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New hybrid method for equilibrium problems and relatively nonexpansive mappings in Banach spaces

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Abstract

In this paper, applying hybrid projection method, a new modified Ishikawa iteration scheme is presented for finding a common element of the solution set of an equilibrium problem and the set of fixed points of relatively nonexpansive mappings in Banach spaces. A numerical example is given and the numerical behaviour of the sequences generated by this algorithm is compared with several existence results in literature to illustrate the usability of obtained results.

Keywords: Equilibrium problems; Fixed point, Hybrid method; Relatively nonexpansive mapping; Weak convergence.
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1. Introduction

Let C be a nonempty closed convex subset of a real Banach space E with the dual space E^*. The equilibrium problem in the sense of Blum and Oettli [6] for a bifunction $f : C \times C \to \mathbb{R}$ is as follow:

$$\text{“ find} \quad x \in E \quad \text{such that} \quad f(x, y) \geq 0, \quad (y \in C) \text{ ”}. \tag{1.1}$$

The solution set of (1.1) is defined by $EP(f) = \{x \in C : \ f(x, y) \geq 0, \ \ \forall y \in C\}$.

Assume that $S : C \to E^*$ is a mapping and let $f(x, y) = \langle Sx, y - x\rangle$ for all $x, y \in C$. Then $z \in EP(f)$ if and only if z is a solution of the variational inequality $\langle Sx, y - x\rangle \geq 0$ for all $y \in C$. So, the formulation (1.1) includes variational inequalities as special cases.

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The equilibrium problem is also known as Ky Fan inequality [13]. Many well-known problems are formulated as an equilibrium problem, such as the optimization problem, the variational inequality problem and nonlinear complementarity problem, the saddle point problem, the generalized Nash equilibrium problem in game theory, the fixed point problem and others; (see [22, 25]). In the other words, numerous problems in applied sciences reduce to find a solution of an equilibrium problem. For this reason, solving the equilibrium problem is very interesting and therefore some methods have been proposed to solve the equilibrium problem; see for instance [6, 12, 15, 18, 21].

Recently, many iteration processes were introduced by mathematicians for finding a common element of the set of fixed points of a nonlinear mapping and the solution set of an equilibrium problem in the framework of Hilbert spaces and Banach spaces, respectively; see for instance, [3], [4], [10, 11], [23]-[27], [31] and the references therein. The most useful processes are Mann [19] and Ishikawa [16] iteration processes.

Ishikawa process is indeed more general than Mann process. In spite of this fact, research has been done on the latter due probably to reasons that the formulation of Mann process is simpler than that of Ishikawa process and that a convergence theorem for Mann process may lead to a convergence theorem for Ishikawa process under appropriate conditions. On the other hand, the Mann process may fail to converge while Ishikawa process can still converge for a Lipschitz pseudocontractive mapping in a Hilbert space [9]. Actually, Mann and Ishikawa iteration processes have only weak convergence, in general (see [14]).

In 2009, Takahashi and Zembayashi [28] for finding an element of $EP(f) \cap F(S)$, introduced the following iterative scheme for a relatively nonexpansive self mapping S of a nonempty, closed convex subset C in a Banach space E:

$$\begin{cases} u_1 \in H \ \text{ chosen arbitrarily}, \\ u_n \in C \text{ such that } \quad f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall\, y \in C, \\ u_{n+1} = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JSx_n), \end{cases}$$

for all $n \in \mathbb{N}$, where $f : C \times C \to \mathbb{R}, \{\alpha_n\}$ and $\{r_n\}$ satisfy appropriate conditions and $F(S)$ is fixed points set of S. They proved that $\{u_n\}$ converges weakly to $w \in F(S) \cap EP(f)$, where $w = \lim_{n\to\infty} \Pi_{F(S)\cap EP(f)} x_n$.

In this paper, motivated by Takahashi-Zembayashi [28], we modify Ishikawa iteration process for finding a common element of the solution set of an equilibrium problem and the set of fixed points of a relatively nonexpansive mapping by applying the hybrid projection method in Banach spaces. We give a numerical example to illustrate the usability of our results.

2. Preliminaries

Let E be a real Banach space with the dual space E^* and C be a nonempty closed convex subset of E. We denote the weak convergence and the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \rightharpoonup x$ and $x_n \to x$, respectively. A Banach space E is said to be strictly convex if $\|\frac{x+y}{2}\| < 1$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $x \neq y$. It is also said to be uniformly convex if for every $\epsilon \in (0, 2]$, there exists a $\delta > 0$, such that $\|\frac{x+y}{2}\| < 1 - \delta$ for all $x, y \in E$ with $\|x\| = \|y\| = 1$ and $\|x - y\| \geq \epsilon$. Furthermore, E is called smooth if the limit

$$\lim_{t\to 0} \frac{\|x + ty\| - \|x\|}{t}, \tag{2.1}$$

exists for all $x, y \in B_E = \{x \in E : \|x\| = 1\}$. It is also said to be uniformly smooth if the limit (2.1) is attained uniformly for all $x, y \in B_E$. It is well known that E is uniformly convex if and only if E^* is uniformly smooth.

