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# Solutions of initial and boundary value problems via $F$-contraction mappings in metric-like space

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**Abstract**

We present sufficient conditions for the existence of solutions of second-order two-point boundary value and fractional order functional differential equation problems in a space where self-distance is not necessarily zero. For this, first we introduce a Ćirić type generalized $F$-contraction and $F$-Suzuki contraction in a metric-like space and give relevance to fixed point results. To illustrate our results, we give throughout the paper some examples.

*Keywords:* Metric-like space; fixed point; F-contraction; boundary value problem.
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## 1. Introduction

Matthews [7] introduced the notion of a partial metric space as a part of the study of denotational semantics of data flow networks. He showed that the Banach contraction mapping theorem can be generalized to the partial metric context for applications in program verification. In partial metric spaces, self-distance of an arbitrary point need not be equal to zero.

Recently, Amini-Harandi [1] introduced the notion of metric-like space which is a new generalization of partial metric space. Amini-Harandi defined $\sigma$-completeness of metric-like spaces. Further, Shukla et al. introduced in [11] the notion of 0-$\sigma$-complete metric-like space and proved some fixed point theorems in such spaces, as improvements of Amini-Harandi's results.

First, we recall some definitions and facts which will be used throughout the paper.

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**Definition 1.1.** [7] A partial metric on a nonempty set $\mathcal{X}$ is a function $p : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ such that for all $x, y, z \in \mathcal{X}$:

($p_1$) $x = y \Longleftrightarrow p(x,x) = p(x,y) = p(y,y)$,

($p_2$) $p(x,x) \leq p(x,y)$,

($p_3$) $p(x,y) = p(y,x)$,

($p_4$) $p(x,y) \leq p(x,z) + p(z,y) - p(z,z)$.

The pair $(\mathcal{X}, p)$ is called a partial metric space.

A basic example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in \mathbb{R}^+$. Other examples of partial metric spaces which are interesting from a computational point of view may be found in [3, 7]. Obviously, one of the main features of this generalization of metric spaces is the so-called "non-zero self-distance". It is also a property of the following generalization.

**Definition 1.2.** [1] A metric-like on a nonempty set $\mathcal{X}$ is a function $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ such that, for all $x, y, z \in \mathcal{X}$,

($\sigma_1$) $\sigma(x,y) = 0 \Rightarrow x = y$;
($\sigma_2$) $\sigma(x,y) = \sigma(y,x)$;
($\sigma_3$) $\sigma(x,y) \leq \sigma(x,z) + \sigma(z,y)$.

A metric-like space is a pair $(\mathcal{X}, \sigma)$ such that $\mathcal{X}$ is a nonempty set and $\sigma$ is a metric-like on $\mathcal{X}$.

Each metric-like $\sigma$ on $\mathcal{X}$ generates a topology $\tau_\sigma$ on $\mathcal{X}$ whose base is the family of open $\sigma$-balls

$$B_\sigma(x, \epsilon) = \{y \in \mathcal{X} : |\sigma(x,y) - \sigma(x,x)| < \epsilon\}, \text{ for all } x \in \mathcal{X} \text{ and } \epsilon > 0.$$

It is obvious that each metric space is a partial metric space and each partial metric space is a metric-like space, but the converse may not be true.

**Example 1.3.** [1] Let $\mathcal{X} = \{0, 1\}$ and $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2, & \text{if } \ x = y = 0, \\ 1, & \text{otherwise.} \end{cases}$$

Then $(\mathcal{X}, \sigma)$ is a metric-like space, but it is neither a metric space nor a partial metric space, since $\sigma(0,0) > \sigma(0,1)$.

**Example 1.4.** [10] Let $\mathcal{X} = [0,1]$, then mapping $\sigma_i : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ defined by $\sigma_1(u,v) = u + v - uv$, is a metric-like on $\mathcal{X}$.

**Example 1.5.** [10] Let $\mathcal{X} = \mathbb{R}$; then the mapping $\sigma_i : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+ (i \in \{2,3,4\})$, defined by

$$\sigma_2(u,v) = |u| + |v| + a, \qquad \sigma_3(u,v) = |u - b| + |v - b|, \qquad \sigma_4(u,v) = (u)^2 + (v)^2,$$

are metric-like on $\mathcal{X}$, where $a \geq 0$ and $b \in \mathbb{R}$.

**Definition 1.6.** [1, 11] Let $(\mathcal{X}, \sigma)$ be a metric-like space. Then:

1. A sequence $\{x_n\}$ in $\mathcal{X}$ converges to a point $x \in \mathcal{X}$ if $\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x)$. The sequence $\{x_n\}$ is said to be $\sigma$-Cauchy if $\lim_{n,m\to\infty} \sigma(x_n, x_m)$ exists and is finite. The space $(\mathcal{X}, \sigma)$ is called complete if for each $\sigma$-Cauchy sequence $\{x_n\}$, there exists $x \in \mathcal{X}$ such that
$$\lim_{n\to\infty} \sigma(x_n, x) = \sigma(x, x) = \lim_{n,m\to\infty} \sigma(x_n, x_m).$$
2. A sequence $\{x_n\}$ in $(\mathcal{X}, \sigma)$ is called a 0-$\sigma$-Cauchy sequence if $\lim_{n,m\to\infty} \sigma(x_n, x_m) = 0$. The space $(\mathcal{X}, \sigma)$ is said to be 0-$\sigma$-complete if every 0-$\sigma$-Cauchy sequence in $\mathcal{X}$ converges (in $\tau_\sigma$) to a point $x \in \mathcal{X}$ such that $\sigma(x, x) = 0$.
3. a mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is continuous, if the following limits exist (finite) and
$$\lim_{n\to\infty} \sigma(x_n, x) = \lim_{n\to\infty} \sigma(\mathcal{T}x_n, \mathcal{T}x).$$

**Remark 1.7.** [1] Let $\mathcal{X} = \{0, 1\}$, let $\sigma(x, y) = 1$ for each $x, y \in \mathcal{X}$, and let $x_n = 1$ for each $n \in \mathbb{N}$. Then it is easy to see that $x_n \to 0$ and $x_n \to 1$, and so in metric-like spaces the limit of a convergent sequence is not necessarily unique.

**Lemma 1.8.** [6] Let $(\mathcal{X}, \sigma)$ be a metric-like space.

(a) If $x, y \in \mathcal{X}$ then $\sigma(x, y) = 0$ implies that $\sigma(x, x) = \sigma(y, y) = 0$.

(b) If a sequence $\{x_n\}$ in $\mathcal{X}$ converges to some $x \in \mathcal{X}$ with $\sigma(x, x) = 0$ then $\lim_{n\to\infty} \sigma(x_n, y) = \sigma(x, y)$ for all $y \in \mathcal{X}$.

**Remark 1.9.** [11] If a metric-like space is $\sigma$-complete, then it is 0-$\sigma$-complete. The following example shows that the converse assertions of these facts do not hold.

