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Numerical resolution of large deflections in cantilever beams by Bernstein spectral method and a convolution quadrature

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Abstract

The mathematical modeling of the large deflections for the cantilever beams leads to a nonlinear differential equation with the mixed boundary conditions. Different numerical methods have been implemented by various authors for such problems. In this paper, two novel numerical techniques are investigated for the numerical simulation of the problem. The first is based on a spectral method utilizing modal Bernstein polynomial basis. This gives a polynomial expression for the beam configuration. To do so, a polynomial basis satisfying the boundary conditions is presented by using the properties of the Bernstein polynomials. In the second approach, we first transform the problem into an equivalent Volterra integral equation with a convolution kernel. Then, the second order convolution quadrature method is implemented to discretize the problem along with a finite difference approximation for the Neumann boundary condition on the free end of the beam. Comparison with the experimental data and the existing numerical and semi–analytical methods demonstrate the accuracy and efficiency of the proposed methods. Also, the numerical experiments show the Bernstein–spectral method has a spectral order of accuracy and the convolution quadrature methods equipped with a finite difference discretization gives a second order of accuracy.

Keywords: Bernstein polynomials; cantilever beam; large deflection; nonlinearity; convolution quadrature.

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1. Introduction

In some problems of science and engineering, the governing equation describing the behaviour of the system, involves a nonlinear term. Depending on the physical features, the nonlinearity may be replaced with a linear term, then a linear solver is used. However, linearization techniques may fail to give a good enough approximation, especially when a Taylor based linearization is used over a nonsmall interval. Among these problems, the large deflection of a cantilever beam modeled by a nonlinear differential equation with the mixed boundary conditions are concerned in this work. Displacement of a structural element like a beam is discussed by the deflection that is associated with the slope of the beam in the current configuration, that is equal to the integral of the slope function. In mechanical engineering, the evaluation of deflection for a cantilever beam made of a linear-elastic isotropic material is a well-known classic problem in the strength of materials and theory of elasticity [6, 12]. In the field of biomechanics, determination of the deflection in vertebrate long bone as a slender cantilever beam with large deflection is also taken into account [4, 19].

In this work, the large deflection of a cantilever beam subjected to a single load at its end is considered. The material model of the beam is linear–elastic. The dominant assumption underlying this work is that our beam behaves as a thin plate with a large deflection that is governed by the theory of plates and sheets. According to this theory, the behavior of the thin plates with relatively low resistance to bending are the same as membranes. Bending of such a thin plate leads to strain in the middle plane. In the case of large deflection for deriving the differential equations, the corresponding stresses should be taken into account that leads to geometric nonlinearity, thus the equations of equilibrium are formulated in the deformed state and are updated with the deformation during the deflection. With respect to the geometric nonlinearity, the bending displacements of a deflected cantilever beam are obtained from the classical beam theory that is a special case of Timoshenko beam theory [14].

Following is a short description of other research studies seeking to evaluate the large deflection of the cantilever beams. Bisshop and Druckerin [3] investigated the large deflection of the cantilever beams for both the rectangular and round cross-sections. They used a Runge–Kutta method at the beginning and their solution is recursively corrected with a predictor–corrector. Lee, [10] solved the same problem for Ludwick type material and combined loading consisting of a uniformly distributed load and one vertical concentrated load at the free end by Butcher's fifth order Runge–Kutta method. Applying the nonlinear shooting method, Banerjee et al. [1] converted the boundary valued problem into an initial value problem, then with some assumptions on the curvature at the fixed point, they applied the Runge–Kutta method to the differential equation. Considering the Ludwick type stress–strain law, a corrected bending moment was introduced by Solano E. Carrillo [18]. It is demonstrated that for some special cases, the differential equation of large deflection can be solved by a semi-analytical method. More recently, Kimiaeifar et al. [9] and Maleki et al. [13] presented homotopy semi–analytical solutions for the large deflection analysis of a cantilever beam under free end and uniform distributed loads. Considering the critical role of laboratory testing, Beléndez et al. [2] presented a comparison between laboratory experimental data and theoretical results. They made a system of a steel ruler of a rectangular section that is fixed at one end and loaded at the free end with a mass.