We denote by J the normalized duality mapping from E into 2^{E^*} which is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\},$$

for all $x \in E$, where $\langle ., . \rangle$ denotes the generalized duality pairing between E and E^*. Some properties of the normalized duality mapping are listed in the following:

1. For every $x \in E$, Jx is nonempty closed convex and bounded subset of E^*.
2. If E is smooth or E^* is strictly convex, then J is single-valued.
3. If E is strictly convex, then J is one-one, i.e., if $x \neq y$ then $Jx \cap Jy = \emptyset$.
4. If E is reflexive, then J is onto.
5. if E is smooth and reflexive, then J is norm-to-weak continuous.
6. If E is smooth, strictly convex and reflexive and $J^* : E^* \to 2^E$ is the normalized duality mapping on E^*, then $J^{-1} = J^*$, $JJ^* = I_{E^*}$ and $J^*J = I_E$, where I_E and I_{E^*} are the identity mapping on E and E^*, respectively.
7. If E is uniformly convex and uniformly smooth, then J is uniformly norm-to-norm continuous on bounded sets of E and $J^{-1} = J^*$ is also uniformly norm-to-norm continuous on bounded sets of E^*.

The duality mapping J is said to be weakly sequentially continuous if $x_n \rightharpoonup x$ implies that $Jx_n \rightharpoonup Jx$ in weak* topology. For more details see [1].

Let E be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined as follows:

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2,$$

for all $x, y \in E$. It is clear from the definition of ϕ that for all $x, y, z \in E$,

1. $(\|y\| - \|x\|)^2 \leq \phi(x, y) \leq (\|y\| + \|x\|)^2$,
2. $\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle$.

It is clear that if E is a Hilbert space, then $\phi(x, y) = \|x - y\|^2$.

In 1996, Alber [2], defined the concept of the generalized projection mapping as sequel. Assume that C is a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E. The generalized projection $\Pi_C : E \to C$ is a mapping that assigns to an arbitrary point $x \in E$ the minimum point of the functional $\phi(y, x)$, i.e., $\Pi_C x = x_0$, where x_0 is the solution to the minimization problem

$$\phi(x_0, x) = \inf_{y \in C} \phi(y, x).$$

Existence and uniqueness of the operator Π_C follow from the properties of the functional $\phi(y, x)$ and strict monotonicity of the mapping J.

Let S be a self-mapping of C. A point p in C is said to be an asymptotic fixed point of S [24], if there exists a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $\|x_n - Sx_n\| \to 0$. We denote by $\hat{F}(S)$ the set of all asymptotic fixed points of S. A self-mapping S of C is said to be relatively nonexpansive [7, 8], if the following conditions are satisfied:

1. $F(S)$ is nonempty;
2. $\phi(u, Sx) \leq \phi(u, x)$, $\forall u \in F(S)$, $\forall x \in C$;

3. $F(S) = \hat{F}(S)$.

Lemma 2.1. [20] Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E and let S be a relatively nonexpansive self-mapping of C. Then $F(S)$ is closed and convex.

Some well-known properties of generalized metric projection are listed below. We will use them in the proof of our main results in next section.

Lemma 2.2. [2] Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E. Then

(i) $\phi(x, \Pi_C y) + \phi(\Pi_C y, y) \leq \phi(x, y)$, for all $x \in C$ and all $y \in E$.
(ii) $z = \Pi_C x \iff \langle y - z, Jx - Jz \rangle \leq 0$, for all $y \in C$.

Lemma 2.3. [30] Let E be a uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and

$$\|tx + (1-t)y\|^2 \leq t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)g(\|x - y\|),$$

for all $x, y \in B_r$ and $t \in [0, 1]$, where $B_r = \{z \in E : \|z\| \leq r\}$.

Lemma 2.4. [17] Let E be a uniformly convex Banach space and $r > 0$. Then there exists a strictly increasing, continuous and convex function $g : [0, 2r] \to \mathbb{R}$ such that $g(0) = 0$ and

$$g(\|x - y\|) \leq \phi(x, y),$$

for all $x, y \in B_r$.

To study the equilibrium problem, for the bifunction $f : C \times C \to \mathbb{R}$, we assume that f satisfies the following conditions:

(A1) $f(x, x) = 0$ for all $x \in C$;
(A2) f is monotone, i.e., $f(x, y) + f(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,
$$\lim_{t \downarrow 0} f(tz + (1-t)x, y) \leq f(x, y);$$
(A4) for each $x \in C$, $y \mapsto f(x, y)$ is convex and lower semicontinuous.

Lemma 2.5. [6] Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $r > 0$ and $x \in E$. Then, there exists $z \in C$ such that

$$f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0,$$

for all $y \in C$.

Lemma 2.6. [28] Let C be a nonempty, closed convex subset of a smooth, strictly convex and reflexive Banach space E, f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and let $r > 0$ and $x \in E$. Define a mapping $T_r : E \to C$ as follows:

$$T_r(x) = \{z \in C : f(z, y) + \frac{1}{r}\langle y - z, Jz - Jx \rangle \geq 0, \ \forall y \in C\},$$

for all $x \in E$. Then, the following statements hold:

(i) T_r is single-valued;

(ii) T_r is firmly nonexpansive-type, i.e., for all $x, y \in E$,

$$\langle T_r x - T_r y, JT_r x - JT_r y\rangle \leq \langle T_r x - T_r y, Jx - Jy\rangle;$$

(iii) $F(T_r) = EP(f)$;

(iv) EP(f) is closed and convex and T_r is relatively nonexpansive mapping.

