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Higher order multi-point fractional boundary value problems with integral boundary conditions

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Abstract

In this paper, we are concerned with positive solutions for higher order m–point nonlinear fractional boundary value problems with integral boundary conditions. We establish the criteria for the existence of at least one, two and three positive solutions for higher order m–point nonlinear fractional boundary value problems with integral boundary conditions by using some results from the theory of fixed point index, Avery–Henderson fixed point theorem and the Legget–Williams fixed point theorem, respectively.

Keywords: boundary value problems; cone; fixed point theorems; positive solutions; Riemann–Liouville fractional derivative; integral boundary conditions.

2010 MSC: Primary 26A33; Secondary 34B10, 34B18.

1. Introduction

In [11, 12], Il'in and Moiseev studied the existence of solutions of multi–point boundary value problems for linear second-order ordinary differential equations. Since then, more general nonlinear multi–point boundary value problems have been studied by several authors. We refer the reader to [2, 6, 10, 14, 17, 18, 19, 28, 29, 33] for some references along this line.

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary noninteger order. The topic of fractional differential equations and inclusions has recently emerged as a popular field of research due to its extensive development and applications in several disciplines such as physics, mechanics, chemistry, engineering, aerodynamics, electrodynamics of complex medium, polymer rheology, see [20, 24, 25, 26].

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Among all the researches on the theory of the fractional differential equations, the study of the boundary value problems for fractional differential equations recently has attracted a great deal of attention from many researchers. There are some papers considering the existence of positive solutions of higher order fractional boundary value problems, for instance, [1, 7, 8, 16, 35] and the existence of positive solutions of the boundary value problems for some specific fractional differential equations with integral boundary conditions (see [4, 5, 13, 15, 23, 27, 30, 31, 32]).

Zhang, Wang and Sun [34] were interested in the following boundary value problem (BVP)

$$\begin{cases} D^{\alpha}_{0+}u(t) + h(t)f(t, u(t)) = 0, \ 0 < t < 1, \ 3 < \alpha \leq 4 \\ u(0) = u'(0) = u''(0) = 0, \\ u(1) = \lambda \int\limits_0^{\eta} u(s)ds. \end{cases}$$

They have studied the existence of at least one positive solution to BVP using the fixed point index theory. Wang and Zhang [30] are concerned with the existence of at least one solution of the following higher order multi–point BVP

$$\begin{cases} D^{\alpha}_{0+}u(t) + h(t)f(t, u(t)) = 0, \ 0 < t < 1, \ n-1 < \alpha \leq n \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u^{(i)}(1) = \lambda \int\limits_0^{\eta} u(s)ds, \end{cases}$$

where i is given constant.

In this paper, we study the existence of positive solutions to multi–point BVP for higher order fractional differential equations:

$$\begin{cases} D^{\alpha}_{0+}u(t) + f(t, u(t)) = 0, \ \ t \in [0, 1], \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \sum\limits_{p=1}^{m-2} a_p \int\limits_0^{\xi_p} u(s)ds, \end{cases} \tag{1.1}$$

where D^{α}_{0+} is the Riemann–Liouville fractional derivative of order α. Throughout the paper we suppose that $n \geq 3$, $n-1 < \alpha \leq n$ where $n \in \mathbb{N}$ and $a_p \geq 0$ are given constants and $0 < \xi_1 < \ldots < \xi_{m-2} < 1$. We assume that $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous.

We note that if $n = 4$ and $m = 3$, then the BVP (1.1) reduces to the BVP in [34]. Besides, we establish the criteria for the existence of at least three positive solutions.

In this article, existence results of at least one positive solution of (1.1) are first established as a result of the theory of fixed point index. Second, we apply the Avery–Henderson fixed–point theorem to prove the existence of at least two positive solutions of (1.1). Finally, we use the Legget–Williams fixed–point theorem to show that the existence of at least three positive solutions of (1.1).

2. Preliminaries

In this section, we preliminarily provide some definitions and lemmas which are useful in the following discussion.