In fluid mechanics, especially the gas dynamics, the typical problems represent singularities in the solutions. However, for the solid mechanics, the solutions naturally represent smooth solutions. This involves both linear and nonlinear cases. Then the solutions are continuous functions of the parameters. Therefore by the Weierstrass theorem, the solutions for these problems may be approximated by polynomials to any desired accuracy. This is the idea behind the spectral methods in which the basis for expanding the solution is chosen from orthogonal polynomials, commonly Jacobi polynomials. In this paper, we choose the Bernstein polynomials with the spectral method to simulate the nonlinear differential equation modeling the cantilever beam. This basis is not orthogonal, however with simple features, it provides a good tool for the approximation of differential equations. Recently many works have used this function along with numerical methods for solving differential, integro-differential and fractional differential equations, see for instance, [5, 8] and the references therein.

In this paper, we present a numerical method for the simulation of a cantilever beam expressed as a boundary value problem with mixed conditions. We first introduce a basis by Bernstein polynomials satisfying homogeneous mixed boundary conditions. Then, a Bernstein–spectral method is presented for the numerical simulation of the problem. Also, a convolution quadrature method combined with a second order backward difference for the handling of the Neumann condition at the free end of the cantilever is presented for the discretization of the problem. It is discussed that the resulting nonlinear system has a special structure that makes it possible to be approximated by a linear system.

This paper is organized as follows. Section 2 describes the physical aspects and modeling of a cantilever beam with regard to the static governing equations of the Euler–Bernoulli beam. Section 3 introduces a basis by Bernstein polynomials in order to use with the spectral method for the discretization of the problem. In section 4, we we the transformation to a Volterra integral equation and the convolution quadrature method is presented. The numerical experiments are provided in Section 5. The paper ends with some concluding remarks.

2. Formulation of the deflection in the cantilever beam

Fig. 1 shows a deflected cantilever beam subjected to one single load and a moment at its end [16, 7]. The Bernoulli–Euler beam equation states at any arbitrary point on the beam, the relationship of the bending moment and the curvature, κ as [6]

$$M = EI\kappa,$$

where E is the modulus of elasticity, I is the second moment of inertia of the cross section of the beam with respect to axis y and $\kappa = \frac{d\theta}{ds}$. EI is the stiffness of the beam.

2.1. Kinematic Equations

The kinematic equations which describe the motion of every point along the beam are written as [6]

$$\theta = \frac{dw}{ds} \quad \kappa = \frac{d^2w}{ds^2} = \frac{d\theta}{ds},$$

where s is the distance between any arbitrary point and the fixed end of the cantilever beam, $w(x)$ is transverse displacement, θ is the rotation or slope and κ is the curvature. It is assumed that during the deflection of the beam, the cross section remains normal to the axis of the beam Fig. 1.

With decomposition of the effect of the end force F and end moment M_0 into a pair of horizontal and a vertical components one obtains,

$$M(x, y) = EI\frac{d\theta}{ds} = P(a - x) + nP(b - y) + M_0, \tag{2.1}$$

where the points a and b be the start and the end points of the deflected beam,P andnP are the horizontal and vertical components and EI is the flexural rigidity of the beam. During deformation, the length of the beam in initial configuration L, is considered to be constant, thus:

$$\frac{dx}{ds} = \cos\theta, \quad \frac{dy}{ds} = \sin\theta.$$

So

$$x = \int_0^s \cos\theta ds, \quad y = \int_0^s \sin\theta ds. \tag{2.2}$$

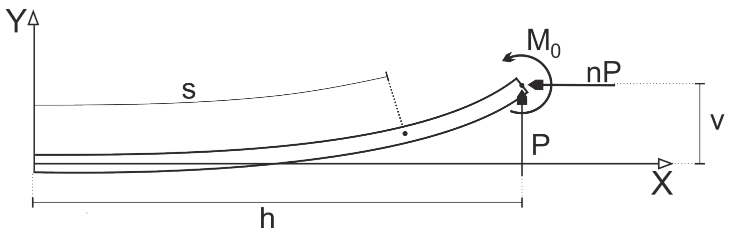

Figure 1: Deflected cantilever beam subjected to one single load and a moment at its end.

