(z))-1}{\Phi(\omega(z))+1}\right)-1\right).\tag{2.8}$$

Using the expansion in (2.2) and (2.5), we have

$$\begin{aligned}&(1+2\alpha)a_2z+[2(1+3\alpha)a_3-(1+2\alpha)a_2^2]z^2+\cdots.\\ &\qquad=\frac{d_0B_1\omega_1}{4}z+\left(\frac{\omega_2B_1d_0}{4}+\frac{(B_2-B_1)\omega_1^2d_0}{16}+\frac{\omega_1B_1d_1}{4}\right)z^2+\cdots.\end{aligned}\tag{2.9}$$

Comparing the like coefficient of z, z^2 from (2.9), we find that

$$(1+2\alpha)a_2=\frac{d_0B_1\omega_1}{4},\tag{2.10}$$

$$2(1+3\alpha)a_3-(1+2\alpha)a_2^2=\frac{\omega_2B_1d_0}{4}+\frac{(B_2-B_1)\omega_1^2d_0}{16}+\frac{\omega_1B_1d_1}{4}.\tag{2.11}$$

Now, (2.10) gives

$$a_2=\frac{d_0B_1\omega_1}{4(1+2\alpha)}.\tag{2.12}$$

From (2.11) and (2.12) we get

$$2(1+3\alpha)a_3=\frac{1}{4}\left(B_1\omega_2d_0+\omega_1B_1d_1\right)+\frac{\omega_1^2}{16}\left((B_2-B_1)d_0+\frac{B_1^2d_0^2}{(1+2\alpha)}\right)\tag{2.13}$$

which yields

$$a_3 = \frac{B_1}{8(1+3\alpha)}\left[\omega_1 d_1 + d_0\left(\omega_2 + \frac{1}{4}\left(\frac{(B_2-B_1)}{B_1} + \frac{B_1 d_0}{(1+2\alpha)}\right)\omega_1^2\right)\right]. \tag{2.14}$$

For some $\mu \in \mathbb{C}$, we obtain from (2.12) and (2.14)

$$a_3 - \mu a_2^2 = \frac{B_1}{8(1+3\alpha)}\left[\omega_1 d_1 + d_0\omega_2 - \frac{1}{4}\left(\frac{2\mu B_1 d_0^2(1+3\alpha)}{(1+2\alpha)^2} - \left(\frac{(B_2-B_1)d_0}{B_1} + \frac{B_1 d_0^2}{(1+2\alpha)}\right)\right)\omega_1^2\right]. \tag{2.15}$$

Since $\psi(z)$ is given by (1.3) is analytic and bounded in $\mathbb{U}$, therefore, on using [11, p. 172], for some $y\,(|y| \le 1)$. We have

$$|d_0| \le 1 \quad \text{and} \quad d_1 = (1-d_0^2)y. \tag{2.16}$$

On substituting the value of d_1 in (2.15), we get

$$a_3 = \frac{B_1}{8(1+3\alpha)}\left[y\omega_1 + d_0\omega_2 + \frac{(B_2-B_1)d_0}{4B_1}\omega_1^2 - \left(\frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{B_1}{(1+2\alpha)}\right)\omega_1^2 + y\omega_1\right)d_0^2\right]. \tag{2.17}$$

From (2.17) if $d_0 = 0$, we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{8(1+3\alpha)}. \tag{2.18}$$

But if $d_0 \ne 0$, then suppose that

$$F(d_0) := y\omega_1 + d_0\omega_2 + \frac{(B_2-B_1)d_0}{4B_1}\omega_1^2 - \left(\frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{B_1}{(1+2\alpha)}\right)\omega_1^2 + y\omega_1\right)d_0^2 \tag{2.19}$$

which is a polynomial in d_0 and hence analytic in $|d_0| \le 1$, and maximum $|F(d_0)|$ is attained at $d_0 = e^{i\theta}, (0 \le \theta < 2\pi)$. We find that

$$\max_{0\le\theta<2\pi}\left|F(e^{i\theta})\right| = |F(1)| \tag{2.20}$$

and

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{8(1+3\alpha)}\left|\omega_2 - \frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{(B_2-B_1)}{B_1} - \frac{B_1}{(1+2\alpha)}\right)\omega_1^2\right|. \tag{2.21}$$