Definition 2.1. The Riemann–Liouville fractional derivative of order $\alpha > 0$ for a function $u : (0, \infty) \to \mathbb{R}$ is defined by

$$D^{\alpha}_{0+}u(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_0^t (t-s)^{n-\alpha-1} u(s)ds,$$

where $\Gamma(\cdot)$ is the Euler gamma function and $n = [\alpha] + 1$. $[\alpha]$ denotes the integer part of the number α, provided that the right–hand side is point wise defined on $(0, \infty)$.

Definition 2.2. The Riemann–Liouville fractional integral of order $\alpha > 0$ of a function $u : (0, \infty) \to \mathbb{R}$ is given by

$$I_{0+}^{\alpha} u(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} u(s) ds.$$

Lemma 2.3. (Kilbas et al. [20]) The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0,1)$.

Lemma 2.4. (Kilbas et al. [20]) If $\alpha > 0$, then the differential equation

$$D_{0+}^{\alpha} u = 0 \tag{2.1}$$

has a unique solution $u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}$, $c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $n - 1 < \alpha \leq n$.

Lemma 2.5. (Kilbas et al. [20]) If $\alpha > 0$, then the following equality holds for $u \in L(0,1)$, $D_{0+}^{\alpha} u \in L(0,1)$;

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}, \tag{2.2}$$

$c_i \in \mathbb{R}$, $i = 1, \ldots, n$, where $n - 1 < \alpha \leq n$.

Lemma 2.6. *If $y \in C[0,1]$, $K := 1 - \sum\limits_{p=1}^{m-2} a_p \frac{\xi_p^{\alpha}}{\alpha} > 0$ and $J_1 = [0, \xi_1]$, $J_2 = [\xi_1, \xi_2]$,..., $J_{m-2} = [\xi_{m-3}, \xi_{m-2}]$, $J_{m-1} = [\xi_{m-2}, 1]$, then BVP*

$$\begin{cases} -D_{0+}^{\alpha} u(t) + y(t) = 0, \ \ t \in [0,1], \\ u(0) = u'(0) = \cdots = u^{(n-2)}(0) = 0, \\ u(1) = \sum\limits_{p=1}^{m-2} a_p \int\limits_0^{\xi_p} u(s) ds, \end{cases} \tag{2.3}$$

has a unique solution

$$u(t) = \int\limits_0^1 G(t,s) y(s) ds, \tag{2.4}$$

where

$$G(t,s) = \frac{1}{K\Gamma(\alpha)} \begin{cases} (1-s)^{\alpha-1} t^{\alpha-1} - \sum\limits_{p=k}^{m-2} \frac{a_p}{\alpha} (\xi_p - s)^{\alpha} t^{\alpha-1}, \ t \leq s, \ s \in J_k \\ k = 1, 2, \ldots, m-1; \\ (1-s)^{\alpha-1} t^{\alpha-1} - K(t-s)^{\alpha-1} - \sum\limits_{p=k}^{m-2} \frac{a_p}{\alpha} (\xi_p - s)^{\alpha} t^{\alpha-1} \\ t \geq s, \ s \in J_k, \ k = 1, 2, \ldots, m-1. \end{cases} \tag{2.5}$$

Proof . Lemma 2.5 yields

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int\limits_0^t (t-s)^{\alpha-1} y(s) ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_n t^{\alpha-n}. \tag{2.6}$$

By boundary conditions of (2.3) we get $c_2 = c_3 = \cdots = c_n = 0$. Then we have

$$u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s)ds + c_1 t^{\alpha-1}$$

and

$$u(1) = -\frac{1}{\Gamma(\alpha)} \int_0^1 (1-s)^{\alpha-1} y(s)ds + c_1. \tag{2.7}$$