2.2. Constitutive differential equation

Differentiating equation (2.2) with respect to s, it can be expressed as

$$EI\frac{d^2\theta}{ds^2} = -P\frac{dx}{ds} - nP\frac{dy}{ds}. \tag{2.3}$$

Substituting equations (2.2) and (2.3) for the second derivative of θ respect to the s we get:

$$\frac{d^2\theta}{ds^2} = -\frac{P}{EI}(\cos\theta + n\sin\theta), \tag{2.4}$$

that is a non–linear second order differential equation.

$$\frac{d^2\theta}{ds^2} = -\frac{P}{EI}(\cos\theta + n\sin\theta). \tag{2.5}$$

2.3. Boundary conditions

To solve the second order differential equation (2.5) two boundary conditions are required. For this special case, the boundary conditions are the mixed one and the boundary value problem is nonhomogeneous, Fig. 2:

$$\theta(0) = 0, \quad \frac{d\theta}{ds}|_{s=L} = \beta = \frac{M_0}{EI}. \tag{2.6}$$

Note that when the value of end moment is zero $\beta = 0$.

However, This is a typical problem with mixed boundary conditions, This type of problems can be solved using some numerical methods that use the same idea by introducing auxiliary parameters, writing the differential equation as an initial value problem and using RK methods [4]. Among them the Shooting method is well–known. Finite element methods use low order polynomials as basis functions can effectively give satisfactory solutions, specially for problems with local features. For example in numerical simulation of car accidents, etc. On the other hand, spectral methods approximate the solution by using high order polynomials. It is known that these methods provide accurate results for problems with smooth solutions over the whole domain. Moreover the order of convergence is spectral, in the sense that it behaves faster than any polynomials order $O(h^p)$ [15].

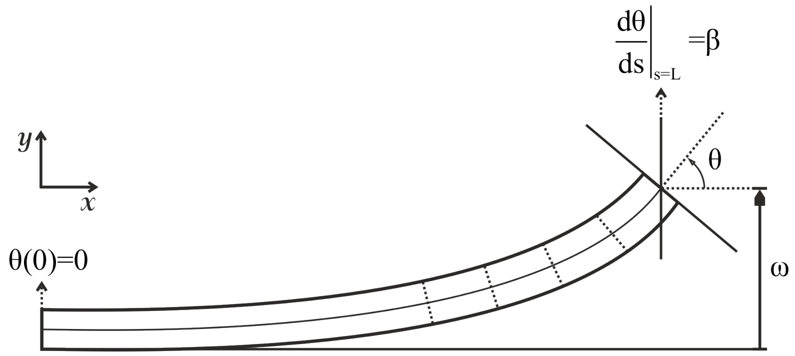

Figure 2: Cross-sections of the beam remain plane during bending.

3. Bernstein basis polynomials

For a positive integer N, the Bernstein polynomials of degree N over the interval $[a, b]$ are defined as

$$B_{i,N}(x) = \binom{N}{i} \frac{(x-a)^i (b-x)^{N-i}}{(b-a)^N} \qquad i = 0, 1 \dots, N. \tag{3.1}$$

Here $\binom{N}{i}$ is the binomial coefficient given as $\binom{N}{i} = \frac{N!}{i!(N-i)!}$ and a and b represent the beginning and end point of the beam. It is convenient to assume that $B_{i,N}(x) \equiv 0$ for $i < 0$ and $i > N$. The set of Bernstein polynomials with degree N, i.e., $\{\phi_i := B_{i,N}(x) : i = 0, \dots, N\}$ forms a basis for $\mathbb{P}_N$, the space of polynomials with degree less than or equal N.

3.1. Some useful properties of the Bernstein basis to use with the Cantilever beams

Based on the mixed boundary conditions (2.6), we present a basis for solving equation (2.5) that its properties are in good accordance with the physical properties of the problem.

It can be easily seen that for the boundaries, the values of the Bernstein polynomials are given by;

$$\phi_i(a) = \delta_{i,0}, \quad \phi_i(b) = \delta_{i,N}, \tag{3.2}$$

$$\frac{d\phi_i}{dx}|_{x=a} = \frac{-N}{b-a}(\delta_{i,0} - \delta_{i,1}), \quad \frac{d\phi_i}{dx}|_{x=a} = \frac{-N}{b-a}(\delta_{i,N-1} - \delta_{i,N}), \tag{3.3}$$

for $0 \le i \le N$, in which δ stands for the Kronecker delta given by

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

Based on (3.2) and (3.3) we present the following result.