**Example 1.10.** [11] Let $\mathcal{X} = [0, 1) \cap \mathbb{Q}$ and $\sigma : \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be defined by
$$\sigma(x, y) = \begin{cases} 2x, & \text{if } x = y, \\ \max\{x, y\}, & \text{otherwise} \end{cases}$$
for all $x, y \in \mathcal{X}$ . Then $(\mathcal{X}, \sigma)$ is a metric-like space. Note that $(\mathcal{X}, \sigma)$ is not a partial metric space, as $\sigma(1, 1) = 2 > 1 = \sigma(1, 0)$. Now, it is easy to see that $(\mathcal{X}, \sigma)$ is a 0-$\sigma$-complete metric-like space, while it is not a $\sigma$-complete metric-like space.

In [14], Wardowski introduced a new type of contractions which he called $F$-contractions. Several authors proved various variants of fixed point results using such contractions. Adapting Wardowski's approach to metric space, the set of functions $\mathfrak{F}$ defined as follows:

**Definition 1.11.** We denote by $\mathfrak{F}$ the family of all functions $F : \mathbb{R}^+ \to \mathbb{R}$ with the following properties:

(F1) $F$ is strictly increasing, that is, for all $\alpha, \beta \in (0, \infty)$ such that $\alpha < \beta$, $F(\alpha) < F(\beta)$.

(F2) For each sequence $\{\alpha_n\}$ of positive numbers,
$$\lim_{n\to\infty} \alpha_n = 0 \text{ if and only if } \lim_{n\to\infty} F(\alpha_n) = -\infty.$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\alpha\to 0^+} \alpha^k F(\alpha) = 0$.

**Example 1.12.** [14]. Let $F_i : \mathbb{R}^+ \to \mathbb{R}$ ( $i = 1, 2, 3, 4$ ) be defined by $F_1(t) = \ln t$, $F_2(t) = \ln t + t$, $F_3(t) = -1/\sqrt{t}$, $F_4(t) = \ln(t^2 + t)$. Then each $F_i$ satisfies the properties (F1)–(F3).

**Definition 1.13.** [14] Let $(\mathcal{X}, d)$ be a metric space. A self-mapping $\mathcal{T}$ on $\mathcal{X}$ is called an $F$-contraction if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\tau + F(d(\mathcal{T}x, \mathcal{T}y)) \le F(d(x, y)), \tag{1.1}$$

for all $x, y \in \mathcal{X}$ with $d(\mathcal{T}x, \mathcal{T}y) > 0$.

**Example 1.14.** [14] Let $F : \mathbb{R}^+ \to \mathbb{R}$ be given by $F(x) = \ln x$. It is clear that $\mathfrak{F}$ satisfies (F1)–(F3) for any $k \in (0, 1)$. Each mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfying (1.1) is an $F$-contraction such that

$$d(\mathcal{T}x, \mathcal{T}y) \le e^{-\tau} d(x, y), \text{ for all } x, y \in \mathcal{X}, \quad \mathcal{T}x \ne \mathcal{T}y.$$

It is clear that for $x, y \in \mathcal{X}$ such that $\mathcal{T}x \ne \mathcal{T}y$ the previous inequality also holds and hence $\mathcal{T}$ is a contraction.

Many work have been done in this direction, (see, for example, [8, 9, 12, 13, 15]). In this paper, we introduce the notion of rational Ćirić type generalized $F$-contraction and $F$- Suzuki contraction in a metric-like space and utilize the same to establish fixed point results. In support we supply some examples to verify the results. Finally, we use the obtained results to derive the solutions of second-order two-point boundary value and fractional order functional differential equation problems.

## 2. The main results

We first introduce the notion of Ćirić type generalized $F$-contraction in a metric-like space.

**Definition 2.1.** Let $(\mathcal{X}, \sigma)$ be a metric-like space. A self-mapping $\mathcal{T}$ on $\mathcal{X}$ is called an Ćirić type generalized $F$-contraction, if there exist $F \in \mathfrak{F}$ and $\tau \in \mathbb{R}^+$ such that

$$\begin{aligned} &\tau + F(\sigma(\mathcal{T}x, \mathcal{T}y)) \\ &\le F\left( \max\left\{ \sigma(x, y), \sigma(x, \mathcal{T}x), \sigma(y, \mathcal{T}y), \frac{\sigma(x, \mathcal{T}y) + \sigma(y, \mathcal{T}x)}{4}, \frac{\sigma(x, \mathcal{T}x)\sigma(y, \mathcal{T}y)}{1 + \sigma(x, y)} \right\} \right), \end{aligned} \tag{2.1}$$

for all $x, y \in \mathcal{X}$ with $\sigma(\mathcal{T}x, \mathcal{T}y) > 0$.

If $F(\alpha) = \ln(\alpha)$ $(\alpha > 0)$ and $\tau = \ln(\frac{1}{\lambda})$, where $\lambda = (0, 1)$, we can say that every Ćirić type generalized contraction is also Ćirić type generalized $F$-contraction in metric-like space.

Our first main result is as follows:

**Theorem 2.2.** *Let $(\mathcal{X}, \sigma)$ be a $0 - \sigma$-complete metric-like space and $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ be a Ćirić type generalized $F$-contraction. If $\mathcal{T}$ or $F$ is continuous, then $\mathcal{T}$ has a unique fixed point in $\mathcal{X}$.*

**Proof .** If $\mathcal{T}x_0 = x_0$, then the proof is completed. Suppose $\mathcal{T}x_0 \ne x_0$. Put $x_n = \mathcal{T}^n x_0$ and so $x_{n+1} = \mathcal{T}x_n$. If there exists $n_0 \in \{1, 2, \dots\}$ such that right-hand side of (2.1) is 0 for $x = x_{n_0-1}$ and $y = x_{n_0}$, then it is clear that $x_{n_0-1} = x_{n_0} = \mathcal{T}x_{n_0-1}$ and so we have finished. Now let $x_{n+1} \ne x_n$ for every $n \in \{0, 1, \ldots\}$ and let $\varrho_n = \sigma(x_{n+1}, x_n)$ for $n \in \{0, 1, \ldots\}$. Then $\varrho_n > 0$ for all $n \in \{0, 1, \ldots\}$. Now using (2.1), we have

$$\begin{aligned}
\tau + F(\varrho_n) &= \tau + F(\sigma(x_{n+1}, x_n)) = \tau + F(\sigma(\mathcal{T}x_n, \mathcal{T}x_{n-1})) \\
&\leq F\left(\max\left\{\begin{array}{c} \sigma(x_n, x_{n-1}), \sigma(x_n, \mathcal{T}x_n), \sigma(x_{n-1}, \mathcal{T}x_{n-1}), \\ \frac{\sigma(x_n, \mathcal{T}x_{n-1}) + \sigma(x_{n-1}, \mathcal{T}x_n)}{4}, \frac{\sigma(x_n, \mathcal{T}x_n)\sigma(x_{n-1}, \mathcal{T}x_{n-1})}{1+\sigma(x_n, x_{n-1})} \end{array}\right\}\right) \\
&= F\left(\max\left\{\begin{array}{c} \sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \sigma(x_{n-1}, x_n), \\ \frac{\sigma(x_n, x_n) + \sigma(x_{n-1}, x_{n+1})}{4}, \frac{\sigma(x_n, x_{n+1})\sigma(x_{n-1}, x_n)}{1+\sigma(x_n, x_{n-1})} \end{array}\right\}\right) \\
&\leq F\left(\max\left\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1}), \frac{3\sigma(x_n, x_{n+1}) + \sigma(x_{n-1}, x_n)}{4}\right\}\right) \\
&\leq F\left(\max\left\{\sigma(x_n, x_{n-1}), \sigma(x_n, x_{n+1})\right\}\right) \\
&\leq F\left(\max\left\{\varrho_{n-1}, \varrho_n\right\}\right). \qquad (2.2)
\end{aligned}$$