Also, for $p = 1, 2, \ldots, m-2$ we obtain

$$\begin{aligned}\int_0^{\xi_p} u(s)ds &= -\frac{1}{\Gamma(\alpha)} \int_0^{\xi_p} \int_0^x (x-s)^{\alpha-1} y(s)dsdx + c_1 \int_0^{\xi_p} s^{\alpha-1} ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^{\xi_p} \int_s^{\xi_p} (x-s)^{\alpha-1} y(s)dxds + c_1 \int_0^{\xi_p} s^{\alpha-1} ds \\ &= -\frac{1}{\Gamma(\alpha)} \int_0^{\xi_p} \frac{(\xi_p - s)^\alpha}{\alpha} y(s)ds + c_1 \frac{\xi_p^\alpha}{\alpha}.\end{aligned}$$

Thus, we get

$$\sum_{p=1}^{m-2} \int_0^{\xi_p} u(s)ds = \sum_{p=1}^{m-2} \Big(-\frac{1}{\Gamma(\alpha)} \int_0^{\xi_p} \frac{(\xi_p - s)^\alpha}{\alpha} y(s)ds + c_1 \frac{\xi_p^\alpha}{\alpha} \Big).$$

When (2.7) combining with

$$u(1) = \sum_{p=1}^{m-2} a_p \int_0^{\xi_p} u(s)ds,$$

we have

$$c_1 = \int_0^1 \frac{(1-s)^{\alpha-1}}{K\Gamma(\alpha)} y(s)ds - \sum_{p=1}^{m-2} \int_0^{\xi_p} \frac{(\xi_p - s)^\alpha}{K\alpha\Gamma(\alpha)} y(s)ds.$$

Hence, we have

$$\begin{aligned}u(t) &= -\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} y(s)ds + \int_0^1 \frac{(1-s)^{\alpha-1} t^{\alpha-1}}{K\Gamma(\alpha)} y(s)ds - \sum_{p=1}^{m-2} \int_0^{\xi_p} \frac{(\xi_p - s)^\alpha t^{\alpha-1}}{K\alpha\Gamma(\alpha)} y(s)ds \\ &= \int_0^1 G(t,s)y(s)ds.\end{aligned}$$

The proof is complete. □

For the convenience, define the following:

$$m = \frac{1}{\alpha K\Gamma(\alpha)} \sum_{p=1}^{m-2} a_p \xi_p^\alpha, \;\; M = \frac{n}{K\Gamma(\alpha)} \Big[1 + \sum_{p=1}^{m-2} \frac{a_p(\xi_p^{\alpha-1} - \xi_p^\alpha)}{\alpha} \Big]. \tag{2.8}$$

Throughout the paper we assume that $M > m$.

Lemma 2.7. *The Green's function $G(t,s)$ defined by (2.5) satisfies*

$$G(t,s) \leq Ms(1-s)^{\alpha-1}$$

for $(t,s) \in [0,1] \times [0,1]$.

Proof . For $t \leq s$, $s \in J_k$, $k = 1, 2, \ldots, m-1$, we have

$$\begin{aligned}
K\Gamma(\alpha)G(t,s) &= (1-s)^{\alpha-1}t^{\alpha-1} - \sum_{p=k}^{m-2} \frac{a_p}{\alpha}(\xi_p - s)^{\alpha} t^{\alpha-1} \\
&= \left(1 - \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha}\right)(1-s)^{\alpha-1}t^{\alpha-1} + \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha}(1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad - \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha}\left(1 - \frac{s}{\xi_p}\right)^{\alpha} t^{\alpha-1} \\
&\leq \left(1 - \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha}\right)(1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad + \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha} t^{\alpha-1}\left[(1-s)^{\alpha-1} - \left(1 - \frac{s}{\xi_p}\right)^{\alpha}(1-s)^{\alpha-1}\right] \\
&\leq \left(1 - \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha}\right)(1-s)^{\alpha-1}t^{\alpha-1} \\
&\quad + \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha} t^{\alpha-1}(1-s)^{\alpha-1}\left[1 - \left(1 - \frac{s}{\xi_p}\right)\right] \\
&\quad \times \left[1 + \left(1 - \frac{s}{\xi_p}\right) + \left(1 - \frac{s}{\xi_p}\right)^2 + \cdots + \left(1 - \frac{s}{\xi_p}\right)^{n-1}\right] \\
&\leq n\left(1 - \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha}\right)(1-s)^{\alpha-1}st^{\alpha-2} + ns\sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha-1}}{\alpha}(1-s)^{\alpha-1}t^{\alpha-1} \\
&\leq ns(1-s)^{\alpha-1}\left[1 - \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha} + \sum_{p=k}^{m-2} \frac{a_p \xi_p^{\alpha-1}}{\alpha}\right] \\
&\leq ns(1-s)^{\alpha-1}\left[1 - \sum_{p=1}^{m-2} \frac{a_p \xi_p^{\alpha}}{\alpha} + \sum_{p=1}^{m-2} \frac{a_p \xi_p^{\alpha-1}}{\alpha}\right] \\
&= ns(1-s)^{\alpha-1}\left[1 + \sum_{p=1}^{m-2} \frac{a_p(\xi_p^{\alpha-1} - \xi_p^{\alpha})}{\alpha}\right].
\end{aligned}$$