Theorem 3.1. Let $N \in \mathbb{N}$, $a, b \in \mathbb{R}$ and $a < b$. The set $\{\Phi_i\}_{i=1}^{N-1}$ defined by

$$\Phi_i(x) = \begin{cases} \phi_i(x), & i = 1, \dots, N-2, \\ \phi_i(x) + \phi_{i+1}(x), & i = N-1, \end{cases} \tag{3.4}$$

forms a basis for $\mathbb{P}_N^0 = \{v \in \mathbb{P}_N,\ v(a) = 0, v'(b) = 0\}$.

Proof . Based on (3.2) and (3.3) we derive that $\Phi_i \in \mathbb{P}_N^0$ for $i = 1, \ldots, N-1$. On the other hand, $\dim \mathbb{P}_N = N+1$ so $\dim \mathbb{P}_N^0 = N-1$. Therefore, it suffices to prove that $\{\Phi_i\}_{i=1}^{N-1}$ are linear independent. To see this, let $\sum_{i=1}^{N-1} c_i \Phi_i(x) \equiv 0$. Putting $x = 1$, we get $c_N = 0$. Since Bernstein polynomials forms a basis, so Φ_i $i = 1, \ldots, N-2$ are linear independence. Thus the other coefficients must be zero. □

3.2. Variational formulation of the problem

Considering the new function

$$\bar{\theta} = \theta - \beta s, \tag{3.5}$$

it is easy to see that the equation (2.5)–(2.6) is written as the following differential equation with homogeneous boundary conditions:

$$\frac{d^2\bar{\theta}}{ds^2} = -\frac{P}{EI}(\cos(\bar{\theta} + \beta s) + n \sin(\bar{\theta} + \beta s)), \tag{3.6}$$

$$\bar{\theta}(0) = 0, \quad \frac{d\bar{\theta}}{ds}|_{s=L} = 0. \tag{3.7}$$

This simplifies the formulation of the spectral method described below.

Let $a = 0$ and $b = L$ and $\Omega = (0, L)$. The deflection of the cantilever beam behaves as a continuous phenomena. The Weierstrass theorem states that the continuous functions can be approximated by polynomials with any desired accuracy. So we can seek the solution in the space of polynomials satisfying the boundary conditions, $\mathbb{P}_N^0$. The Galerkin weak formulation of the problem (2.5)–(2.6) is to find $\bar{\Theta} \in \mathbb{P}_N^0$ such that

$$(\frac{d\bar{\Theta}}{ds}, \frac{dv}{ds}) = \frac{P}{EI}(\cos \bar{\Theta} + n \sin \bar{\Theta}, v) \quad \forall v \in \mathbb{P}_N^0, \tag{3.8}$$

in which the inner product is the standard inner product of the vector space $L^2(\Omega)$, i.e., $(f, g) = \int_\Omega f g ds$.

Based on the basis functions introduced in Theorem 3.1, let us write

$$\bar{\Theta}(s) = \sum_{j=1}^{N-1} c_i \Phi_i(s), \tag{3.9}$$

as an approximate for $\theta(s)$, in which the coefficients c_i are determined by taking $v = \Phi_i$, $i = 1, \ldots, N-1$ and solving the resulting algebraic system. For the ease of computations, we use the Galerkin method with numerical integration, i.e.,

$$< \frac{d\bar{\Theta}}{ds}, \frac{d\Phi_i}{ds} >= \frac{P}{EI} < \cos \bar{\Theta} + n \sin \bar{\Theta}, \Phi_i >, \quad i = 1, \ldots, N-1. \tag{3.10}$$

in which $< \cdot, \cdot >$ stands for the discrete L^2-product. A Gauss quadrature is used to perform the numerical integration.

4. Second approach: convolution quadrature

In this section, we first present briefly the convolution quadrature method. Then, writing the differential equation with mixed conditions (3.6)–(3.7) as an equivalent Volterra integral equation, it is discretized with the second order convolution quadrature method.