If $\varrho_{n-1} \leq \varrho_n$ for some $n \in \{1, 2, 3, \ldots\}$, then from (2.2) we have $\tau + F(\varrho_n) \leq F(\varrho_n)$, which is a contradiction since $\tau > 0$. Thus $\varrho_{n-1} > \varrho_n$ for all $n \in \{1, 2, 3, \ldots\}$ and so from (2.2) we have

$$F(\varrho_n) \leq F(\varrho_{n-1}) - \tau.$$

Therefore we derive

$$F(\varrho_n) \leq F(\varrho_{n-1}) - \tau \leq F(\varrho_{n-2}) - 2\tau \leq \ldots \leq F(\varrho_0) - n\tau \text{ for all } n \in \mathbb{N},$$

that is,

$$F(\varrho_n) \leq F(\varrho_0) - n\tau \text{ for all } n \in \mathbb{N}. \qquad (2.3)$$

From (2.3), we get $F(\varrho_n) \to -\infty$ as limit $n \to \infty$. Thus, from (F2), we have

$$\lim_{n\to\infty} \varrho_n = 0. \qquad (2.4)$$

Now by property (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (\varrho_n)^k F(\varrho_n) = 0. \qquad (2.5)$$

By (2.5), the following holds for all $n \in \mathbb{N}$:

$$(\varrho_n)^k F(\varrho_n) - (\varrho_n)^k F(\varrho_0) \leq (\varrho_n)^k(-n\tau) \leq 0. \qquad (2.6)$$

Passing to limit as $n \to \infty$ in (2.6), using (2.4)-(2.5) we obtain

$$\lim_{n\to\infty} n(\varrho_n)^k = 0 \qquad (2.7)$$

From (2.7), there exits $n_1 \in \mathbb{N}$ such that $n(\varrho_n)^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$\varrho_n \leq \frac{1}{n^{\frac{1}{k}}}. \qquad (2.8)$$

In order to show that $\{x_n\}$ is a 0-Cauchy sequence, consider $m, n \in \mathbb{N}$ such that $m > n \geq n_1$. Using property ($\sigma3$) and (2.8), we have

$$\begin{aligned}\sigma(x_n, x_m) &\leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \ldots + \sigma(x_{m-1}, x_m)\\ &= \varrho_n + \varrho_{n+1} + \ldots + \varrho_{m-1}\\ &= \Sigma_{i=n}^{m-1} \varrho_i \leq \Sigma_{i=n}^{\infty} \varrho_i \leq \Sigma_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}.\end{aligned}$$

By the convergence of the series $\Sigma_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$, passing to limit $n \to \infty$, we get $\sigma(x_n, x_m) \to 0$ and hence $\{x_n\}$ is a 0-Cauchy sequence in $(\mathcal{X}, \sigma)$. Since $\mathcal{X}$ is 0-complete metric-like space, there exists a $v \in \mathcal{X}$ such that $\lim_{n\to+\infty} x_n \to v$; equivalently,

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n\to\infty} \sigma(x_n, v) = \sigma(v, v) = 0. \tag{2.9}$$

Now, if $\mathcal{T}$ is $\sigma$-continuous, we obtain from (2.9) that

$$\lim_{n,m\to\infty} \sigma(\mathcal{T}x_n, \mathcal{T}v) = \lim_{n\to\infty} \sigma(x_{n+1}, \mathcal{T}v) = \sigma(v, \mathcal{T}v) = 0. \tag{2.10}$$

This proves that $v$ is a fixed point of $\mathcal{T}$; that is, $v = \mathcal{T}v$.

Now, suppose $F$ is continuous. In this case, we claim that $v = \mathcal{T}v$. Assume the contrary, that is, $v \neq \mathcal{T}v$. In this case, there exist an $n_0 \in \mathbb{N}$ and a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $\sigma(\mathcal{T}x_{n_k}, \mathcal{T}z) > 0$ for all $n_k \geq n_0$. (Otherwise, there exists $n_1 \in \mathbb{N}$ such that $x_n = \mathcal{T}v$ for all $n \geq n_1$, which implies that $x_n \to \mathcal{T}v$. This is a contradiction, since $v \neq \mathcal{T}v$.) Since $\sigma(\mathcal{T}x_{n_k}, \mathcal{T}z) > 0$ for all $n_k \geq n_0$, then from (2.1), we have

$$\begin{aligned}&\tau + F(\sigma(x_{n_k+1}, \mathcal{T}v))\\ &= \tau + F(\sigma(\mathcal{T}x_{n_k}, \mathcal{T}v))\\ &\leq F\left(\max\left\{\sigma(x_{n_k}, v), \sigma(x_{n_k}, \mathcal{T}x_{n_k}), \sigma(v, \mathcal{T}v), \frac{\sigma(x_{n_k}, \mathcal{T}v) + \sigma(v, \mathcal{T}x_{n_k})}{4}, \frac{\sigma(x_{n_k}, \mathcal{T}x_{n_k})\sigma(z, \mathcal{T}z)}{1 + \sigma(x_{n_k}, v)}\right\}\right)\\ &= F\left(\max\left\{\sigma(x_{n_k}, v), \sigma(x_{n_k}, x_{n_k+1}), \sigma(v, \mathcal{T}v), \frac{\sigma(x_{n_k}, \mathcal{T}v) + \sigma(v, x_{n_k+1})}{4}, \frac{\sigma(x_{n_k}, x_{n_k+1})\sigma(v, \mathcal{T}v)}{1 + \sigma(x_{n_k}, v)}\right\}\right).\end{aligned}$$

Passing to the limit $k \to \infty$ and using the continuity of $F$ we have $\tau + F(\sigma(v, \mathcal{T}v)) \leq F(\sigma(v, \mathcal{T}v))$, a contradiction. Therefore we claim is true, that is $v = \mathcal{T}v$. The uniqueness of the fixed follows easily from (2.1). □

If we consider the different types of function $F$ on the condition (2.1) of Theorem 2.2, then we obtain the variety of contractions.

Put

$$\Theta(x, y) = \max\left\{\sigma(x, y), \sigma(x, \mathcal{T}x), \sigma(y, \mathcal{T}y), \frac{\sigma(x, \mathcal{T}y) + \sigma(y, \mathcal{T}x)}{4}, \frac{\sigma(x, \mathcal{T}x)\sigma(y, \mathcal{T}y)}{1 + \sigma(x, y)}\right\},$$

(I) Take $F(\alpha) = \ln \alpha$ ($\alpha > 0$ ) and $\tau = \ln(\frac{1}{\lambda})$ where $\lambda \in (0, 1)$, then

$$\sigma(\mathcal{T}x, \mathcal{T}y) \leq \lambda\Theta(x, y) \tag{2.11}$$

for all $x, y \in \mathcal{X}$ with $\mathcal{T}x \neq \mathcal{T}y$.

