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Fractional Hermite-Hadamard type inequalities for n-times $\log$-convex functions

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Abstract

In this paper, we establish some Hermite–Hadamard type inequalities for function whose n-th derivatives are logarithmically convex by using Riemann–Liouville integral operator.

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2010 MSC: Primary 26D15; Secondary 26D20, 26A51.

1. Introduction

One of the most well–known inequalities in mathematics for convex functions is the so called Hermite–Hadamard integral inequality

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}, \tag{1.1}$$

where f is a real convex function on the finite interval $[a, b]$. If the function f is concave, then (1.1) holds in the reverse direction (see [16]).

The Hermite–Hadamard inequality play an important role in nonlinear analysis and optimization. The above inequality has attracted many researchers, various generalizations, refinements, extensions

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and variants have appeared in the literature, one can mention [1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 19, 20, 21] and references cited therein.

We first recall some definitions and lemmas.

Definition 1.1. (Pečarić et al. [18]) A positive function $f : I \to \mathbb{R}$ is said to be logarithmically convex, if $f(tx + (1-t)y) \leq [f(x)]^t [f(y)]^{1-t}$ holds for all $x, y \in I$ and $t \in [0, 1]$.

Definition 1.2. (Kilbas et al. [9]) Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J^{\alpha}_{a^+} f$ and $J^{\alpha}_{b^-} f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a^+} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt \qquad (x > a),$$

$$J^{\alpha}_{b^-} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt \qquad (b > x),$$

respectively, where $\Gamma(\alpha) = \int\limits_0^{\infty} e^{-t} t^{\alpha-1} dt$ is the Gamma function and $J^0_{a^+} f(x) = J^0_{b^-} f(x) = f(x)$.

Lemma 1.3. (Wang et al. [21]) For $\alpha > 0$ and $k > 0$, $z > 0$,

$$J(\alpha, k) = \int_0^1 (1-t)^{\alpha-1} k^t dt = \sum_{i=1}^{\infty} \frac{(\ln k)^{i-1}}{(\alpha)_i} < \infty,$$

$$H(\alpha, k, z) = \int_0^z t^{\alpha-1} k^t dt = z^{\alpha} k^z \sum_{i=1}^{\infty} \frac{(-z \ln k)^{i-1}}{(\alpha)_i} < \infty,$$

where $(\alpha)_i = \prod\limits_{j=0}^{i-1} (\alpha + j)$.

Lemma 1.4. (Wang and Qi [20]) Let $n \in \mathbb{N}$ and $\alpha > 0$, and let $f : [a, b] \subset \mathbb{R}_0 \to \mathbb{R}$ be an n–times differentiable function on (a, b). If $f^{(n)} \in L([a, b])$, then we have

$$\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}} (J^{\alpha}_{a^+} f(b) - J^{\alpha}_{b^-} f(a)) = \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right)$$
$$- \frac{(b-a)^n}{2} \int_0^1 ((-1)^{n-1} (1-t)^{\alpha+n-1} - t^{\alpha+n-1}) f^{(n)}(at + (1-t)b)\, dt. \tag{1.2}$$

The main purpose of this paper is to establishing a new Hermite–Hadamard type inequalities for functions whose n^{th} derivatives are logarithmically convex and via Riemann–Liouville integral operators.