4.1. The convolution quadrature

Convolution quadrature method for approximating integrals of the convolution form, including especially the fractional derivatives and integrals, was first discovered and analyzed by Lubich [11]. For the functions k and f, the convolution quadrature approximates the continuous convolution integral

$$\int_0^s k(s-t)f(t)dt \quad s>0, \tag{4.1}$$

at nodes $s=s_n=mh$ for $m=1,2,\dots$, with a step size $h>0$ with a discrete convolution given by

$$\sum_{j=0}^{m} \bar{w}_{m-j} f_j, \quad m=1,\dots,M, \tag{4.2}$$

where $f_j=f(jh)$ and the convolution quadrature weights are given as the coefficients of the generating power series

$$K(\frac{\delta(\xi)}{h}) = \sum_{j=0}^{\infty} w_j \xi^j, \tag{4.3}$$
$$\bar{w}_m = \frac{w_m}{2}, \quad \bar{w}_j = w_j,\, j\neq m,$$

in which K is the Laplace transform of k and $\delta(\xi)=(1-\xi)+(1-\xi)^2/2$ is the based on the second order backward difference formula (BDF). The following theorem states the method with the second order BDF gives a second order accuracy (see for instance [17]).

Theorem 4.1. Let $K(s)$ is analytic and bounded as $|K(s)|\le \bar{M}|s-\sigma|^{-\mu}$ for $|arg(s-\sigma)|<\phi$ with $\phi>\frac{\pi}{2}$ for some real $\mu>0,\bar{M}$ and σ and let $f(t)=ct^{\gamma-1},\gamma>0$. Then, the error of the convolution quadrature approximation (4.2) is written as

$$|\int_0^s k(s-t)f(t)dt - \sum_{j=0}^{n} \bar{w}_{n-j} f_j| \le Ct^{\mu-2}h^2.$$

4.2. Application to the deflection of the cantilever beam

Consider the problem (3.6) with mixed boundary conditions (3.7). Let $\frac{d\bar{\theta}}{ds}|_{s=0}=z_0$. Integrating (3.6) from 0 to s gives

$$\frac{d\bar{\theta}}{ds} = z_0 - \frac{P}{EI}\int_0^s (\cos(\bar{\theta}+\beta v) + n\sin(\bar{\theta}+\beta v))dv. \tag{4.4}$$

Integrating again from 0 to s, with changing the order of integration, we get

$$\bar{\theta}(s) = z_0 s - \frac{P}{EI}\int_0^s (s-v)\left(\cos(\bar{\theta}(v)+\beta v) + n\sin(\bar{\theta}(v)+\beta v)\right)dv. \tag{4.5}$$

This is the convolution integral (4.1) with the kernel $k(t)=t$. Considering (4.5) at $s=s_m=mh, m=1,\dots,M$ with $h=L/M$ and applying the discrete approximation (4.2), we have the following system of algebraic equations

$$\bar{\theta}_m = z_0 s_m - \frac{P}{EI}\left(\sum_{j=0}^{m} \bar{w}_{m-j}\left(\cos(\bar{\theta}_j+\beta s_j) + n\sin(\bar{\theta}_j+\beta s_j)\right)\right) \quad m=1,\dots,M.$$

Now by (3.5), we get M equations with $M+1$ unknowns, z_0 and $\theta_m, m=1,\ldots,M$ as

$$\theta_m = z_0 s_m + \beta s_m - \frac{P}{EI}\left(\sum_{j=0}^{m} \bar{w}_{m-j}\left(\cos\theta_j + n\sin\theta_j\right)\right) \quad m=1,\ldots,M. \tag{4.6}$$

We impose an extra equation by using a second order difference approximation to (3.7) as

$$\frac{\bar{\theta}_M - 4\bar{\theta}_{M-1} + \bar{\theta}_{M-2}}{2h} = 0.$$

Equivalently, by (3.5), we get

$$\theta_M - 4\theta_{M-1} + \theta_{M-2} + \beta(s_M - 4s_{M-1} + s_{M-2}) = 0. \tag{4.7}$$

Since for the convolution integral (4.5), $K(s) = \frac{1}{s^2}$, based on Theorem 4.1, the method has a second order convergence rate as it is verified by the numerical examples in the following section. When $\theta_m, m=1,\ldots,M$ are obtained by solving (4.6)-(4.7), then by (2.2), x and y are obtained for instance by Legendre quadrature.