(II) Take $F(\alpha) = \ln\alpha + \alpha$ ($\alpha > 0$ ) and $\tau = \ln(\frac{1}{\lambda})$ where $\lambda \in (0,1)$, then

$$\sigma(\mathcal{T}x,\mathcal{T}y))e^{\sigma(\mathcal{T}x,\mathcal{T}y)-\Theta(x,y)} \leq \lambda\Theta(x,y), \tag{2.12}$$

for all $x, y \in \mathcal{X}$ with $\mathcal{T}x \neq \mathcal{T}y$.

(III) Take $F(\alpha) = -\frac{1}{\sqrt{\alpha}}$ ($\alpha > 0$ ) and $\tau = \lambda$ where $\lambda > 0$, then

$$\sigma(\mathcal{T}x,\mathcal{T}y)) \leq \frac{1}{(1+\lambda\sqrt{\Theta(x,y)})^2}\Theta(x,y), \tag{2.13}$$

for all $x, y \in \mathcal{X}$ with $\mathcal{T}x \neq \mathcal{T}y$.

(IV) Take $F(\alpha) = \ln(\alpha^2 + \alpha)$ ($\alpha > 0$ ) and $\tau = \ln(\frac{1}{\lambda})$ where $\lambda > 0$, then

$$\sigma(\mathcal{T}x,\mathcal{T}y))[\sigma(\mathcal{T}x,\mathcal{T}y)+1] \leq \lambda\Theta(x,y)[\Theta(x,y)+1], \tag{2.14}$$

for all $x, y \in \mathcal{X}$ with $\mathcal{T}x \neq \mathcal{T}y$.

The following example can be used to illustrate the usage of Theorem 2.2.

**Example 2.3.** Let $\mathcal{X} = [0,1] \cap \mathbb{Q}$ and $\sigma\colon \mathcal{X} \times \mathcal{X} \to \mathbb{R}^+$ be defined by

$$\sigma(x,y) = \begin{cases} 2x, & \text{if } x = y; \\ \max\{x,y\}, & \text{otherwise} \end{cases}$$

for all $x, y \in \mathcal{X}$. Then $(\mathcal{X}, \sigma)$ is a 0-$\sigma$-complete metric-like space which is not $\sigma$-complete ([11, Example 5]). Let $\mathcal{T}\colon \mathcal{X} \to \mathcal{X}$ be mapping given by $\mathcal{T}x = \frac{x}{4}$. Take $F(\alpha) = \ln(\alpha) + \alpha$ and $\tau = \ln 4$, where $\alpha > 0$. Then for $x > y$,

$$\sigma(\mathcal{T}x,\mathcal{T}y) = \sigma\left(\frac{x}{4},\frac{y}{4}\right) = \frac{x}{4} > 0$$

and

$$\begin{aligned} &\max\left\{\sigma(x,y), \sigma(x,\frac{x}{4}), \sigma(y,\frac{y}{4}), \frac{\sigma(x,\frac{y}{4})+\sigma(y,\frac{x}{4})}{4}, \frac{\sigma(x,\frac{x}{4}),\sigma(y,\frac{y}{4})}{1+\sigma(x,y)}\right\} \\ &= \max\left\{x, x, y, \frac{x+\max\{y,\frac{x}{4}\}}{4}, \frac{xy}{1+x}\right\} \\ &= x. \end{aligned}$$

Hence,

$$\begin{aligned} \tau + F(\sigma(\mathcal{T}x,\mathcal{T}y)) &= \ln 4 + \frac{x}{4} + \ln(\frac{x}{4}) \leq x + \ln x \\ &= F\left(\max\left\{\begin{array}{c} \sigma(x,y), \sigma(x,\mathcal{T}x), \sigma(y,\mathcal{T}y), \frac{\sigma(x,\mathcal{T}y)+\sigma(y,\mathcal{T}x)}{4}, \\ \frac{\sigma(x,\mathcal{T}x)\sigma(y,\mathcal{T}y)}{1+\sigma(x,y)} \end{array}\right\}\right). \end{aligned}$$

Similarly, for $x = y \neq 0$ ( otherwise $\sigma(\mathcal{T}x,\mathcal{T}y) = 0$ ) one gets that

$$\begin{aligned} \tau + F(\sigma(\mathcal{T}x,\mathcal{T}y)) &= \ln 4 + \frac{x}{2} + \ln(\frac{x}{2}) \leq 2x + \ln 2x \\ &= F\left(\max\left\{\begin{array}{c} \sigma(x,y), \sigma(x,\mathcal{T}x), \sigma(y,\mathcal{T}y), \frac{\sigma(x,\mathcal{T}y)+\sigma(y,\mathcal{T}x)}{4}, \\ \frac{\sigma(x,\mathcal{T}x)\sigma(y,\mathcal{T}y)}{1+\sigma(x,y)} \end{array}\right\}\right). \end{aligned}$$

Thus, all the conditions of Theorem 2.2 are satisfied. Then $\mathcal{T}$ has a unique fixed point (which is 0).

Following figures (Figs. 1,2) show that R.H.S. expression dominates the L.H.S expression in $[0,1]\cap\mathbb{Q}$, which validates our inequalities in the example 2.3.