2. Main results

Theorem 2.1. *let $f:[a,b]\subset\mathbb{R}_0\to\mathbb{R}$ be an n–times differentiable function on $[a,b]$ where n is a positive integer such that $f^{(n)}\in L([a,b])$ with $f^{(n)}(b)\neq 0$. If $\left|f^{(n)}\right|$ is log–convex, then the following fractional inequality holds*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq\begin{cases}\dfrac{(b-a)^n\left|f^{(n)}(b)\right|}{\alpha+n}, & n\text{ is even}, \lambda=1\\[2mm] \dfrac{(b-a)^n\left(2^{\alpha+n-1}-1\right)\left|f^{(n)}(b)\right|}{(\alpha+n)2^{\alpha+n-1}}, & n\text{ is odd}, \lambda=1\\[2mm] \dfrac{(b-a)^n\left|f^{(n)}(b)\right|}{2}\left(\sum\limits_{i=1}^{\infty}\dfrac{(\ln\lambda)^{i-1}}{(\alpha+n)_i}+\lambda\sum\limits_{i=1}^{\infty}\dfrac{(-\ln\lambda)^{i-1}}{(\alpha+n)_i}\right), & n\text{ is even}, \lambda\neq 1\\[2mm] \dfrac{(b-a)^n\left|f^{(n)}(b)\right|}{2}\left(\sum\limits_{i=1}^{\infty}\dfrac{(\ln\lambda)^{i-1}}{(\alpha+n)_i}-\dfrac{\sqrt{\lambda}}{2^{\alpha+n}}\sum\limits_{i=1}^{\infty}\dfrac{\left(\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+n)_i}-\sqrt{\lambda}\sum\limits_{i=1}^{\infty}\dfrac{\left(-\ln\sqrt{\lambda}\right)^{i-1}}{(\alpha+n)_i}\right.\\ \quad\left.+\lambda\sum\limits_{i=1}^{\infty}\dfrac{(-\ln\lambda)^{i-1}}{(\alpha+n)_i}-\sqrt{\lambda}\sum\limits_{i=1}^{\infty}\dfrac{\left(-\ln\sqrt{\lambda}\right)^{i-1}}{(\alpha+n)_i}-\dfrac{\sqrt{\lambda}}{2^{\alpha+n}}\sum\limits_{i=1}^{\infty}\dfrac{\left(\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+n)_i}\right), & n\text{ is odd}, \lambda\neq 1\end{cases}\tag{2.1}$$

where

$$\lambda=\left|\frac{f^{(n)}(a)}{f^{(n)}(b)}\right|.\tag{2.2}$$

Proof . Using Lemma 1.4, and modulus, we have

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq\frac{(b-a)^n}{2}\int_0^1\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|\left|f^{(n)}(at+(1-t)b)\right|dt.\tag{2.3}$$

Clearly

$$\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|=(1-t)^{\alpha+n-1}+t^{\alpha+n-1},\tag{2.4}$$

if n is an even number and if n is an odd number, we have

$$\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|=\begin{cases}(1-t)^{\alpha+n-1}-t^{\alpha+n-1} & \text{if } 0\leq t<\frac{1}{2}\\ t^{\alpha+n-1}-(1-t)^{\alpha+n-1} & \text{if } \frac{1}{2}\leq t\leq 1.\end{cases}\tag{2.5}$$

Assume that n is an even number, from (2.3), (2.4), and log–convexity of $\left|f^{(n)}\right|$ we get

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq\frac{(b-a)^n}{2}\int_0^1\left((1-t)^{\alpha+n-1}+t^{\alpha+n-1}\right)\left|f^{(n)}(a)\right|^t\left|f^{(n)}(b)\right|^{1-t}dt$$

$$= \frac{(b-a)^n}{2} \left| f^{(n)}(b) \right| \int_0^1 \left((1-t)^{\alpha+n-1} + t^{\alpha+n-1} \right) \lambda^t dt$$

and so

$$\begin{aligned} &\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right| \\ &\leq \frac{(b-a)^n}{2} \left| f^{(n)}(b) \right| \left(\int_0^1 (1-t)^{\alpha+n-1} \lambda^t dt + \int_0^1 t^{\alpha+n-1} \lambda^t dt \right), \end{aligned} \tag{2.6}$$

where λ is defined by (2.2). There are two cases. If $\lambda = 1$, then (2.6) becomes

$$\begin{aligned} &\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right| \\ &\leq \frac{(b-a)^n}{\alpha+n} \left| f^{(n)}(b) \right|. \end{aligned} \tag{2.7}$$

In the case where $\lambda \neq 1$, an application of Lemma 1.3 for $\int_0^1 (1-t)^{\alpha+n-1} \lambda^t dt$ and $\int_0^1 t^{\alpha+n-1} \lambda^t dt$ gives

$$\int_0^1 (1-t)^{\alpha+n-1} \lambda^t dt = \sum_{i=1}^{\infty} \frac{(\ln \lambda)^{i-1}}{(\alpha+n)_i} \tag{2.8}$$

and

$$\int_0^1 t^{\alpha+n-1} \lambda^t dt = \lambda \sum_{i=1}^{\infty} \frac{(-\ln \lambda)^{i-1}}{(\alpha+n)_i}. \tag{2.9}$$