Hence by Lemma 2.1 we get

$$\left|a_3 - \mu a_2^2\right| \le \frac{B_1}{8(1+3\alpha)} max\left\{1, \left|\frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{(B_2-B_1)}{B_1} - \frac{B_1}{(1+2\alpha)}\right)\right|\right\} \tag{2.22}$$

and this last above inequality together with (2.18) establishes the result in Theorem 2.4. This completes the proof. □

Theorem 2.5. *Let $f \in \mathcal{A}$ of the form (1.1) belong to the class $S^*(\alpha, \Phi)$. Then*

$$|a_2| \le \frac{B_1}{4(1+2\alpha)}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \le \frac{B_1}{8(1+3\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{B_1}{1+2\alpha} + \frac{B_2-B_1}{B_1} - \frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2}\right)\right|\right\}.$$

Proof . Let $f \in S^*(\alpha, \Phi)$. Similar to the proof of Theorem 2.4, if $\psi(z) = 1$, then (1.3) evidently implies that $d_0 = 1$ and $d_n = 0$, $n > 1$. Hence, in view of (2.12), (2.15) and Lemma 2.1, and 2.2 we obtain the desired result of Theorem 2.5. □

Theorem 2.6. *Let $f \in \mathcal{A}$ be of the form* (1.1) *and if it satisfies the condition*

$$\frac{zf'(z)}{f(z)} + \alpha\frac{z^2 f''(z)}{f(z)} - 1 \ll \phi(\Phi(\omega(z))) - 1. \tag{2.23}$$

Then

$$|a_2| \leq \frac{B_1}{4(1+2\alpha)}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{8(1+3\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{B_1}{1+2\alpha} + \frac{B_2 - B_1}{B_1} - \frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2}\right)\right|\right\}.$$

Proof . Following the proof of Theorem 2.4, if $\omega(z) \equiv z$ in (2.1) and by Lemma 2.3 then in view of (2.12) and (2.15), we get

$$|a_2| \leq \frac{B_1}{4(1+2\alpha)}$$

and

$$a_3 - \mu a_2^2 = \frac{B_1}{8(1+3\alpha)}\left[d_1 + \frac{(B_2 - B_1)d_0}{4B_1} - \frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{B_1}{(1+2\alpha)}\right)d_0^2\right]. \tag{2.24}$$

Substituting the value of d_1 from (2.16) in (2.24), we get

$$a_3 - \mu a_2^2 = \frac{B_1}{8(1+3\alpha)}\left[y + \frac{(B_2 - B_1)d_0}{4B_1} - \left[\frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{B_1}{(1+2\alpha)}\right] + y\right)d_0^2\right]. \tag{2.25}$$

If $d_0 = 0$, then we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{8(1+3\alpha)} \tag{2.26}$$

and if $d_0 \neq 0$, let

$$G(d_0) = y + \frac{(B_2 - B_1)d_0}{4B_1} - \left[\frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{B_1}{(1+2\alpha)}\right] + y\right)d_0^2,$$

which being a polynomial in d_0 is analytic in $|d_0| \leq 1$, and maximum of $|G(d_0)|$ is attained at $d_0 = e^{i\theta}(0 \leq \theta \leq 2\pi)$. Thus, we have

$$\max_{0 \leq \theta < 2\pi}\left|G(e^{i\theta})\right| = |G(1)|$$

and consequently

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{8(1+3\alpha)}\left|\frac{1}{4}\left(\frac{2\mu B_1(1+3\alpha)}{(1+2\alpha)^2} - \frac{B_1}{(1+2\alpha)} - \frac{(B_2 - B_1)}{B_1}\right)\right|,$$

which together with (2.26) establishes the desired result of Theorem 2.6. □

3. The Function Class $M_q(\alpha, \Phi)$

Theorem 3.1. *Let $f \in M_q(\alpha, \Phi)$. Then*

$$|a_2| \leq \frac{1}{4(1+\alpha)} \tag{3.1}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{1}{8(1+2\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{B_1(1+3\alpha)}{(1+\alpha)^2} + \frac{B_2 - B_1}{B_1} - \frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2}\right)\right|\right\}. \tag{3.2}$$