Lemma 2.7. [28] Let C be a nonempty closed convex subset of a smooth, strictly convex and reflexive Banach space E, let f be a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and let $r > 0$. Then for $x \in E$ and $p \in F(T_r)$,

$$\phi(p, T_r x) + \phi(T_r x, x) \leq \phi(p, x).$$

3. Main Results

In this section, we prove some weak convergence theorems for finding an element of the solution set of an equilibrium problem which is a fixed point of a relatively nonexpansive mapping.

Proposition 3.1. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E. Assume that f is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1)-(A4)$ and S is a relatively nonexpansive self-mapping of C with $F(S) \cap EP(f) \neq \emptyset$. Suppose that $0 < a \leq \alpha_n \leq 1$ and $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$. If $\{x_n\}$ is a sequence generated by $u_1 \in E$ and*

$$\begin{cases} x_n \in C \text{ such that } \quad f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall\, y \in C, \\ y_n = J^{-1}((1-\beta_n)Jx_n + \beta_n JSx_n), \\ u_{n+1} = J^{-1}((1-\alpha_n)Jx_n + \alpha_n JSy_n), \end{cases}$$

for all $n \in \mathbb{N}$, then $\{\Pi_{F(S)\cap EP(f)} x_n\}$ converges strongly to $w \in F(S) \cap EP(f)$, where $\Pi_{F(S)\cap EP(f)}$ is the generalized projection of E onto $F(S) \cap EP(f)$.

Proof . Suppose that $u \in F(S) \cap EP(f)$. Letting $x_n = T_{r_n} u_n$ for all $n \in \mathbb{N}$, since T_{r_n} and also S are relatively nonexpansive, we have

$$\begin{aligned} \phi(u, y_n) &= \phi(u, J^{-1}(1-\beta_n)Jx_n + \beta_n JSx_n) \\ &= \|u\|^2 - 2\langle u, (1-\beta_n)Jx_n + \beta_n JSx_n\rangle + \|(1-\beta_n)Jx_n + \beta_n JSx_n\|^2 \\ &\leq \|u\|^2 - 2(1-\beta_n)\langle u, Jx_n\rangle - 2\beta_n\langle u, JSx_n\rangle + (1-\beta_n)\|x_n\|^2 + \beta_n\|Sx_n\|^2 \\ &\leq \phi(u, x_n) \end{aligned}$$

and therefore

$$\begin{aligned} \phi(u, x_{n+1}) &= \phi(u, T_{r_{n+1}} u_{n+1}) \\ &\leq \phi(u, u_{n+1}) \\ &= \phi(u, J^{-1}((1-\alpha_n)Jx_n + \alpha_n Jy_n) \\ &\leq \|u\|^2 - 2(1-\alpha_n)\langle u, Jx_n\rangle - 2\alpha_n\langle u, Ju_n\rangle + (1-\alpha_n)\|x_n\|^2 + \alpha_n\|Sy_n\|^2 \\ &\leq \phi(u, x_n). \end{aligned} \tag{3.1}$$

Then, we can conclude that $\lim_{n\to\infty} \phi(u, x_n)$ exists. Therefore $\phi(u, x_n)$ is bounded and so $\{x_n\}$ and $\{y_n\}$ are bounded. On the other hand, since S is relatively nonexpansive, so $\{Sx_n\}$ and $\{Sy_n\}$ are bounded. Put $z_n = \Pi_{F(S)\cap EP(f)}x_n$ for all $n \in \mathbb{N}$. Therefore from $z_n \in F(S) \cap EP(f)$ and (3.1), we get

$$\phi(z_n, x_{n+1}) \leq \phi(z_n, x_n). \tag{3.2}$$

Since $\Pi_{F(S)\cap EP(f)}$ is the generalized projection, from Lemma 2.2 (i), we obtain

$$\begin{aligned}\phi(z_{n+1}, x_{n+1}) &= \phi(\Pi_{F(S)\cap EP(f)}x_{n+1}, x_{n+1})\\ &\leq \phi(z_n, x_{n+1}) - \phi(z_n, \Pi_{F(S)\cap EP(f)}x_{n+1})\\ &= \phi(z_n, x_{n+1}) - \phi(z_n, z_{n+1})\\ &\leq \phi(z_n, x_{n+1}).\end{aligned}$$

So, using (3.2) we derive that

$$\phi(z_{n+1}, x_{n+1}) \leq \phi(z_n, x_n).$$

Hence, $\{\phi(z_n, x_n)\}$ is a convergent sequence. Also, from (3.2) we can conclude that

$$\phi(z_n, z_{n+m}) \leq \phi(z_n, x_n),$$

for all $n \in \mathbb{N}$. Utilizing Lemma 2.2 (i), we get

$$\phi(z_n, z_{n+m}) + \phi(z_{n+m}, x_{n+m}) \leq \phi(z_n, x_{n+m}) \leq \phi(z_n, x_n),$$

because of $z_{n+m} = \Pi_{F(S)\cap EP(f)}x_{n+m}$ and therefore

$$\phi(z_n, z_{n+m}) \leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}).$$