Substituting (2.8) and (2.9) in (2.6), we get

$$\begin{aligned} &\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right| \\ &\leq \frac{(b-a)^n}{2} \left| f^{(n)}(b) \right| \left(\sum_{i=1}^{\infty} \frac{(\ln \lambda)^{i-1}}{(\alpha+n)_i} + \lambda \sum_{i=1}^{\infty} \frac{(-\ln \lambda)^{i-1}}{(\alpha+n)_i} \right). \end{aligned} \tag{2.10}$$

Now, we treat the case where n is an odd number. It follows from (2.3), (2.5), and log–convexity of $\left| f^{(n)} \right|$

$$\begin{aligned} &\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right| \\ &\leq \frac{(b-a)^n}{2} \left| f^{(n)}(b) \right| \left(\int_0^{\frac{1}{2}} \left((1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right) \lambda^t dt \right. \\ &\quad \left. + \int_{\frac{1}{2}}^1 \left(t^{\alpha+n-1} - (1-t)^{\alpha+n-1} \right) \lambda^t dt \right), \end{aligned} \tag{2.11}$$

where λ is defined by (2.2). For $\lambda = 1$, (2.11) gives

$$\begin{aligned} &\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\ \le\ & \frac{\left(2^{\alpha+n-1}-1\right)(b-a)^n}{(\alpha+n)\,2^{\alpha+n-1}}\left|f^{(n)}(b)\right|. \end{aligned} \tag{2.12}$$

For $\lambda \neq 1$, we have

$$\begin{aligned} &\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\ \le\ & \tfrac{(b-a)^n\left|f^{(n)}(b)\right|}{2}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\lambda^t dt\right. \\ &\left.+\int_{\frac{1}{2}}^{1}\left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right)\lambda^t dt\right) \\ =\ & \tfrac{(b-a)^n\left|f^{(n)}(b)\right|}{2}\left(\sum_{i=1}^{\infty}\tfrac{(\ln\lambda)^{i-1}}{(\alpha+n)_i}-\frac{\sqrt{\lambda}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\tfrac{\left(\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+n)_i}-\sqrt{\lambda}\sum_{i=1}^{\infty}\tfrac{\left(-\ln\sqrt{\lambda}\right)^{i-1}}{(\alpha+n)_i}\right. \\ &\left.+\lambda\sum_{i=1}^{\infty}\tfrac{(-\ln\lambda)^{i-1}}{(\alpha+n)_i}-\sqrt{\lambda}\sum_{i=1}^{\infty}\tfrac{\left(-\ln\sqrt{\lambda}\right)^{i-1}}{(\alpha+n)_i}-\frac{\sqrt{\lambda}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\tfrac{\left(\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+n)_i}\right), \end{aligned} \tag{2.13}$$

where we have used the fact that

$$\begin{aligned} \int_0^{\frac{1}{2}}(1-t)^{\alpha+n-1}\lambda^t dt &= \int_0^1(1-t)^{\alpha+n-1}\lambda^t dt-\int_{\frac{1}{2}}^1(1-t)^{\alpha+n-1}\lambda^t dt \\ &= \int_0^1(1-t)^{\alpha+n-1}\lambda^t dt-\frac{\lambda}{2^{\alpha+n}}\int_0^1 t^{\alpha+n-1}\left(\lambda^{-\frac{1}{2}}\right)^t dt \\ &= \sum_{i=1}^{\infty}\frac{(\ln\lambda)^{i-1}}{(\alpha+n)_i}-\frac{\sqrt{\lambda}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+n)_i}, \end{aligned} \tag{2.14}$$

$$\int_0^{\frac{1}{2}}t^{\alpha+n-1}\lambda^t dt=\frac{1}{2^{\alpha+n}}\int_0^1 t^{\alpha+n-1}\sqrt{\lambda}^t dt=\sqrt{\lambda}\sum_{i=1}^{\infty}\frac{\left(-\ln\sqrt{\lambda}\right)^{i-1}}{(\alpha+n)_i}, \tag{2.15}$$