Proof . Let $f \in M_q(\alpha, \Phi)$. Then from (1.12), we have

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 = \psi(z)\left(\phi\left(\frac{\Phi(\omega(z)) - 1}{\Phi(\omega(z)) + 1}\right) - 1\right). \tag{3.3}$$

Using the expansion in (2.3) and (2.5), we obtain

$$\begin{aligned}(1+\alpha)a_2 z &+ [2(1+2\alpha)a_3 - (1+3\alpha)a_2^2]z^2 + \cdots \\ &= \frac{d_0 B_1 \omega_1}{4}z + \left(\frac{\omega_2 B_1 d_0}{4} + \frac{(B_2 - B_1)\omega_1^2 d_0}{16} + \frac{\omega_1 B_1 d_1}{4}\right)z^2 + \cdots.\end{aligned} \tag{3.4}$$

Comparing the coefficient in (3.4), we find that

$$(1+\alpha)a_2 = \frac{d_0 B_1 \omega_1}{4}, \tag{3.5}$$

$$2(1+2\alpha)a_3 - (1+3\alpha)a_2^2 = \frac{\omega_2 B_1 d_0}{4} + \frac{(B_2 - B_1)\omega_1^2 d_0}{16} + \frac{\omega_1 B_1 d_1}{4}. \tag{3.6}$$

Now, (3.5) gives

$$a_2 = \frac{d_0 B_1 \omega_1}{4(1+\alpha)}, \tag{3.7}$$

which in view of (3.6) yields

$$a_3 = \frac{B_1}{8(1+2\alpha)}\left[\omega_1 d_1 + d_0\left(\omega_2 + \frac{\omega_1^2}{4}\left(\frac{(B_2 - B_1)}{B_1} + \frac{(1+3\alpha)B_1 d_0}{(1+\alpha)^2}\right)\right)\right]. \tag{3.8}$$

For some $\mu \in \mathbb{C}$, and from (3.7) and (3.8) we obtain

$$\begin{aligned}&a_3 - \mu a_2^2 \\ &= \frac{B_1}{8(1+2\alpha)}\left[\omega_1 d_1 + \omega_2 d_0 - \frac{1}{4}\left(\frac{2\mu B_1 d_0^2(1+2\alpha)}{(1+\alpha)^2} - \left(\frac{(B_2 - B_1)d_0}{B_1} + \frac{(1+3\alpha)B_1 d_0^2}{(1+\alpha)^2}\right)\right)\omega_1^2\right].\end{aligned} \tag{3.9}$$

Since $\psi(z)$ is given by (1.3) is analytic and bounded in $\mathbb{U}$, therefore, on using [11, p. 172], we have for some $y\,(|y| \leq 1)$. On putting the value of d_1 from (2.16) into (3.9), we get

$$\begin{aligned}a_3 - \mu a_2^2 = \frac{B_1}{8(1+2\alpha)}&\left[y\omega_1 + \omega_2 d_0 + \frac{(B_2 - B_1)d_0\omega_1^2}{4B_1}\right. \\ &\left. - \left(\frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2}\right)\omega_1^2 + y\omega_1\right)d_0^2\right].\end{aligned} \tag{3.10}$$

If $d_0 = 0$ in (3.10), we at once get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{8(1+2\alpha)}. \tag{3.11}$$

But if $d_0 \neq 0$, let us then suppose that

$$F(d_0) = y\omega_1 + \omega_2 d_0 + \frac{(B_2 - B_1)d_0\omega_1^2}{4B_1} - \left(\frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2}\right)\omega_1^2 + y\omega_1\right)d_0^2,$$

which is a polynomial in d_0 and hence analytic in $|d_0| \leq 1$, and maximum $|F(d_0)|$ is attained at $d_0 = e^{i\theta}, (0 \leq \theta < 2\pi)$. We find that

$$\max_{0 \leq \theta < 2\pi} \left|F(e^{i\theta})\right| = |F(1)|$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{8(1+2\alpha)}\left|\omega_2 - \frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2} - \frac{(B_2 - B_1)d_0}{B_1}\right)\omega_1^2\right|.$$