Also, for $t \geq s$, $s \in J_k$, $k = 1, 2, \ldots, m-1$, we obtain

$$K\Gamma(\alpha)G(t,s) = (1-s)^{\alpha-1}t^{\alpha-1} - K(t-s)^{\alpha-1} - \sum_{p=k}^{m-2} \frac{a_p}{\alpha}(\xi_p - s)^{\alpha} t^{\alpha-1}$$

$$
\begin{aligned}
&\le (1-s)^{\alpha-1}t^{\alpha-1} - \sum_{p=k}^{m-2} \frac{a_p}{\alpha}(\xi_p - s)^{\alpha} t^{\alpha-1} \\
&\le ns(1-s)^{\alpha-1}\Bigg[1 - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha} + \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha-1}}{\alpha}\Bigg] \\
&= ns(1-s)^{\alpha-1}\Bigg[1 + \sum_{p=1}^{m-2} \frac{a_p(\xi_p^{\alpha-1} - \xi_p^{\alpha})}{\alpha}\Bigg].
\end{aligned}
$$

Hence, we get $G(t,s) \le Ms(1-s)^{\alpha-1}$. □

Lemma 2.8. *The Green's function $G(t,s)$ defined by (2.5) satisfies*

$$G(t,s) \ge ms(1-s)^{\alpha-1}t^{\alpha-1}$$

and $G(t,s) \ge 0$ for $(t,s) \in [0,1] \times [0,1]$.

Proof . For $t \le s$, $s \in J_k$, $k = 1, 2, \ldots, m-1$, we have

$$
\begin{aligned}
K\Gamma(\alpha)G(t,s) &= (1-s)^{\alpha-1}t^{\alpha-1} - \sum_{p=k}^{m-2} \frac{a_p}{\alpha}(\xi_p - s)^{\alpha} t^{\alpha-1} \\
&\ge (1-s)^{\alpha-1}t^{\alpha-1} - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}\left(1 - \frac{s}{\xi_p}\right)^{\alpha} t^{\alpha-1} \\
&\ge (1-s)^{\alpha-1}t^{\alpha-1} - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}(1-s)^{\alpha} t^{\alpha-1} \\
&= (1-s)^{\alpha-1}t^{\alpha-1}\Bigg[1 - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}(1-s)\Bigg] \\
&= (1-s)^{\alpha-1}t^{\alpha-1}\Bigg[1 - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha} + \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}s\Bigg] \\
&\ge \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha} s(1-s)^{\alpha-1}t^{\alpha-1}.
\end{aligned}
$$