Note that (4.6) can be written as

$$\theta_m + \lambda_0\left(\cos\theta_m + n\sin\theta_m\right) - z_0 s_m = \beta s_m - \frac{P}{EI}\sum_{j=0}^{m-1}\bar{w}_{m-j}\left(\cos\theta_j + n\sin\theta_j\right),$$

for $m=1,\ldots,M, \lambda_0 = -\frac{P}{EI}\bar{w}_0$. This is a Now by (4.3), we have $\bar{w}_0 = K(\frac{\delta(0)}{h}) = \frac{2h^2}{3} \to 0$ as $h \to 0$ with $O(h^2)$, so the nonlinear system can be explicitly solved by removing the second term in the LHS of the above system. Otherwise, it may be solved by using an iterative solver such as successive iteration method.

5. Numerical results

In this section, we present some numerical experiments based on both Bernstein–spectral method and the convolution quadrature method for varying parameters.

In Table 1, the numerical results obtained by both collocation and Galerkin Bernstein methods are reported and compared with the elliptic integral solutions and the Adomian method [1] at $\bar{s}=1$. It is seen that both the collocation and Galerkin methods give a better accuracy.

[!htb]

Table 1: Comparison of the numerical results for different schemes at $\bar{s}=1$

Loads	Elliptic solution		Shooting [1]		Bernstein Collocation		Bernstein Galerkin	
	$\bar{x}$	$\bar{y}$	$\bar{y}$	$\max(e_{\bar{x}}, e_{\bar{y}})$	$\bar{y}$	$\max(e_{\bar{x}}, e_{\bar{y}})$	$\bar{y}$	$\max(e_{\bar{x}}, e_{\bar{y}})$
$\alpha=1, \kappa=0, n=1$	0.879	0.4292	0.4295	3.20E-04	0.4292	5.73E-05	0.4292	8.48E-06
$\alpha=1, \kappa=0.2, n=1$	0.817	0.5139	0.5142	3.90E-04	0.5139	6.16E-05	0.5139	8.46E-06
$\alpha=1, \kappa=-0.6, n=1$	0.997	0.0456	0.4560	4.10E-01	0.0456	3.12E-05	0.0456	2.00E-06
$\alpha=0.2, \kappa=-0.6, n=0.5$	0.958	-0.2418	-0.2421	2.50E-04	-0.2418	8.96E-06	-0.2418	8.70E-06

Table 2 presents the errors $\max(|x-\bar{x}|, |y-\bar{y}|)$ for $\alpha=1.4, \kappa=0.0$ and $n=1$ at $\bar{s}=1$ for the collocation and Galerkin methods. In this table, N represents the number of basis functions used in the spectral method and the degree of the Adomian polynomials, respectively.

Table 2: The convergence of the collocation and Galerkin methods and comparison with the Adomian method.

N	Collocation			Galerkin			Adomian [1]		
	$\bar{x}$	$\bar{y}$	$\max(e_{\bar{x}}, e_{\bar{y}})$	$\bar{x}$	$\bar{y}$	$\max(e_{\bar{x}}, e_{\bar{y}})$	$\bar{x}$	$\bar{y}$	$\max(e_{\bar{x}}, e_{\bar{y}})$
2	0.75493	0.58849	9.80E-03	0.76417	0.57924	5.54E-04	0.78308	0.55860	2.01E-02
3	0.75743	0.58545	6.76E-03	0.76393	0.57868	9.35E-05	0.76760	0.57387	4.82E-03
4	0.75743	0.58545	6.76E-03	0.76384	0.57868	6.98E-06	0.75247	0.58839	1.14E-02
5	0.76728	0.57554	3.44E-03	0.76383	0.57869	1.11E-06	0.77050	0.57118	7.50E-03
6	0.76308	0.57940	7.48E-04	0.76383	0.57869	1.19E-06	0.76326	0.57820	5.72E-04
7	0.76375	0.57876	7.96E-05	0.76383	0.57869	1.19E-06	0.76471	0.57681	1.87E-03
8	0.76389	0.57863	5.81E-05	0.76383	0.57869	1.19E-06	0.76454	0.57611	2.57E-03
9	0.76382	0.57870	1.23E-05	0.76383	0.57869	1.19E-06	0.76461	0.57691	1.77E-03