On using Lemma 2.1 shows that

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{8(1+2\alpha)} max\left\{1, \left|\frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2} - \frac{(B_2 - B_1)d_0}{4B_1}\right)\right|\right\} \tag{3.12}$$

and this last above inequality together with (3.11) thus establishes the result in Theorem 3.1. This completes the proof. □

Theorem 3.2. *Let $f \in M(\alpha, \Phi)$. Then*

$$|a_2| \leq \frac{1}{4(1+\alpha)}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{8(1+2\alpha)} max\left\{1, \left|\frac{1}{4}\left(\frac{B_1(1+3\alpha)}{(1+\alpha)^2} - \frac{B_2 - B_1}{B_1} - \frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2}\right)\right|\right\}.$$

Proof . Let $f \in M(\alpha, \Phi)$. Similar to the proof of Theorem 3.1, if $\psi(z) = 1$, then (1.3) evidently implies that $d_0 = 1$ and $d_n = 0, n = 2, 3, 4, \ldots$. Hence, in view of (3.7), (3.9) and Lemma 2.1, and 2.2 we obtain the desired result Theorem 3.2. □

The next result is devoted to the majorization and the result pertaining to it is contained in the following:

Theorem 3.3. *If a function $f \in \mathcal{A}$ of the form (1.1) satisfies*

$$(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - 1 \ll \phi(\Phi(\omega(z))) - 1, \tag{3.13}$$

then

$$|a_2| \leq \frac{1}{4(1+\alpha)}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{1}{8(1+2\alpha)} \max\left\{1, \left|\frac{1}{4}\left(\frac{B_1(1+3\alpha)}{(1+\alpha)^2} + \frac{B_2 - B_1}{B_1} - \frac{2\mu B_1(1+2\alpha)}{(1+2\alpha)^2}\right)\right|\right\}.$$

The result is sharp.

Proof . Following the proof of Theorem 3.1, if $\omega(z) \equiv z$ in (2.1) so that $\omega_1 = 1$ and by Lemma 2.3, then in view of (3.7) and (3.9), we get

$$|a_2| \leq \frac{B_1}{4(1+\alpha)},$$

$$a_3 - \mu a_2^2 = \frac{B_1}{8(1+2\alpha)}\left[d_1 + \frac{(B_2-B_1)d_0}{4B_1} - \frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2}\right)d_0^2\right]. \tag{3.14}$$

On putting the value of d_1 from (2.16) in (3.14), it is seen that

$$a_3 - \mu a_2^2 = \frac{B_1}{8(1+2\alpha)}\left[y + \frac{(B_2-B_1)d_0}{4B_1} - \left[\frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2}\right) + y\right]d_0^2\right]. \tag{3.15}$$

If $d_0 = 0$ in (3.15), we obtain

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{8(1+2\alpha)} \tag{3.16}$$

and if $d_0 \neq 0$, let

$$G(d_0) := y + \frac{(B_2-B_1)d_0}{4B_1} - \left[\frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2}\right) + y\right]d_0^2, \tag{3.17}$$

which is a polynomial in d_0 and hence analytic in $|d_0| \leq 1$, and maximum $|F(d_0)|$ is attained at $d_0 = e^{i\theta}, (0 \leq \theta < 2\pi)$. We find that

$$\max_{0\leq\theta<2\pi}\left|F(e^{i\theta})\right| = |F(1)| \tag{3.18}$$

and consequently

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{8(1+2\alpha)} max\left\{1, \left|\frac{1}{4}\left(\frac{2\mu B_1(1+2\alpha)}{(1+\alpha)^2} - \frac{(1+3\alpha)B_1}{(1+\alpha)^2} - \frac{(B_2-B_1)}{B_1}\right)\right|\right\}, \tag{3.19}$$

which together with (3.17) establishes the desired result of Theorem 3.3. □

4. The Function Class $L_q(\alpha, \Phi)$

Theorem 4.1. *Let $f \in \mathcal{A}$ of the form (1.1) belong to the class $L_q(\alpha, \Phi)$. Then*

$$|a_2| \leq \frac{B_1}{|4(2-\alpha)|} \tag{4.1}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|8(3-2\alpha)|}\max\left\{1, \left|\frac{1}{4}\left(\frac{B_1\left(3(4-3\alpha)-(2-\alpha)^2\right)}{2(2-\alpha)^2} + \frac{B_2-B_1}{B_1} - \frac{2\mu B_1(3-2\alpha)}{(2-\alpha)^2}\right)\right|\right\}. \tag{4.2}$$