Put $r = \sup_{n\in\mathbb{N}} \|z_n\|$. By Lemma 2.4, there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to \mathbb{R}$ whit $g(0) = 0$ such that

$$g(\|x - y\|) \leq \phi(x, y),$$

for all $x, y \in B_r$. Then, we have

$$g(\|z_n - z_{n+m}\|) \leq \phi(z_n, z_{n+m}) \leq \phi(z_n, x_n) - \phi(z_{n+m}, x_{n+m}).$$

Using the convergence of $\{\phi(z_n, x_n)\}$ and the property of g, we can conclude that $\{z_n\}$ is a Cauchy sequence. From closedness of $F(S) \cap EP(f)$, we derive that $\{z_n\}$ is convergent strongly to $w \in F(S) \cap EP(f)$. □

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E. Assume that f is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and S is a relatively nonexpansive self-mapping of C with $F(S)\cap EP(f) \neq \emptyset$. Assume that $0 < a \leq \alpha_n \leq 1$ and $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Suppose that $\{x_n\}$ is a sequence generated by $u_1 \in E$ and*

$$\begin{cases} x_n \in C \text{ such that } \quad f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall\, y \in C,\\ y_n = J^{-1}((1 - \beta_n)Jx_n + \beta_n JSx_n),\\ u_{n+1} = J^{-1}((1 - \alpha_n)Jx_n + \alpha_n JSy_n),\end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E. If J is weakly sequentially continuous, then $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(f)$, where $w = \lim_{n\to\infty} \Pi_{F(S)\cap EP(f)}x_n$.

Proof . Similar to the proof of Proposition 3.1, we can conclude that $\{x_n\}$ and $\{Sx_n\}$ are bounded. Put $r = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|Sx_n\|\}$. Since E is a uniformly smooth Banach space, we can conclude that E^* is a uniformly convex Banach space. So, by Lemma 2.3, there exists a continuous, strictly increasing and convex function $g : [0, 2r] \to \mathbb{R}$ whit $g(0) = 0$ such that for $u \in F(S) \cap EP(f)$ we have

$$\begin{aligned}\phi(u, y_n) &= \phi(u, J^{-1}(1-\beta_n)Jx_n + \beta_n JSx_n)\\ &\le \|u\|^2 - 2(1-\beta_n)\langle u, Jx_n\rangle - 2\beta_n\langle u, JSx_n\rangle + (1-\beta_n)\|x_n\|^2 + \beta_n\|Sx_n\|^2\\ &\qquad - (1-\beta_n)\beta_n g(\|Jx_n - JSx_n\|)\\ &= \beta_n\phi(u, x_n) + (1-\beta_n)\phi(u, Sx_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|)\\ &\le \phi(u, x_n) - \beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|),\end{aligned}$$

so, using last inequality and convexity of $\|.\|^2$, we get

$$\begin{aligned}\phi(u, x_{n+1}) &= \phi(u, T_{r_{n+1}}u_{n+1})\\ &\le \phi(u, u_{n+1})\\ &= \phi(u, J^{-1}((1-\alpha_n)Jx_n + \alpha_n JSy_n)\\ &\le \|u\|^2 - 2(1-\alpha_n)\langle u, Jx_n\rangle - 2\alpha_n\langle u, JSy_n\rangle + (1-\alpha_n)\|x_n\|^2 + \alpha_n\|Sy_n\|^2 \qquad (3.3)\\ &= (1-\alpha_n)\phi(u, x_n) + \alpha_n\phi(u, Sy_n)\\ &\le (1-\alpha_n)\phi(u, x_n) + \alpha_n\phi(u, x_n) - \alpha_n\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|)\\ &\le \phi(u, x_n) - \alpha_n\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|),\end{aligned}$$

hence

$$\alpha_n\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|) \le \phi(u, x_n) - \phi(u, x_{n+1}).$$

Since $0 < a \le \alpha_n \le 1$, it is easy to see that

$$a\beta_n(1-\beta_n)g(\|Jx_n - JSx_n\|) \le \phi(u, x_n) - \phi(u, x_{n+1}). \qquad (3.4)$$

Since $\{\phi(u, x_n)\}$ is convergent and $\liminf_{n\to\infty} \beta_n(1-\beta_n) > 0$, it follows from (3.4) that

$$\lim_{n\to\infty} g(\|Jx_n - JSx_n\|) = 0.$$

Hence, from property of g, we get

$$\lim_{n\to\infty} \|JSx_n - Jx_n\| = 0. \qquad (3.5)$$

Utilizing uniformly norm-to-norm continuity of J^{-1} on bounded sets, we conclude that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \qquad (3.6)$$

Also, boundedness of $\{x_n\}$ implies that the existence of a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x} \in C$. Since S is relatively nonexpansive, we can conclude from (3.6) that $\hat{x} \in \hat{F}(S) = F(S)$.