$$\begin{aligned} \int_{\frac{1}{2}}^1 t^{\alpha+n-1}\lambda^t dt &= \int_0^1 t^{\alpha+n-1}\lambda^t dt-\int_0^{\frac{1}{2}}t^{\alpha+n-1}\lambda^t dt \\ &= \lambda\sum_{i=1}^{\infty}\frac{(-\ln\lambda)^{i-1}}{(\alpha+n)_i}-\sqrt{\lambda}\sum_{i=1}^{\infty}\frac{\left(-\ln\sqrt{\lambda}\right)^{i-1}}{(\alpha+n)_i}, \end{aligned} \tag{2.16}$$

and

$$\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+n-1}\lambda^{t}dt = \int_{0}^{1}(1-t)^{\alpha+n-1}\lambda^{t}dt-\int_{0}^{\frac{1}{2}}(1-t)^{\alpha+n-1}\lambda^{t}dt$$
$$= \frac{\sqrt{\lambda}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\lambda\right)^{i-1}}{(\alpha+n)_{i}}. \tag{2.17}$$

The desired result follows from (2.7), (2.10), (2.12), and (2.13). □

Theorem 2.2. *let $f:[a,b]\subset\mathbb{R}_0\to\mathbb{R}$ be an n–times differentiable function on $[a,b]$ where n is a positive integer such that $f^{(n)}\in L([a,b])$ with $f^{(k)}(b)\neq 0$. If $\left|f^{(n)}\right|^{q}$ is log–convex for some $q>1$, then the following fractional inequality holds*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|$$
$$\leq\begin{cases}\frac{(b-a)^{n}}{\alpha+n}\left|f^{(n)}(b)\right|, & n \text{ is even}, \lambda=1\\[2mm] \frac{\left(2^{\alpha+n-1}-1\right)(b-a)^{n}}{(\alpha+n)2^{\alpha+n-1}}\left|f^{(n)}(b)\right|, & n \text{ is odd}, \lambda=1\\[2mm] \frac{(b-a)^{n}\left|f^{(n)}(b)\right|}{2^{\frac{1}{q}}(\alpha+n)^{1-\frac{1}{q}}}\left(\sum_{i=1}^{\infty}\frac{(\ln\lambda^{q})^{i-1}}{(\alpha+n)_{i}}+\lambda^{q}\sum_{i=1}^{\infty}\frac{(-\ln\lambda^{q})^{i-1}}{(\alpha+n)_{i}}\right)^{\frac{1}{q}}, & n \text{ is even}, \lambda\neq 1\\[2mm] \frac{(b-a)^{n}}{2}\left|f^{(n)}(b)\right|\left(\frac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-2}}\right)^{1-\frac{1}{q}}\left(\sum_{i=1}^{\infty}\frac{(\ln\lambda^{q})^{i-1}}{(\alpha+n)_{i}}-\frac{\lambda^{\frac{q}{2}}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\lambda^{q}\right)^{i-1}}{(\alpha+n)_{i}}\right.\\ \quad-\lambda^{\frac{q}{2}}\sum_{i=1}^{\infty}\frac{\left(-\ln\lambda^{\frac{q}{2}}\right)^{i-1}}{(\alpha+n)_{i}}+\lambda^{q}\sum_{i=1}^{\infty}\frac{(-\ln\lambda^{q})^{i-1}}{(\alpha+n)_{i}}-\lambda^{\frac{q}{2}}\sum_{i=1}^{\infty}\frac{\left(-\ln\lambda^{\frac{q}{2}}\right)^{i-1}}{(\alpha+n)_{i}}\\ \quad\left.-\frac{\lambda^{\frac{q}{2}}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\lambda^{q}\right)^{i-1}}{(\alpha+n)_{i}}\right)^{\frac{1}{q}}, & n \text{ is odd}, \lambda\neq 1\end{cases} \tag{2.18}$$

where λ is defined as in (2.2).