Proof . Let $f \in L_q(\alpha, \Phi)$. Then from (1.18), we have

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha}\left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 = \psi(z)\left(\phi\left(\frac{\Phi(\omega(z))-1}{\Phi(\omega(z))+1}\right) - 1\right). \tag{4.3}$$

Using the expansion in (2.3) and (2.5) we obtain

$$(2-\alpha)a_2 z + \frac{1}{2}[4(3-2\alpha)a_3 + ((2-\alpha)^2 - 3(4-3\alpha))a_2^2]z^2 + \cdots$$
$$= \frac{d_0 B_1 \omega_1}{4} z + \left(\frac{\omega_2 B_1 d_0}{4} + \frac{(B_2 - B_1)\omega_1^2 d_0}{16} + \frac{\omega_1 B_1 d_1}{4}\right) z^2 + \cdots . \quad (4.4)$$

Comparing the coefficient in (4.4), and proceeding as in above theorems 2.4 and 3.1 we get

$$a_2 = \frac{d_0 B_1 \omega_1}{4(2-\alpha)}, \quad (4.5)$$

$$a_3 = \frac{B_1}{8(3-2\alpha)}\left[\omega_1 d_1 + d_0\left(\omega_2 + \frac{1}{4}\left(\frac{(B_2-B_1)}{B_1} + \frac{(3(4-3\alpha)-(2-\alpha)^2)B_1 d_0}{2(2-\alpha)^2}\right)\omega_1^2\right)\right]. \quad (4.6)$$

For some $\mu \in \mathbb{C}$, we obtain from (4.5) and (4.6)

$$a_3 - \mu a_2^2 = \frac{B_1}{8(3-2\alpha)}\left[\omega_1 d_1 + \omega_2 d_0 - \frac{1}{4}\left(\frac{2\mu B_1 d_0^2(3-2\alpha)}{(2-\alpha)^2} - \left(\frac{(B_2-B_1)d_0}{B_1} + \frac{(3(4-3\alpha)-(2-\alpha)^2)B_1 d_0^2}{2(2-\alpha)^2}\right)\right)\omega_1^2\right]. \quad (4.7)$$

Proceeding as on similar arguments in Theorems 2.4 and 3.1 we get Since $\psi(z)$ is given by (1.3) is analytic and bounded in $\mathbb{U}$, therefore, on using [11, p. 172], we have for some $y\,(|y| \leq 1)$ upon putting the value of d_1 from (2.16) into (4.7), we get

$$a_3 - \mu a_2^2 = \frac{B_1}{8(3-2\alpha)}\left[y\omega_1 + \omega_2 d_0 + \frac{(B_2-B_1)d_0\omega_1^2}{4B_1} - \left[\frac{1}{4}\left(\frac{2\mu B_1(3-2\alpha)}{(2-\alpha)^2} - \frac{(3(4-3\alpha)-(2-\alpha)^2)B_1}{2(2-\alpha)^2}\right)\omega_1^2 + y\omega_1\right]d_0^2\right]. \quad (4.8)$$

If $d_0 = 0$ in (4.8), we at once get

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{|8(3-2\alpha)|}. \quad (4.9)$$

But if $d_0 \neq 0$, let us then suppose that

$$F(d_0) := y\omega_1 + \omega_2 d_0 + \frac{(B_2-B_1)d_0\omega_1^2}{4B_1} - \left[\frac{1}{4}\left(\frac{2\mu B_1(3-2\alpha)}{(2-\alpha)^2} - \frac{(3(4-3\alpha)-(2-\alpha)^2)B_1}{2(2-\alpha)^2}\right)\omega_1^2 + y\omega_1\right]d_0^2. \quad (4.10)$$

Again by similar arguments in Theorems 2.4 and 3.1 and using Lemma 2.1 shows that