Now, We prove that $\hat{x} \in EP(f)$. Put $s = \sup_{n\in\mathbb{N}}\{\|x_n\|, \|u_n\|\}$. Utilizing Lemma 2.4, there exists a continuous, strictly increasing and convex function $g_1 : [0, 2r] \to \mathbb{R}$ whit $g_1(0) = 0$ such that

$$g_1(\|x - y\|) \le \phi(x, y),$$

for all $x, y \in B_s$. Letting $x_n = T_{r_n} u_n$ it follows from Lemma 2.7 and (3.3) that for $u \in F(S) \cap EP(f)$,

$$g_1(\|x - y\|) \leq \phi(x_n, u_n) \leq \phi(u, u_n) - \phi(u, x_n) \leq \phi(u, x_{n-1}) - \phi(u, x_n).$$

The convergence of $\{\phi(u, x_n)\}$ implies that

$$\lim_{n\to\infty} g_1(\|x_n - u_n\|) = 0.$$

Using the property of g_1, we derive that

$$\lim_{n\to\infty} \|x_n - u_n\| = 0.$$

Utilizing uniformly norm-to-norm continuity of J on bounded sets, we obtain

$$\lim_{n\to\infty} \|Jx_n - Ju_n\| = 0.$$

Since $\liminf\limits_{n\to\infty} r_n > 0$, we get

$$\lim_{n\to\infty} \frac{\|Jx_n - Ju_n\|}{r_n} = 0. \tag{3.7}$$

From $x_n = T_{r_n} u_n$, we have

$$f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \tag{3.8}$$

for all $y \in C$. Substituting n by n_k in (3.8) and using condition $(A2)$, we obtain

$$\frac{1}{r_{n_k}}\langle y - x_{n_k}, Jx_{n_k} - Ju_{n_k}\rangle \geq -f(x_{n_k}, y) \geq f(y, x_{n_k}),$$

for all $y \in C$. Letting $k \to \infty$ in last inequality, it follows from (3.7) and condition $(A4)$ that

$$0 \geq f(y, \hat{x}),$$

for all $y \in C$. Suppose that $t \in (0, 1]$, $y \in C$ and $y_t = ty + (1-t)\hat{x}$. Therefore, $y_t \in C$ and $f(y_t, \hat{x}) \leq 0$. Hence

$$0 = f(y_t, y_t) \leq tf(y_t, y) + (1-t)f(y_t, \hat{x}) \leq tf(y_t, y),$$

so, $f(y_t, y) \geq 0$, for all $y \in E$. Taking the limit as $t \downarrow 0$ and using $(A3)$, we get $\hat{x} \in EP(f)$. Then,

$$\hat{x} \in F(S) \cap EP(f). \tag{3.9}$$

Put $z_n = \Pi_{F(S)\cap EP(f)} x_n$. From Lemma 2.2 (ii) and (3.9), we have

$$\langle z_{n_k} - \hat{x}, Jx_{n_k} - Jz_{n_k}\rangle \geq 0.$$

Taking the limit as $k \to \infty$ in last inequality, from Proposition 3.1 we conclude that

$$\langle w - \hat{x}, J\hat{x} - Jw\rangle \geq 0,$$

since $z_n \to w \in F(S) \cap EP(f)$ and J is weakly sequentially continuous. On the other hand, since J is monotone, we get

$$\langle w - \hat{x}, J\hat{x} - Jw\rangle \leq 0.$$

Therefore

$$\langle w - \hat{x}, J\hat{x} - Jw\rangle = 0.$$

Using strictly convexity of E, we derive that $w = \hat{x}$. Hence, $x_n \rightharpoonup \hat{x} \in F(S) \cap EP(f)$, where $\hat{x} = \lim\limits_{n\to\infty} \Pi_{F(S)\cap EP(f)} x_n$. □

Corollary 3.3. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E. Suppose that S is a relatively nonexpansive self-mapping of C with $F(S) \neq \emptyset$. Assume that $0 < a \leq \alpha_n \leq 1$ and $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Suppose that $\{x_n\}$ is a sequence generated by $u_1 \in E$ and*

$$\begin{cases} x_n \in C \text{ such that } \quad \langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall\, y \in C, \\ y_n = J^{-1}((1 - \beta_n)Jx_n + \beta_n JSx_n), \\ u_{n+1} = J^{-1}((1 - \alpha_n)Jx_n + \alpha_n JSy_n), \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E. Suppose that J is weakly sequentially continuous, then, $\{x_n\}$ converges weakly to $w \in F(S)$, where $w = \lim_{n\to\infty} \Pi_{F(S)}x_n$.

Proof . Letting $f(x, y) = 0$ for all $x, y \in C$ and $r_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2, we get the desired result. □

Corollary 3.4. *Let C be a nonempty closed convex subset of a uniformly smooth and uniformly convex Banach space E. Suppose that f is a bifunction from $C \times C$ to $\mathbb{R}$ satisfying $(A1) - (A4)$ and S is a relatively nonexpansive self-mapping of C with $F(S) \cap EP(f) \neq \emptyset$. Assume that $\{r_n\} \subset (0, \infty)$ and $\{\beta_n\}$ is a sequence in $[0, 1]$ such that $\liminf_{n\to\infty} \beta_n(1 - \beta_n) > 0$. Suppose that $\{x_n\}$ is a sequence generated by $u_1 \in E$ and*

$$\begin{cases} x_n \in C \text{ such that } \quad f(x_n, y) + \frac{1}{r_n}\langle y - x_n, Jx_n - Ju_n\rangle \geq 0, \quad \forall\, y \in C, \\ y_n = J^{-1}((1 - \beta_n)Jx_n + \beta_n JSx_n), \\ u_{n+1} = Sy_n, \end{cases}$$

for all $n \in \mathbb{N}$, where J is the duality mapping on E. If J is weakly sequentially continuous, then, $\{x_n\}$ converges weakly to $w \in F(S) \cap EP(f)$, where $w = \lim_{n\to\infty} \Pi_{F(S)\cap EP(f)}x_n$.