Proof . Like in Theorem 2.1, assume that n is an even number, using 1.4, modulus, (2.4), power mean inequality, and log–convexity of $\left|f^{(n)}\right|^{q}$, we get

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^{k}}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^{k}f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^{n}}{2}\left(\int_{0}^{1}\left((1-t)^{\alpha+n-1}+t^{\alpha+n-1}\right)dt\right)^{1-\frac{1}{q}}$$
$$\times\left(\int_{0}^{1}\left((1-t)^{\alpha+n-1}+t^{\alpha+n-1}\right)\left|f^{(n)}(at+(1-t)b)\right|^{q}dt\right)^{\frac{1}{q}}$$

$$= \frac{(b-a)^n \left|f^{(n)}(b)\right|}{2^{\frac{1}{q}}(\alpha+n)^{1-\frac{1}{q}}} \left(\int_0^1 \left((1-t)^{\alpha+n-1} + t^{\alpha+n-1} \right) \lambda^{qt} dt \right)^{\frac{1}{q}}, \tag{2.19}$$

where λ is defined as in (2.2). We distinguish two cases:
if $\lambda = 1$, then (2.19) gives

$$\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right|$$
$$\leq \frac{(b-a)^n \left|f^{(n)}(b)\right|^q}{\alpha+n}. \tag{2.20}$$

In the case where $\lambda \neq 1$, (2.19) becomes

$$\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right|$$
$$\leq \frac{(b-a)^n \left|f^{(n)}(b)\right|}{2^{\frac{1}{q}}(\alpha+n)^{1-\frac{1}{q}}} \left(\int_0^1 (1-t)^{\alpha+n-1} \lambda^{qt} dt + \int_0^1 t^{\alpha+n-1} \lambda^{qt} dt \right)^{\frac{1}{q}}. \tag{2.21}$$

Using (2.8) and (2.9) by replacing λ by λ^q substituting the result in (2.21), we obtain

$$\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right|$$
$$\leq \frac{(b-a)^n \left|f^{(n)}(b)\right|}{2^{\frac{1}{q}}(\alpha+n)^{1-\frac{1}{q}}} \left(\sum_{i=1}^{\infty} \frac{(\ln \lambda^q)^{i-1}}{(\alpha+n)_i} + \lambda^q \sum_{i=1}^{\infty} \frac{(-\ln \lambda^q)^{i-1}}{(\alpha+n)_i} \right)^{\frac{1}{q}}. \tag{2.22}$$

Now, suppose that n is an odd number, using 1.4, modulus, (2.5), power mean inequality, and log–convexity of $\left|f^{(n)}\right|^q$, we get

$$\left| \frac{\Gamma(\alpha+n)}{2(b-a)^\alpha} \left(J_{a^+}^\alpha f(b) - J_{b^-}^\alpha f(a) \right) - \sum_{k=0}^{n-1} \frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)} \left(f^{(k)}(a) + (-1)^k f^{(k)}(b) \right) \right|$$
$$\leq \frac{(b-a)^n}{2} \left(\int_0^{\frac{1}{2}} \left((1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right) dt + \int_{\frac{1}{2}}^1 \left(t^{\alpha+n-1} - (1-t)^{\alpha+n-1} \right) dt \right)^{1-\frac{1}{q}}$$
$$\times \left(\int_0^{\frac{1}{2}} \left((1-t)^{\alpha+n-1} - t^{\alpha+n-1} \right) \left| f^{(n)}(at + (1-t)b) \right|^q dt \right.$$
$$\left. + \int_{\frac{1}{2}}^1 \left(t^{\alpha+n-1} - (1-t)^{\alpha+n-1} \right) \left| f^{(n)}(at + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}$$

$$= \frac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(\frac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-2}}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)\lambda^{qt}dt\right.$$
$$\left.+\int_{\frac{1}{2}}^{1}\left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right)\lambda^{qt}dt\right)^{\frac{1}{q}}, \tag{2.23}$$

where λ is defined as in (2.2).