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{|8(3-2\alpha)|}\max\left\{1, \left|\frac{1}{4}\left(\frac{2\mu B_1(3-2\alpha)}{(2-\alpha)^2} - \frac{(3(4-3\alpha)-(2-\alpha)^2)B_1}{2(2-\alpha)^2} - \frac{(B_2-B_1)}{B_1}\right)\right|\right\} \quad (4.11)$$

and this last above inequality together with (4.9) thus establishes the result in Theorem 4.1. This completes the proof. □

Theorem 4.2. *Let $f \in \mathcal{A}$ of the form* (1.1) *belong to the class $L_q(\alpha, \Phi)$. Then*

$$|a_2| \leq \frac{B_1}{|4(2-\alpha)|}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|8(3-2\alpha)|} \max\left\{1, \left|\frac{1}{4}\left(\frac{B_1\left(3(4-3\alpha)-(2-\alpha)^2\right)}{2(2-\alpha)^2} + \frac{B_2 - B_1}{B_1} - \frac{2\mu B_1(3-2\alpha)}{(2-\alpha)^2}\right)\right|\right\}.$$

Proof . Let $f \in L(\alpha, \Phi)$. Similar to the proof of Theorem 4.1, if $\psi(z) = 1$, then (1.3) evidently implies that $d_0 = 1$ and $d_n = 0$, $n = 2, 3, 4, \ldots$ Hence, in view of (4.10), (4.11) and Lemma 2.1 and 2.2, we obtain the desired result of Theorem 4.2. □

Theorem 4.3. *If a function $f \in \mathcal{A}$ of the form* (1.1) *satisfies*

$$\left(\frac{zf'(z)}{f(z)}\right)^{\alpha} + \left(1 + \frac{zf''(z)}{f'(z)}\right)^{1-\alpha} - 1 \ll \phi(\Phi(\omega(z))) - 1, \tag{4.12}$$

then

$$|a_2| \leq \frac{B_1}{|4(2-\alpha)|}$$

and for some $\mu \in \mathbb{C}$

$$|a_3 - \mu a_2^2| \leq \frac{B_1}{|8(3-2\alpha)|} max\left\{1, \left|\frac{1}{4}\left(\frac{B_1\left(3(4-3\alpha)-(2-\alpha)^2\right)}{2(2-\alpha)^2} + \frac{B_2 - B_1}{B_1} - \frac{2\mu B_1(3-2\alpha)}{(2-\alpha)^2}\right)\right|\right\}.$$

Proof . Following the proof of Theorem 4.1 if $\omega(z) \equiv z$ in (2.1) so that $\omega_1 = 1$ and by Lemma 2.3, then in view of (4.10) and (4.11), we get

$$|a_2| \leq \frac{B_1}{|4(2-\alpha)|}$$

and

$$\left|a_3 - \mu a_2^2\right| \leq \frac{B_1}{|8(3-2\alpha)|} \max\left\{1, \left|\frac{1}{4}\left(\frac{2\mu B_1(3-2\alpha)}{(2-\alpha)^2} - \frac{3(4-3\alpha)-(2-\alpha)^2 B_1}{2(2-\alpha)^2} - \frac{(B_2 - B_1)}{4B_1}\right)\right|\right\}. \tag{4.13}$$

Substituting the value of d_1 from (2.16) in (4.13), it is seen that

$$\begin{aligned} a_3 - \mu a_2^2 = \frac{B_1}{8(3-2\alpha)} \Bigg[y + \omega_2 d_0 + \frac{(B_2 - B_1)d_0}{4B_1} \\ - \left(\frac{1}{4}\left(\frac{2\mu B_1 d_0^2(3-2\alpha)}{(2-\alpha)^2} - \frac{3(4-3\alpha)-(2-\alpha)^2 B_1}{2(2-\alpha)^2}\right) + y\right) d_0^2 \Bigg]. \end{aligned} \tag{4.14}$$

Proceeding on lines similar to Theorem 2.6 and 3.3 and the arguments, we get the desired results given in Theorem 4.3. □

5. Concluding Remark

Obviously, various other interesting consequences of our general results (which are asserted in above Theorems) can be derived by appropriately specializing the parameters in these results.

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