Proof . Letting $\alpha_n = 1$ for all $n \in \mathbb{N}$ in Theorem 3.2, we get the desired result. □

4. Numerical example

In this section, we present a numerical example to illustrate our algorithm which is given in Theorem 3.2. Also, we compare the numerical behaviour of the sequences generated by our algorithm with several existence results in literature to demonstrate the usability of our results.

Example 4.1. Let $E = \mathbb{R}$ and $C = [-3, 3]$. Define $f(x, y) := -4x^2 + 3xy + y^2$, so the conditions $(A1) - (A4)$ are satisfied as follows:

(A1) $f(x, x) = -4x^2 + 3x^2 + x^2 = 0$ for all $x \in [-3, 3]$,
(A2) $f(x, y) + f(y, x) = -3(x - y)^2 \leq 0$ for all $x, y \in [-3, 3]$, i.e., f is monotone,
(A3) for each $x, y, z \in [-3, 3]$,

$$\begin{aligned} \lim_{t\downarrow 0} f(tz + (1 - t)x, y) &= \lim_{t\downarrow 0}(-4(tz + (1 - t)x)^4 + 3(tz + (1 - t)x)y + y^2) \\ &= -4x^2 + 3xy + y^2 \\ &= f(x, y), \end{aligned}$$

(A4) It is easily seen that for each $x \in [-3,3]$, $y \to (-4x^2+3xy+y^2)$ is convex and lower semicontinuous.

Moreover,

$$\frac{1}{r}\langle y-x, x-u\rangle = \frac{1}{r}(y-u)(x-u) = \frac{1}{r}(yx-uy-x^2+ux).$$

It follows from condition (i) of Lemma 2.6 that T_r is Single-valued. Let $u = T_r x$, for any $y \in [-3,3]$ and $r > 0$, we have

$$f(x,y) + \frac{1}{r}\langle y-u, x-u\rangle \geq 0.$$

Thus

$$\begin{aligned} -4rx^2 + 3rxy + ry^2 + yx - uy - x^2 + ux \\ &= ry^2 + (3rx + x - u)y - 4ru^2 - x^2 + ux \\ &\geq 0. \end{aligned}$$

Now, let $a = r$, $b = 3rx + x - u$ and $c = -4ru^2 - x^2 + ux$. Hence, we should have $\Delta = b^2 - 4ac \leq 0$, i.e.,

$$\begin{aligned} \Delta &= ((3r+1)x-u)^2 - 4rx(4rx - x + u) \\ &= 25r^2x^2 + 10rx^2 + x^2 + u^2 - 10rxu - 2xu \\ &= ((5r+1)x-u)^2 \\ &\leq 0. \end{aligned}$$

So, it follows that $x = \frac{u}{5r+1}$ and $T_r u = \frac{u}{5r+1}$. This implies that in Theorem 3.2, $x_n = T_{r_n} u_n = \frac{u_n}{5r_n+1}$. Since $F(T_{r_n}) = 0$, from condition (iii) of Lemma 2.6, we get $EP(f) = \{0\}$.

Now, define $S : C \to C$ by $Sx = \frac{1}{4}x$ for all $x \in C$, then $F(S) = \{0\}$ and

$$\phi(0, Sx) = \phi(0, \frac{1}{4}x) = \left|0 - \frac{1}{4}x\right|^2 \leq |x|^2 = \phi(0, x),$$

for all $x \in C$. Let $x_n \rightharpoonup p$ such that $\lim_{n\to\infty} |Sx_n - x_n| = 0$, this yields that $\hat{F}(S) = \{0\}$. Hence, $\hat{F}(S) = F(S)$, i.e., S is a relatively nonexpansive mapping.

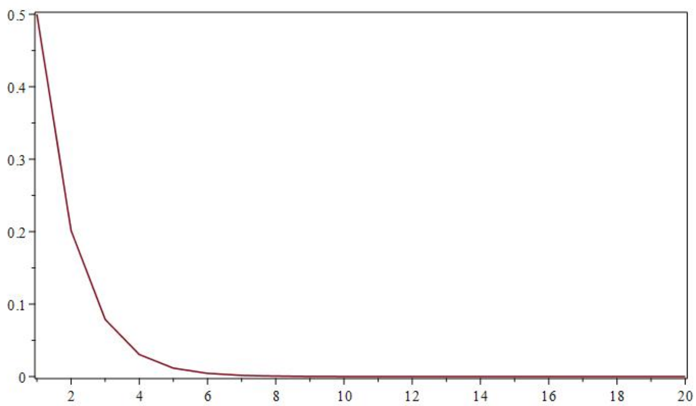

Figure 1: The convegence behavior of the sequence $\{u_n\}$ with starting point $u_1 = 0.5$.