The case where $\lambda = 1$, from (2.23) it yields

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{\left(2^{\alpha+n-1}-1\right)(b-a)^n}{(\alpha+n)2^{\alpha+n-1}}\left|f^{(n)}(b)\right|. \tag{2.24}$$

Now, assume that $\lambda \neq 1$, from (2.23) we have

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$
$$\leq \frac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(\frac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-2}}\right)^{1-\frac{1}{q}}\left(\int_0^{\frac{1}{2}}(1-t)^{\alpha+n-1}\lambda^{qt}dt\right.$$
$$\left.-\int_0^{\frac{1}{2}}t^{\alpha+n-1}\lambda^{qt}dt+\int_{\frac{1}{2}}^{1}t^{\alpha+n-1}\lambda^{qt}dt-\int_{\frac{1}{2}}^{1}(1-t)^{\alpha+n-1}\lambda^{qt}dt\right)^{\frac{1}{q}}$$
$$= \frac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(\frac{2^{\alpha+n-1}-1}{(\alpha+n)2^{\alpha+n-2}}\right)^{1-\frac{1}{q}}$$
$$\times\left(\sum_{i=1}^{\infty}\frac{(\ln\lambda^q)^{i-1}}{(\alpha+n)_i}-\frac{\lambda^{\frac{q}{2}}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\lambda^q\right)^{i-1}}{(\alpha+n)_i}-\lambda^{\frac{q}{2}}\sum_{i=1}^{\infty}\frac{\left(-\ln\lambda^{\frac{q}{2}}\right)^{i-1}}{(\alpha+n)_i}\right.$$
$$\left.+\lambda^{q}\sum_{i=1}^{\infty}\frac{(-\ln\lambda^q)^{i-1}}{(\alpha+n)_i}-\lambda^{\frac{q}{2}}\sum_{i=1}^{\infty}\frac{\left(-\ln\lambda^{\frac{q}{2}}\right)^{i-1}}{(\alpha+n)_i}-\frac{\lambda^{\frac{q}{2}}}{2^{\alpha+n}}\sum_{i=1}^{\infty}\frac{\left(\frac{1}{2}\ln\lambda^q\right)^{i-1}}{(\alpha+n)_i}\right)^{\frac{1}{q}}, \tag{2.25}$$

where we have used (2.14)–(2.17) by replacing λ by λ^q. Thus, the desired result follows from (2.20), (2.22), (2.24), and (2.25). □

Theorem 2.3. *Assume that all the assumptions of Theorem 2.2 are satisfied, then the following fractional inequality holds*

$$\left|\frac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$\leq \begin{cases} \frac{2^{\frac{1}{p}}(b-a)^n\left|f^{(n)}(b)\right|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}}, & n \text{ is even}, \lambda=1 \\ \frac{\left(2^{p(\alpha+n-1)}-1\right)^{\frac{1}{p}}(b-a)^n}{(p(\alpha+n-1)+1)^{\frac{1}{p}}2^{\alpha+n-\frac{1}{p}}}\left|f^{(n)}(b)\right|, & n \text{ is odd}, \lambda=1 \\ \frac{2^{\frac{1}{p}}(b-a)^n\left|f^{(n)}(b)\right|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}, & n \text{ is even}, \lambda\neq 1 \\ -\frac{\left(2^{p(\alpha+n-1)}-1\right)^{\frac{1}{p}}(b-a)^n}{(p(\alpha+n-1)+1)^{\frac{1}{p}}2^{\alpha+n-\frac{1}{p}}}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}, & n \text{ is odd}, \lambda\neq 1 \end{cases} \tag{2.26}$$

where λ is defined as in (2.2) *and* $\frac{1}{p}+\frac{1}{q}=1$.

Proof . From 1.4, modulus, Hölder inequality, and log–convexity of $\left|f^{(n)}\right|^q$, we get

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left(J_{a^+}^\alpha f(b)-J_{b^-}^\alpha f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\ \leq\ &\frac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(\int_0^1\left|(-1)^{n-1}(1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}}. \end{aligned} \tag{2.27}$$