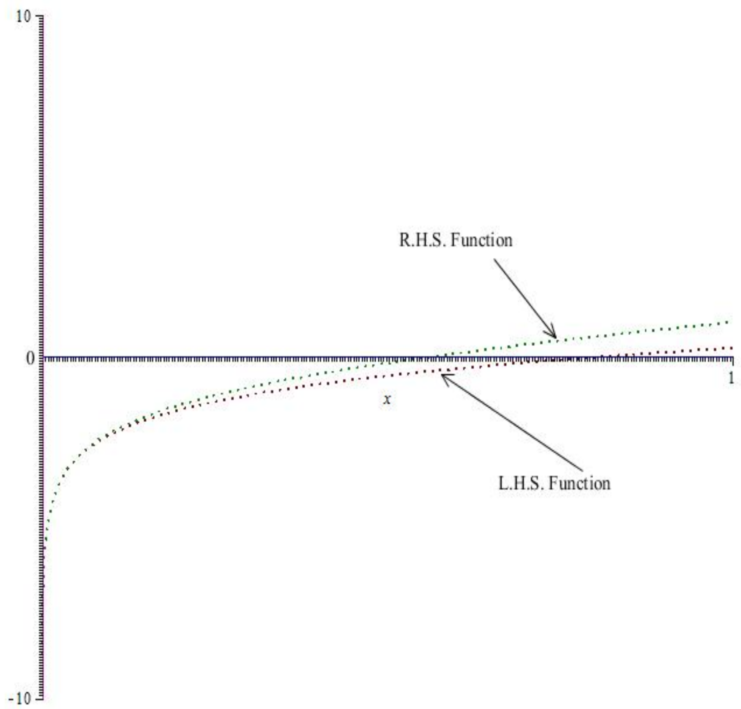


FIGURE 1. Plot of Inequality, 2D view

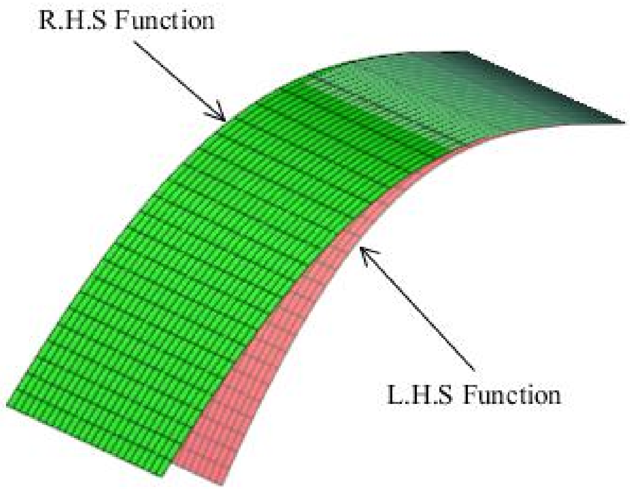


FIGURE 2. Plot of Inequality, 3D view

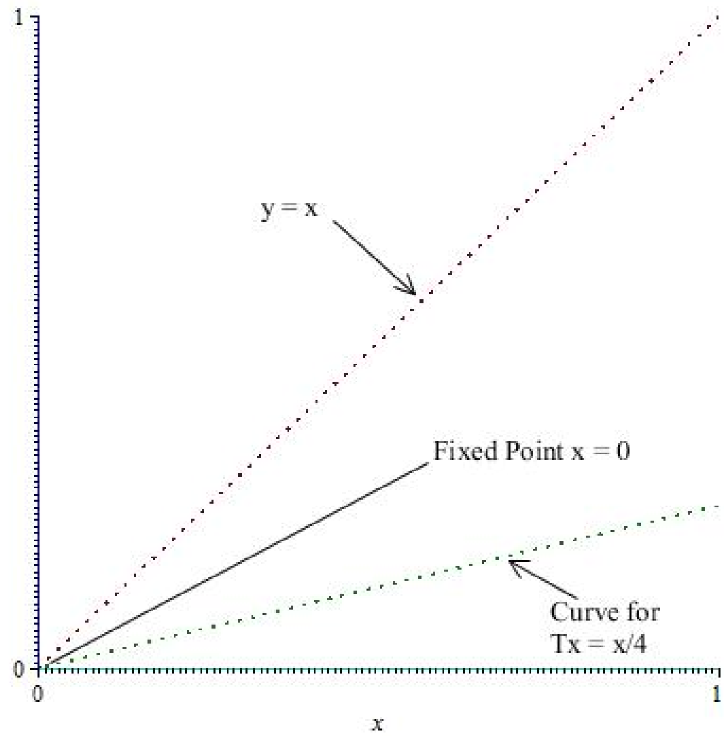


FIGURE 3. Plot showing fixed point of T

Now, we introduce the notion of $F$-Suzuki contraction in metric like space and prove a corresponding fixed point theorem.

**Definition 2.4.** Let $(X, \sigma)$ be a metric like space. A mapping $T : X \to X$ is said to be a $F$-Suzuki contraction if there exists $\tau > 0$ such that for all $x, y \in X$ with $Tx \neq Ty$

$$\frac{1}{2}\sigma(x, Tx) < \sigma(x, y) \Longleftrightarrow \tau + F(\sigma(Tx, Ty)) \leq F(\sigma((x, y)) \tag{2.15}$$

where $F \in \mathfrak{F}$.

**Theorem 2.5.** *Let $(X, \sigma)$ be a $0 - \sigma$ complete metric like space and $T : X \to X$ be a $F$-Suzuki contraction. Then $T$ has a unique fixed point in $X$.*

**Proof .** Choose $x_0 \in X$ and define a sequence $\{x_n\}_{n=1}^{\infty}$ by

$$x_1 = Tx_0, x_2 = Tx_1 = T^2x_0, \ldots, x_{n+1} = Tx_n = T^{n+1}x_0, \ \forall \ n \in \mathbb{N} \tag{2.16}$$

If there exists $n \in \mathbb{N}$ such that $\sigma(x_n, Tx_n) = 0$, the proof is complete.
So, we assume that $0 < \sigma(x_n, Tx_n) = \sigma(x_n, x_{n+1}) = \sigma_n, \ \forall \ n \in \mathbb{N}$.
Therefore

$$\frac{1}{2}\sigma(x_n, Tx_n) < \sigma(x_n, Tx_n), \ \forall n \in \mathbb{N} \tag{2.17}$$

which implies that

$$\tau + F(\sigma(Tx_n, T^2x_n)) \leq F(\sigma(x_n, Tx_n)) \text{ for any } \ n \in \mathbb{N}.$$

i.e.

$$F(\sigma(Tx_n, T^2x_n)) \leq F(\sigma(x_n, Tx_n)) - \tau.$$

Continuing this process, we get

$$\begin{aligned} F(\sigma(x_n, Tx_n)) &\leq F(\sigma(x_{n-1}, Tx_{n-1})) - \tau \\ &\leq F(\sigma(x_{n-2}, Tx_{n-2})) - 2\tau \\ &\vdots \\ &\leq F(\sigma(x_0, Tx_0)) - n\tau. \end{aligned} \tag{2.18}$$

From (2.18), we get $\sigma_n = F(\sigma(x_n, Tx_n)) \to -\infty$ as limit $n \to \infty$.
Thus, from (F2), we have $\lim_{n\to\infty} \sigma_n = \lim_{n\to\infty} \sigma(x_n, Tx_n) = 0$
Now, by property (F3) there exists $k \in (0, 1)$ such that

$$\lim_{n\to\infty} (\sigma_n)^k F(\sigma_n) = 0. \tag{2.19}$$

By (2.19), the following holds for all $n \in \mathbb{N}$ :

$$(\sigma_n)^k F(\sigma_n) - (\sigma_n)^k F(\sigma_0) \leq (\sigma_n)^k(-n\tau) \leq 0 \tag{2.20}$$

passing to limit as $n \to \infty$ in (2.19), using (2.18) and (2.17), we obtain

$$\lim_{n\to\infty} n(\sigma_n)^k = 0 \tag{2.21}$$

From, (2.21) there exists $n_1 \in \mathbb{N}$ such that $n(\sigma_n)^k \leq 1$ for all $n \geq n_1$. So, we have, for all $n \geq n_1$

$$\sigma_n \leq \frac{1}{n^{1/k}}. \tag{2.22}$$

Now, using property $(\sigma_3)$ and (2.22), we have for $m > n \geq n_1$

$$\begin{aligned} \sigma(x_n, x_m) &\leq \sigma(x_n, x_{n+1}) + \sigma(x_{n+1}, x_{n+2}) + \ldots + \sigma(x_{m-1}, x_m) \\ &= \varrho_n + \varrho_{n+1} + \ldots + \varrho_{m-1} \\ &= \Sigma_{i=n}^{m-1} \varrho_i \leq \Sigma_{i=n}^{\infty} \varrho_i \leq \Sigma_{i=n}^{\infty} \frac{1}{n^{\frac{1}{k}}}. \end{aligned}$$