Define $\alpha_n = \frac{1}{3} - \frac{1}{5n}$, $\beta_n = \frac{1}{4} + \frac{1}{3n}$ and $r_n = \frac{1}{5}$, then $\{\alpha_n\}$, $\{\beta_n\}$ and $\{r_n\}$ satisfy in the conditions of Theorem 3.2. Since in a Hilbert space the mapping J is identity and $x_n = \frac{1}{2}u_n$, we get

$$y_n = \frac{1}{2}\left(\frac{3}{4} - \frac{1}{3n}\right)u_n + \frac{1}{8}\left(\frac{1}{4} + \frac{1}{3n}\right)u_n = \left(\frac{13}{32} - \frac{1}{8n}\right)u_n,$$

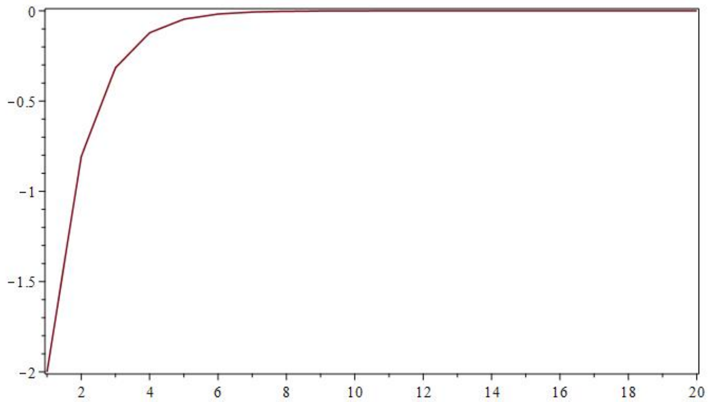

Figure 2: The convegence behavior of the sequence $\{u_n\}$ with starting point $u_2 = -2$.

Table1	Numerical Results for $u_1 = 0.5$ and $u_2 = -2$		
n	u_n	n	u_n
1	0.5	1	-2
2	0.20169	2	-0.80677
3	0.07886	3	-0.31542
4	0.03035	4	-0.12141
5	0.01157	5	-0.04629
$\vdots$	$\vdots$	$\vdots$	$\vdots$
18	3.20783×10^{-8}	18	-1.28313×10^{-7}
19	1.18963×10^{-8}	19	-4.75851×10^{-8}
20	4.40955×10^{-9}	20	-1.76382×10^{-8}
$\vdots$	$\vdots$	$\vdots$	$\vdots$
48	3.40246×10^{-21}	48	-1.36098×10^{-20}
49	1.25416×10^{-21}	49	-5.01663×10^{-21}
50	4.62252×10^{-22}	50	-1.84901×10^{-21}

Table 1: Numerical results for the sequence $\{u_n\}$ with two different starting points: $u_1 = 0.5$ and $u_2 = -2$.

also

$$\begin{aligned} u_{n+1} =& (1-\alpha_n)x_n + \frac{1}{4}\alpha_n y_n \\ =& \frac{1}{2}\left(\frac{2}{3}+\frac{1}{5n}\right)u_n + \frac{1}{4}\left(\frac{1}{3}-\frac{1}{5n}\right)\left(\frac{13}{32}-\frac{1}{8n}\right)u_n \\ =& \left(\frac{141}{384}+\frac{133}{1920n}+\frac{1}{160n^2}\right)u_n. \end{aligned} \tag{4.1}$$

Therefore $P_{F(S)\cap EP(f)}(x_n) = 0$ for all $n \geq 1$, because of $F(S) \cap EP(f) = \{0\}$. Taking the limit as $n \to \infty$ in (4.1), we obtain $\lim_{k\to\infty} u_n = 0$. Since $x_n = \frac{1}{2}u_n$, so $\lim_{k\to\infty} x_n = 0$. See Figure 1 and Figure 2 for investigation of the convergence behavior of the sequence $\{u_n\}$ and also, see Table1 for the values of this sequence with starting points $u_1 = 0.5$ and $u_2 = -2$. The computations associated with example were performed using Maple software.

Now, we compare the numerical behavior of our algorithm with the methods introduced by Alizadeh and Moradlou [[5], Theorem 3.1] and by Tada and Takahashi [[26], Theorem 4.1]. We

Comparing Results for $\{u_n\}$			
n	A_1	A_2	A_3
2	0.20169	0.41250	0.39792
3	0.07886	0.33000	0.30507
4	0.03035	0.25988	0.22944
5	0.01157	0.20270	0.17055
6	0.00438	0.15709	0.12578
$\vdots$	$\vdots$	$\vdots$	$\vdots$
$STOP$	10	33	28

Table 2: Comparing results for the sequence $\{u_n\}$ generated by the algorithms A_1, A_2 and A_3 with starting point $u_1 = 0.5$.

assume that α_n, β_n and r_n are defined as Example 4.1. We denote our algorithm by A_1 and Alizadeh and Moradlou's algorithm by A_2. Similarly, Tada and Takahashi's algorithm is denoted by A_3.

In Table 2, the numerical results of this comparison are reported for the sequence $\{u_n\}$ with starting point $u_1 = 0.5$ and stopping criterion $|u_n| < 10^{-4}$. It is easy to see that, convergence of the iterates which have been generated by our algorithm is faster than two other ones.

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References

[1] R.P. Agarwal, D. O'Regan and D.R. Sahu, *Fixed Point Theory for Lipschitzian-type Mappings with Applications*, Springer, New York, 2009.

[2] Y.I. Alber, *Metric and generalized projection operators in Banach spaces: Properties and applications,* in: A. G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, pp. 15–50, 1996.