Assume that n is an even number, then (2.27) becomes

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left(J_{a^+}^\alpha f(b)-J_{b^-}^\alpha f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\ \leq\ &\frac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(\int_0^1\left((1-t)^{\alpha+n-1}+t^{\alpha+n-1}\right)^p dt\right)^{\frac{1}{p}}\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}} \\ \leq\ &\frac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(2^p\int_0^1\left((1-t)^{p(\alpha+n-1)}+t^{p(\alpha+n-1)}\right)dt\right)^{\frac{1}{p}}\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}} \\ =\ &\frac{2^{\frac{1}{p}}(b-a)^n\left|f^{(n)}(b)\right|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}}\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}}, \end{aligned} \tag{2.28}$$

where we have used the following algebraic inequalities $(v+w)^\beta \leq 2^{\beta-1}\left(v^\beta+w^\beta\right)$ for $\beta\geq 1$ and $v,w\geq 0$.
If $\lambda=1$, (2.28) gives

$$\begin{aligned} &\left|\frac{\Gamma(\alpha+n)}{2(b-a)^\alpha}\left(J_{a^+}^\alpha f(b)-J_{b^-}^\alpha f(a)\right)-\sum_{k=0}^{n-1}\frac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right| \\ \leq\ &\frac{2^{\frac{1}{p}}(b-a)^n\left|f^{(n)}(b)\right|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}}. \end{aligned} \tag{2.29}$$

If $\lambda \neq 1$, (2.28) gives

$$\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$= \frac{2^{\frac{1}{p}}(b-a)^n\left|f^{(n)}(b)\right|}{(p(\alpha+n-1)+1)^{\frac{1}{p}}}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}. \tag{2.30}$$

In the case where n is an odd number, then (2.27) becomes

$$\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$\leq \tfrac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}}$$

$$\times\left(\int_0^{\frac{1}{2}}\left((1-t)^{\alpha+n-1}-t^{\alpha+n-1}\right)^p dt+\int_{\frac{1}{2}}^1\left(t^{\alpha+n-1}-(1-t)^{\alpha+n-1}\right)^p dt\right)^{\frac{1}{p}}$$

$$= \tfrac{(b-a)^n}{2}\left|f^{(n)}(b)\right|\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}}\left(\int_0^{\frac{1}{2}}(1-t)^{p(\alpha+n-1)}dt\right.$$

$$\left.-\int_0^{\frac{1}{2}}t^{p(\alpha+n-1)}dt+\int_{\frac{1}{2}}^1 t^{p(\alpha+n-1)}dt-\int_{\frac{1}{2}}^1(1-t)^{p(\alpha+n-1)}dt\right)^{\frac{1}{p}}$$

$$= \frac{\left(2^{p(\alpha+n-1)}-1\right)^{\frac{1}{p}}(b-a)^n}{(p(\alpha+n-1)+1)^{\frac{1}{p}}2^{\alpha+n-\frac{1}{p}}}\left|f^{(n)}(b)\right|\left(\int_0^1\lambda^{qt}dt\right)^{\frac{1}{q}}, \tag{2.31}$$

where we have used the following algebraic inequalities $(v-w)^{\beta} \leq \left(v^{\beta}-w^{\beta}\right)$ for fixed $\beta \geq 1$ and $0 \leq w < v$.
If $\lambda = 1$, (2.31) gives

$$\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$\leq \frac{\left(2^{p(\alpha+n-1)}-1\right)^{\frac{1}{p}}(b-a)^n}{(p(\alpha+n-1)+1)^{\frac{1}{p}}2^{\alpha+n-\frac{1}{p}}}\left|f^{(n)}(b)\right|. \tag{2.32}$$

If $\lambda \neq 1$, (2.28) gives

$$\left|\tfrac{\Gamma(\alpha+n)}{2(b-a)^{\alpha}}\left(J_{a^+}^{\alpha}f(b)-J_{b^-}^{\alpha}f(a)\right)-\sum_{k=0}^{n-1}\tfrac{\Gamma(\alpha+n)(b-a)^k}{2\Gamma(\alpha+k+1)}\left(f^{(k)}(a)+(-1)^k f^{(k)}(b)\right)\right|$$

$$\leq \frac{\left(2^{p(\alpha+n-1)}-1\right)^{\frac{1}{p}}(b-a)^n}{(p(\alpha+n-1)+1)^{\frac{1}{p}}2^{\alpha+n-\frac{1}{p}}}\left(\frac{\lambda^q-1}{q\ln\lambda}\right)^{\frac{1}{q}}. \tag{2.33}$$

The desired result follows from (2.29), (2.30), (2.32), and (2.33). □

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