By the convergence of the series $\Sigma_{n=1}^{\infty} \frac{1}{n^{\frac{1}{k}}}$, passing to limit $n \to \infty$, we get $\sigma(x_n, x_m) \to 0$ and hence $\{x_n\}$ is a 0-Cauchy sequence in $(\mathcal{X}, \sigma)$. Since $\mathcal{X}$ is 0-complete metric-like space, there exists a $u \in \mathcal{X}$ such that $\lim_{n\to+\infty} x_n \to u$; equivalently,

$$\lim_{n,m\to\infty} \sigma(x_n, x_m) = \lim_{n\to\infty} \sigma(x_n, u) = \sigma(u, u) = 0. \tag{2.23}$$

Now, we claim that there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that

$$\frac{1}{2}\sigma(x_{n_k}, Tx_n) < \sigma(x_{n_k}, u) \text{ or } \frac{1}{2}\sigma(Tx_{n_k}, T^2x_{n_k}) < \sigma(Tx_{n_k}, u), \ \forall k \in \mathbb{N}. \tag{2.24}$$

Suppose contrary that there exists $k \in \mathbb{N}$ such that

$$\frac{1}{2}\sigma(x_{n_k}, Tx_{n_k}) \geq \sigma(x_{n_k}, u) \text{ and } \frac{1}{2}\sigma(Tx_{n_k}, T^2x_{n_k}) \geq \sigma(Tx_{n_k}, u). \tag{2.25}$$

Therefore,

$$2\sigma(x_{n_k}, u) \le \sigma(x_{n_k}, Tx_{n_k}) \le \sigma(x_{n_k}, u) + \sigma(u, Tx_{n_k})$$

which implies that

$$\sigma(x_{n_k}, u) \le \sigma(u, Tx_{n_k}). \tag{2.26}$$

It follows from (2.24) and (2.26) that

$$\sigma(x_{n_k}, u) \le \sigma(u, Tx_{n_k}) \le \frac{1}{2}\sigma(Tx_{n_k}, T^2x_{n_k})$$

Since $\frac{1}{2}\sigma(x_{n_k}, Tx_{n_k}) < \sigma(x_{n_k}, Tx_{n_k})$, then by 2.15, we have

$$\tau + F(\sigma(Tx_{n_k}, T^2(x_{n_k}))) \le F(\sigma(x_{n_k}, Tx_{n_k})).$$

Since $\tau > 0$, this imply that

$$F(\sigma(Tx_{n_k}, T^2x_{n_k})) < F(\sigma(x_{n_k}, Tx_{n_k})).$$

So, from (F1), we get

$$\sigma(Tx_{n_k}, T^2x_{n_k}) < \sigma(x_{n_k}, Tx_{n_k}) \tag{2.27}$$

It follows from (2.24), (2.26) and (2.27) that

$$\begin{aligned}\sigma(Tx_{n_k}, T^2x_{n_k}) < \sigma(x_{n_k}, Tx_{n_k}) &\le \sigma(x_{n_k}, u) + \sigma(u, Tx_{n_k})\\ &\le \frac{1}{2}\sigma(Tx_{n_k}, T^2x_{n_k}) + \frac{1}{2}\sigma(Tx_{n_k}, T^2x_{n_k})\\ &= \sigma(Tx_{n_k}, T^2x_{n_k}).\end{aligned}$$

This is a contradiction. Hence (2.24) holds for every $k \in \mathbb{N}$, i.e. either $\tau + F(\sigma(Tx_{n_k}, Tu)) \le F(\sigma(x_{n_k}, u))$ is true for every $k \in \mathbb{N}$ or $\tau + F(\sigma(T^2x_{n_k}, Tu)) \le F(\sigma(Tx_{n_k}, u)) = F(\sigma(x_{n_k+1}, u))$ is true for every $k \in \mathbb{N}$.
In first case from (2.23) and (F2), we obtain

$$\lim_{n\to\infty} \sigma(Tx_{n_k}, Tu) = 0.$$

Therefore $d(u, Tu) = \lim_{n\to\infty} \sigma(x_{n_k+1}, Tu) = \lim_{n\to\infty} \sigma(Tx_{n_k}, Tu) = 0$.
Also in the second case using (2.23) and (F2), we get

$$\lim_{n\to\infty} \sigma(T^2x_{n_k}, Tu) = 0.$$

Therefore, $d(u, Tu) = \lim_{n\to\infty} \sigma(x_{n_k+2}, Tu) = \lim_{n\to\infty} \sigma(T^2x_{n_k}, Tu) = 0$.
Hence $u$ is a fixed point of $T$. The uniqueness of fixed point follows from (2.15). □

From (F1) and (2.15), we get the following corollary:

**Corollary 2.6.** Let $(X, \sigma)$ be a $0-\sigma$ complete metric like space and $T : X \to X$ be a mapping such that

$$\frac{1}{2}\sigma(x, Tx) < \sigma(x, y) \Longleftrightarrow \sigma(Tx, Ty)) < \sigma(x, y) \tag{2.28}$$

Then $T$ has a unique fixed point in $X$.

**Example 2.7.** Let $\mathcal{X}$ and $\sigma$ be as in Example (2.3). Define $T : X \to X$ as follows:

$$T(x) = \begin{cases} \frac{1}{2}, & \text{if } x = 1,\\ 0, & \text{otherwise.}\end{cases}$$

It can be easily verify that $T$ satisfies all the conditions of Corollary 2.6 and hence it has a unique fixed point (which is 0).

## 3. An application to second order differential equations

Consider the boundary value problem for second order differential equation of the form

$$\begin{cases} x''(t) = -f(t, x(t)), & t \in I \\ x(0) = x(1) = 0. \end{cases} \tag{3.1}$$

where $I = [0,1]$, $f \in C(I \times \mathbb{R}, \mathbb{R})$.

In this section we are going to apply Theorem 2.2 to the study of existence and uniqueness of solutions for a type of second order differential equations. Our approach is inspired by Section 5 of [2].

It is known, and easy to check, that the problem (3.1) is equivalent to the integral equation

$$x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds, \text{ for } t \in I, \tag{3.2}$$

where $G$ is the Green function defined by

$$G(t,s) = \begin{cases} (1-t)s & \text{if} \quad 0 \le s \le t \le 1, \\ (1-s)t & \text{if} \quad 0 \le t \le s \le 1. \end{cases}$$

That is, if $x \in C^2(I, \mathbb{R})$, then $x$ is a solution of problem (3.1) if and only if it is a solution of the integral equation (3.2).