[3] S. Alizadeh and F. Moradlou, *Strong convergence theorems for m-generalized hybrid mappings in Hilbert spaces,* Topol. Methods. Nonlinear Anal., 46 (2015) 315–328.

[4] S. Alizadeh and F. Moradlou, *A strong convergence theorem for equilibrium problems and generalized hybrid mappings,* Medit. J. Math., 13 (2016) 379–390.

[5] S. Alizadeh and F. Moradlou, *Weak convergence theorems for 2-generalized hybrid mappings and equilibrium problems*, Commun. Korean Math. Soc., 31 (2016) 765–777.

[6] E. Blum and W. Oettli, *From optimization and variational inequalities to equilibrium problems*, Math. Students, 63 (1994) 123–145.

[7] D. Butnariu, S. Reich and A.J. Zaslavski, *Asymptotic behavior of relatively nonexpansive operators in Banach spaces,* J. Appl. Anal., 7 (2001) 151–174.

[8] D. Butnariu, S. Reich and A.J. Zaslavski, *Weak convergence of orbits of nonlinear operators in reflexive Banach spaces,* Numer. Funct. Anal. Optim., 24 (2003) 489–508.

[9] C.E. Chidume and S.A. Mutangadura, *An example on the Mann iteration method for Lipschitz pseudocontractions,* Proc. Am. Math. Soc., 129 (2001) 2359-–2363.

[10] L.C. Ceng and J.C. Yao, *A hybrid iterative scheme for mixed equilibrium problems and fixed point problems,* J. Comput. Appl. Math., 214 (2008) 186–201.

[11] L.C. Ceng, S. Al-Homidan, Q.H. Ansari and J.C. Yao, *An iterative scheme for equilibrium problems and fixed point problems of strict pseudo-contraction mappings,* J. Comput. Appl. Math., 223 (2009) 967–974.

[12] P.L. Combettes and S.A. Hirstoaga, *Equilibrium programming in Hilbert spaces,* J. Nonlinear Convex Anal., 6 (2005) 117–136.

[13] K. Fan, *A minimax inequality and applications*, In: Shisha, O. (ed.) Inequality III, Academic press, New York, pp. 103–113, 1972.

[14] A. Genel and J. Lindenstrass, *An example concerning fixed points,* Israel J. Math., 22 (1975) 81—86.

[15] F. Giannessi, A. Maugeri and P.M. Pardalos, *Equilibrium Problems: Nonsmooth Optimization and Variational Inequality Models,* Kluwer Academics Publishers, Dordrecht, Holland, 2001.

[16] S. Ishikawa, *Fixed points by a new iteration method*, Proc. Amer. Math. Soc., 40 (1974) 147–150.

[17] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space,* SIAM J. Optim., 13 (2002) 938–945.

[18] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their Applications,* Academic Press, New York, 1980.

[19] W. R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc., 4 (1953) 506–510.

[20] S. Matsushita and W. Takahashi, *A strong convergence for relatively nonexpansive mappings in a Banach space,* J. Approx. Theory, 134 (2005) 257–266.

[21] A. Moudafi and M. Thera, *Proximal and dynamical approaches to equilibrium problems*, in: Lecture Note in Economics and Mathematical Systems, vol. 477, Springer-Verlag, New York, pp. 187–201, 1999.

[22] L.D. Muu and T.D. Quoc, *Regularization algorithms for solving monotone Ky Fan inequalities with application to a Nash-cournot equilibrium model*, J. Optim. Theory Appl., 142 (2009) 185–204.

[23] S. Plubtieng and P. Kumam, *Weak convergence theorem for monotone mappings and a countable family of nonexpansive mappings,* J. Comput. Appl. Math., 224 (2009) 614–621.

[24] S. Reich, *A weak convergence theorem for the alternating method with Bregman distance,* in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, Marcel Dekker, New York, pp. 313–318, 1996.

[25] J.J. Strodiot, T.T.V. Neguyen and V.H. Neguyen, *A new hybrid extragradient algorithms for solving quasi-equilibrium problems*, J. Glob. Optim., 56 (2013) 373–397.

[26] A. Tada and W. Takahashi, *Weak and strong convergence theorems for a nonexpansive mapping and an equilibrium problem*, J. Optim. Theory Appl., 133 (2007) 359–370.

[27] W. Takahashi and K. Zembayashi, *Strong convergence theorems by hybrid methods for equilibrium problems and relatively nonexpansive mapping,* Fixed Point Theory Appl. (2008) Article ID 528476, 11 pages.

[28] W. Takahashi and K. Zembayashi, *Strong and weak convergence theorems for equilibrium problems and relatively nonexpansive mappings in Banach spaces,* Nonlinear Anal., 70 (2009) 45–57.

[29] K.K. Tan and H.K. Xu, *Approximating fixed points of nonexpansive mappings by the Ishikawa iteration process*, J. Math. Anal. Appl., 178 (1993) 301–308.

[30] H.K. Xu, *Inequalities in Banach spaces with application,* Nonlinear Anal., 16 (1991) 1127–1138.

[31] G. Zamani Eskandani and M. Raeisi, *A new algorithm for finding fixed points of Bregman quasi-nonexpansive mappings and zeros of maximal monotone operators by using products of resolvents*, Results Math., 71 (2017) 1307–1326.