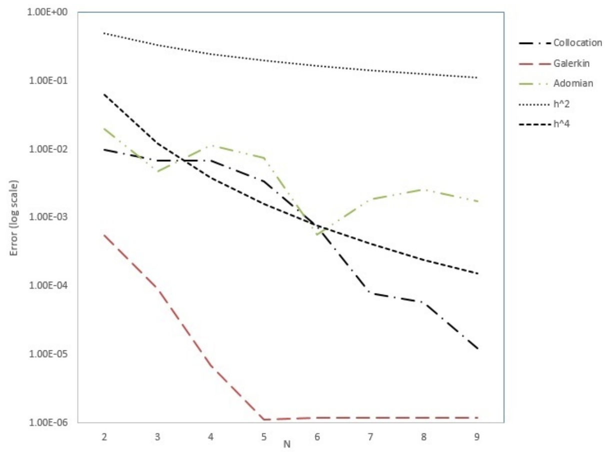

Figure 3: Comparison of the convergence of the collocation and Galerkin method with Adomian, second order and fourth order methods.

Fig. 3 shows the spectral methods converges faster than the Adomian method, and methods with polynomial order of convergence $O(h^2)$ and $O(h^4)$ in general.
In Fig. 4 the deformed configuration of the contilever beam under free–end load and moment beam configuration obtained by the collocation method is shown for $\alpha = 0.8$ and $\alpha = 1.7$.
Table 3 provides the numerical results obtained from the convolution quadrature method (4.6)–(4.7) with two different configuration. The exact solution is considered with a very fine mesh, $h = 0.005$. It is seen that the scheme preserves the second order of accuracy for both cases as it is expected from Theorem 4.1.

Table 3: Errors with convolution quadrature method at $s = 1$.

	$n = 1, \kappa = 0.4, \alpha = 1$						$n = 0.5, \kappa = -0.4, \alpha = 0.2$					
M	$\bar{x}$	$\bar{y}$	$e_{\bar{x}}$	rate	$e_{\bar{y}}$	rate	$\bar{x}$	$\bar{y}$	$e_{\bar{x}}$	rate	$e_{\bar{y}}$	rate
5	0.98710	0.11793	1.79E-03		1.51E-02		0.87273	-0.41989	2.11E-03		3.88E-03	
10	0.98848	0.10693	4.06E-04	2.14	4.13E-03	1.87	0.87116	-0.42270	5.48E-04	1.95	1.07E-03	1.86
20	0.98879	0.10387	9.60E-05	2.08	1.08E-03	1.94	0.87075	-0.42349	1.36E-04	2.01	2.78E-04	1.95
40	0.98887	0.10307	2.29E-05	2.07	2.67E-04	2.01	0.87064	-0.42370	3.29E-05	2.05	6.87E-05	2.01
80	0.98888	0.10286	4.97E-06	2.20	5.93E-05	2.17	0.87062	-0.42376	7.17E-06	2.20	1.52E-05	2.18

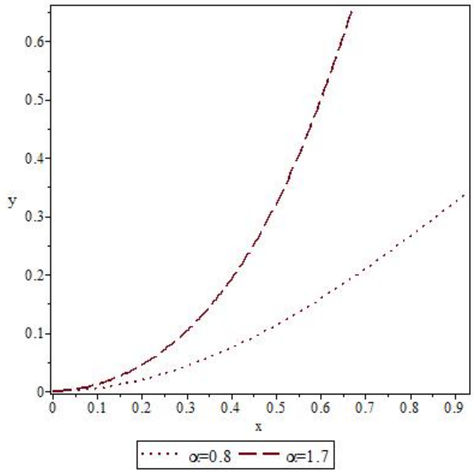

Figure 4: Beam configuration

6. Conclusion

We considered the model of a cantilever beam made of a linear–elastic isotropic material, fixed at one end subjected to a concentrated load at free end. The corresponding mixed nonlinear boundary value problem was solved using Bernstein polynomials method. For this case the main idea was to set a basis satisfying the prescribed mixed boundary conditions. Another approach was proposed based on the convolution quadrature method implemented to an equivalent form of the equation in terms of a Volterra integral equation. Numerical experiments are carried out for both approaches and the accuracy of the methods were compared with the elliptic integral solutions and the Adomian decomposition method. The results show the first approach has a spectral order of accuracy while the second approach converges with a second order rate. The methods may be used to design cantilever beams with desired configurations.

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