Let $\mathcal{X} = C(I)$ be the space of all continuous functions defined on $I$ and $\|u\|_\infty = \max_{t \in I} |u(t)|$ for each $u \in \mathcal{X}$. Consider the metric-like $\sigma$ on $\mathcal{X}$ given by

$$\sigma(x,y) = \|x-y\|_\infty + \|x\|_\infty + \|y\|_\infty \text{ for all } x, y \in X.$$

Note that $\sigma$ is also a partial metric on $\mathcal{X}$ and that

$$d_\sigma(x,y) := 2\sigma(x,y) - \sigma(x,x) - \sigma(y,y) = 2\|x-y\|_\infty.$$

Hence, $(\mathcal{X}, \sigma)$ is complete as the metric space $(\mathcal{X}, \|\cdot\|_\infty)$ is complete.

**Theorem 3.1.** *Assume the following conditions:*

1. *there exist continuous functions* $\alpha : I \to \mathbb{R}^+$ *and* $\beta : I \to \mathbb{R}^+$ *such that*

$$|f(s,a) - f(s,b)| \le 8\alpha(s)|a-b|, \text{ for } s \in I \text{ and } a, b \in \mathbb{R},$$
$$|f(s,a)| \le 8\beta(s)|a|, \text{ for } s \in I \text{ and } a \in \mathbb{R};$$

2. $\max_{s \in I} \alpha(s) = \lambda_1 < \frac{1}{3}$ *and* $\max_{s \in I} \beta(s) = \lambda_2 < \frac{1}{3}$.

*Then the problem* (3.1) *has a unique solution* $u \in \mathcal{X} = C(I, \mathbb{R})$.

**Proof .** Define the self-map $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ by

$$\mathcal{T}x(t) = \int_0^1 G(t,s) f(s, x(s))\, ds,$$

for all $x \in \mathcal{X}$ and $t \in I$. Then, the problem (3.1) is equivalent to finding a fixed point $u$ of $\mathcal{T}$ in $\mathcal{X}$. Let $x, y \in \mathcal{X}$. We have

$$\begin{aligned}|\mathcal{T}x(t) - \mathcal{T}y(t)| &= \left| \int_0^1 G(t,s) f(s, x(s))\, ds - \int_0^1 G(t,s) f(s, y(s))\, ds \right| \\ &\le \int_0^1 G(t,s) |f(s, x(s)) - f(s, y(s))|\, ds \\ &\le 8 \int_0^1 G(t,s) \alpha(s) |x(s) - y(s)|\, ds \\ &\le 8\lambda_1 \|x - y\|_\infty \sup_{t \in I} \int_0^1 G(t,s)\, ds \\ &= \lambda_1 \|x - y\|_\infty.\end{aligned}$$

Next, we recall that for each $t \in I$ one has $\int_0^1 G(t,s)\, ds = \frac{t(1-t)}{2}$, and then

$$\sup_{t \in I} \int_0^1 G(t,s)\, ds = \frac{1}{8}.$$

Therefore,

$$\|\mathcal{T}x - \mathcal{T}y\|_\infty \le \lambda_1 \|x - y\|_\infty. \tag{3.3}$$

Moreover, we have

$$\begin{aligned}|\mathcal{T}x(t)| &= \left| \int_0^1 G(t,s) f(s, x(s))\, ds \right| \le \int_0^1 G(t,s) |f(s, x(s))|\, ds \\ &\le 8 \int_0^1 G(t,s) \beta(s) |x(s)|\, ds \le 8\lambda_2 \|x\|_\infty \sup_{t \in I} \int_0^1 G(t,s)\, ds \\ &\le \lambda_2 \|x\|_\infty.\end{aligned}$$

Thus

$$\|\mathcal{T}x\|_\infty \le \lambda_2 \|x\|_\infty, \tag{3.4}$$

and also

$$\|\mathcal{T}y\|_\infty \le \lambda_2 \|y\|_\infty. \tag{3.5}$$

Assuming $e^{-\tau} = \lambda_1 + 2\lambda_2 < 1$ ($\tau \in \mathbb{R}_+$). Using (3.3)–(3.5), we obtain

$$\begin{aligned}&\sigma(\mathcal{T}x, \mathcal{T}y) = \|\mathcal{T}x - \mathcal{T}y\|_\infty + \|\mathcal{T}x\|_\infty + \|\mathcal{T}y\|_\infty \\ &\le \lambda_1 \|x - y\|_\infty + \lambda_2 \|x\|_\infty + \lambda_2 \|y\|_\infty \\ &\le (\lambda_1 + 2\lambda_2)(\|x - y\|_\infty + \|x\|_\infty + \|y\|_\infty) \\ &= e^{-\tau} \sigma(x, y) \le e^{-\tau} \left( \max \left\{ \sigma(x, y), \sigma(x, \mathcal{T}x), \sigma(y, \mathcal{T}y), \frac{\sigma(x, \mathcal{T}y) + \sigma(y, \mathcal{T}x)}{4}, \frac{\sigma(x, \mathcal{T}x)\sigma(y, \mathcal{T}y)}{1 + \sigma(x, y)} \right\} \right).\end{aligned}$$

Taking the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(\alpha) = \ln \alpha$, belonging to $\mathfrak{F}$ we get

$$\tau + F(\sigma(\mathcal{T}x, \mathcal{T}y)) \le F\left( \max \left\{ \sigma(x, y), \sigma(x, \mathcal{T}x), \sigma(y, \mathcal{T}y), \frac{\sigma(x, \mathcal{T}y) + \sigma(y, \mathcal{T}x)}{4}, \frac{\sigma(x, \mathcal{T}x)\sigma(y, \mathcal{T}y)}{1 + \sigma(x, y)} \right\} \right).$$

Therefore all hypotheses of Theorem 2.2 are satisfied, and so $\mathcal{T}$ has a unique fixed point $u \in \mathcal{X}$, that is, the problem (3.1) has a unique solution $u \in C^2(I)$. $\square$

## 4. An application to fractional differential equations

In this section, we apply Theorem 2.2 to establish the existence of solution of fractional order functional differential equation.
Consider the following initial value problem (IVP for short) of the form

$$D^{\alpha}y(t) = f(t, y_t), \text{ for each } t \in J = [0, b], 0 < \alpha < 1, \tag{4.1}$$

$$y(t) = \phi(t), t \in (-\infty, 0] \tag{4.2}$$

where $D^{\alpha}$ is the standard Riemann-Liouville fractional derivative, $f : J \times B \to \mathbb{R}$, $\phi \in B, \phi(0) = 0$ and $B$ is called a phase space or state space satisfying some fundamental axioms (H-1, H-2, H-3) given below which were introduced by Hale and Kato in [5].
For any function $y$ defined on $(-\infty, b]$ and any $t \in J$ , we denote by $y_t$ the element of $B$ defined by

$$y_t(\theta) = y(t + \theta), \theta \in (-\infty, 0].$$

Here $y_t(\cdot)$ represents the history of the state from $-\infty$ up to present time $t$.

By $C(J, \mathbb{R})$ we denote the Banach space of all continuous functions from $J$ into $\mathbb{R}$ with the norm

$$||y||_{\infty} := \sup\{|y(t)| : t \in J\}$$

where $|\cdot|$ denotes a suitable complete norm on $\mathbb{R}$.

Now consider the metric like space $\sigma$ on $\mathcal{X}$ given by

$$\sigma(x, y) = 2d(x, y) \text{ for all } x, y \in X.$$

Then, $(\mathcal{X}, \sigma)$ is complete as the metric space $(\mathcal{X}, d)$ is complete.

(H-1) If $y : (-\infty, b] \to \mathbb{R}$, and $y_0 \in B$, then for every $t \in [0, b]$ the following conditions hold:
- (i) $y_t$ is in $B$,
- (ii) $||y_t||_B \le K(t) \sup\{|y(s)| : 0 \le s \le t\} + M(t)||y_0||_B$,
- (iii) $|y(t)| \le H||y_t||_B$,
  where $H \ge 0$ is a constant, $K : [0, b] \to [0, \infty)$ is continuous, $M : [0, \infty) \to [0, \infty)$ is locally bounded and $H, K, M$ are independent of $y(\cdot)$.

(H-2) For the function $y(\cdot)$ in (H-1), $y_t$ is a $B$-valued continuous function on $[0, b]$.

(H-3) The space $B$ is complete.

By a solution of problem (4.1)-(4.2), we mean a space $\Omega = \{y : (-\infty, b] \to \mathbb{R} : y|_{(-\infty,0]} \in B$ and $y|_{[0,b]}$ is continuous$\}$. Thus a function $y \in \Omega$ is said to be a solution of (1)-(2) if $y$ satisfies the equation $D^{\alpha}y(t) = f(t, y_t)$ on $J$, and the condition $y(t) = \phi(t)$ on $(-\infty, 0]$.

The following lemma is crucial to prove our existence theorem for the problem (4.1)-(4.2).

**Lemma 4.1.** (See [4].) Let $0 < \alpha < 1$ and let $h : (0, b] \to \mathbb{R}$ be continuous and $\lim_{t \to 0^+} h(t) = h(0^+) \in \mathbb{R}$. Then $y$ is a solution of the fractional integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - s)^{\alpha - 1} h(s) ds,$$

if and only if $y$ is a solution of the initial value problem for the fractional differential equation

$$D^{\alpha}y(t) = h(t), t \in (0, b],$$

$$y(0) = 0.$$

Now we are ready to prove following existence theorem.

**Theorem 4.2.** *Let $f : J \times B \to \mathbb{R}$. Assume*
*(H) there exists $q > 0$ such that*

$$|f(t, u) - f(t, v)| \leq q||u - v||_B, \quad \textit{for } t \in J \quad \textit{and every } u, v \in B.$$

*If $\frac{b^{\alpha}K_b q}{\Gamma(\alpha+1)} = \lambda < 1$ where*

$$K_b = \sup\{|K(t)| : t \in [0, b]\},$$

*then there exists a unique solution for the IVP (4.1)-(4.2) on the interval $(-\infty, b]$.*

**Proof .** To prove the existence of solution for the IVP (4.1)-(4.2), we transform it into a fixed point problem. For this, consider the operator $N : \Omega \to \Omega$ defined by

$$N(y)(t) = \begin{cases} \phi(t) & t \in (-\infty, 0], \\ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds & t \in [0, b]. \end{cases}$$

Let $x(\cdot) : (-\infty, b] \to \mathbb{R}$ be the function defined by

$$x(t) = \begin{cases} \phi(t) & t \in (-\infty, 0], \\ 0 & t \in [0, b]. \end{cases}$$

Then $x_0 = \phi$. For each $z \in C([0, b], \mathbb{R})$ with $z(0) = 0$, we denote by $\bar{z}$ the function defined by

$$\bar{z}(t) = \begin{cases} 0 & \text{if } t \in (-\infty, 0], \\ z(t) & \text{if } t \in [0, b]. \end{cases}$$

If $y(\cdot)$ satisfies the integral equation

$$y(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, y_s) ds,$$

we can decompose $y(\cdot)$ as $y(t) = \bar{z}(t) + x(t), 0 \leq t \leq b$, which implies $y_t = \bar{z}_t + x_t$, for every $0 \leq t \leq b$, and the function $z(\cdot)$ satisfies

$$z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds$$

Set

$$C_0 = \{z \in C([0, b], \mathbb{R}) : z_0 = 0\},$$

and let $|| \cdot ||_b$ be the seminorm in $C_0$ defined by

$$||z||_b = ||z_0||_B + \sup\{|z(t)|; 0 \leq t \leq b\} = \sup\{|z(t)|; 0 \leq t \leq b\}, \ z \in C_0.$$

$C_0$ is a Banach space with norm $||\cdot||_b$. Let the operator $P : C_0 \to C_0$ be defined by

$$(Pz)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s, \bar{z}_s + x_s) ds, \ t \in [0, b]. \tag{4.3}$$

That the operator $N$ has a fixed point is equivalent to $P$ has a fixed point, and so we turn to proving that $P$ has a fixed point. Indeed, consider $z, z^* \in C_0$. Then we have for each $t \in [0, b]$

$$\begin{aligned} |P(z)(t) - P(z^*)(t)| &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} |f(s, \bar{z}_s + x_s) - f(s, \bar{z}^*_s + x_s)| \ ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q ||\bar{z}_s - \bar{z}^*_s||_B \ ds \\ &\leq \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q K_b \sup_{s\in[o,t]} ||z(s) - z^*(s)|| \ ds \\ &\leq \frac{K_b}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} q \ ds \ ||z - z^*||_b. \end{aligned}$$

Therefore

$$||P(z) - P(z^*)||_b \leq \frac{q b^\alpha K_b}{\Gamma(\alpha+1)} ||z - z^*||_b,$$

i.e.

$$\sigma(P(z), P(z^*)) \leq \lambda \sigma(z, z^*).$$

By passing through logarithm, we write $\ln \sigma(P(z), P(z^*)) \leq \ln(e^{-\tau}\sigma(z, z^*))$ (where $\lambda = e^{-\tau} > 0$) and hence

$$\begin{aligned} &\tau + F((\sigma(P(z), P(z^*))) \leq F((\sigma(z, z^*)) \\ &\leq F\left( \max \left\{ \begin{array}{c} \sigma(z, z^*), \sigma(z, P(z)), \sigma(z^*, P(z^*)), \frac{\sigma(z,P(z^*)+\sigma(z^*,P(z))}{4}, \\ \frac{\sigma(z,P(z))\sigma(z^*,P(z^*))}{1+\sigma(z,z^*)} \end{array} \right\} \right). \end{aligned}$$

Now we observe that the function $F : \mathbb{R}^+ \to \mathbb{R}$ defined by $F(u) = \ln u$ for each $u \in \mathbb{R}^+$, then $F \in \mathcal{F}$ and so we deduce that the operator $P$ satisfies all the hypothesis of Theorem 2.2. Thus $P$ has unique fixed point. □

## 5. Conclusion

Taking into account its interesting applications, searching for fixed point theorems involving new contractive type conditions in abstract spaces has received considerable attention through the last few decades. In this connection, the main aim of our paper is to present new concepts of rational Ćirić type generalized $F$-contraction and $F$-Suzuki contraction in a metric-like space and utilize the same to establish fixed point results. Two applications to initial and boundary value problems are illustrated to the usability of the obtained fixed point results.


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