Besides, for $t \ge s$, $s \in J_k$, $k = 1, 2, \ldots, m-1$, we find

$$
\begin{aligned}
K\Gamma(\alpha)G(t,s) &= (1-s)^{\alpha-1}t^{\alpha-1} - K(t-s)^{\alpha-1} - \sum_{p=k}^{m-2} \frac{a_p}{\alpha}(\xi_p - s)^{\alpha} t^{\alpha-1} \\
&\ge (1-s)^{\alpha-1}t^{\alpha-1} - \left(1 - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}\right)(t-s)^{\alpha-1} - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}(1-s)^{\alpha} t^{\alpha-1} \\
&= (1-s)^{\alpha-1}t^{\alpha-1}\Bigg[1 - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}(1-s)\Bigg] - \left(1 - \sum_{p=1}^{m-2} \frac{a_p\xi_p^{\alpha}}{\alpha}\right)(t-s)^{\alpha-1}
\end{aligned}
$$

$$\begin{aligned}
&= \left(1-\sum_{p=1}^{m-2}\frac{a_p\xi_p^\alpha}{\alpha}\right)\left[(1-s)^{\alpha-1}t^{\alpha-1}-(t-s)^{\alpha-1}\right]+\sum_{p=1}^{m-2}\frac{a_p\xi_p^\alpha}{\alpha}s(1-s)^{\alpha-1}t^{\alpha-1}\\
&\geq \left(1-\sum_{p=1}^{m-2}\frac{a_p\xi_p^\alpha}{\alpha}\right)(1-s)^{\alpha-2}t^{\alpha-2}[(1-s)t-(t-s)]+\sum_{p=1}^{m-2}\frac{a_p\xi_p^\alpha}{\alpha}s(1-s)^{\alpha-1}t^{\alpha-1}\\
&\geq \left(1-\sum_{p=1}^{m-2}\frac{a_p\xi_p^\alpha}{\alpha}\right)(1-s)^{\alpha-2}t^{\alpha-2}s(1-t)+\sum_{p=1}^{m-2}\frac{a_p\xi_p^\alpha}{\alpha}s(1-s)^{\alpha-1}t^{\alpha-1}\\
&\geq \sum_{p=1}^{m-2}\frac{a_p\xi_p^\alpha}{\alpha}s(1-s)^{\alpha-1}t^{\alpha-1}.
\end{aligned}$$

Thus, we obtain $G(t,s)\geq ms(1-s)^{\alpha-1}t^{\alpha-1}$ for $(t,s)\in[0,1]\times[0,1]$. Clearly $G(t,s)\geq 0$ for $(t,s)\in[0,1]\times[0,1]$. The proof is complete. □

Let $\mathbb{B}$ denote the Banach space $C[0,1]$ with the norm $\|u\|=\max_{t\in[0,1]}|u(t)|$. Define the cone $P\subset\mathbb{B}$ by

$$P=\{u\in\mathbb{B}: u(t)\geq 0 \text{ for } \forall t\in[0,1], u(t)\geq\frac{m}{M}t^{\alpha-1}\|u\|\}, \tag{2.9}$$

where m and M are given in (2.8). By Lemma 2.6 the solutions of the BVP (1.1) are the fixed points of the operator $A:P\to\mathbb{B}$ defined by

$$Au(t)=\int_0^1 G(t,s)f(s,u(s))ds. \tag{2.10}$$

Lemma 2.9. *$A:P\to P$ is completely continuous.*

Proof . If $u\in P$, then we have $Au(t)\geq 0$ on $[0,1]$ by using Lemma 2.7. On the other hand, we obtain

$$\|Au\|=\max_{t\in[0,1]}\int_0^1 G(t,s)f(s,u(s))ds\leq\int_0^1 Ms(1-s)^{\alpha-1}f(s,u(s))ds,$$

by Lemma 2.7. Therefore, we get

$$Au(t)=\int_0^1 G(t,s)f(s,u(s))ds\geq\int_0^1 ms(1-s)^{\alpha-1}t^{\alpha-1}f(s,u(s))ds\geq\frac{m}{M}t^{\alpha-1}\|Au\|,$$

by Lemma 2.8. Thus $Au\in P$ and therefore we have $AP\subset P$. And by a standard argument, we know that $A:P\to P$ is completely continuous. □

Definition 2.10. Remember that a subset $K\neq\emptyset$ of X is called a retract of X if there is a continuous map $R:X\to K$, a retraction, such that $Rx=x$ on K. Let X be a Banach space, $K\subset X$ retract, $\Omega\subset K$ open and $F:\overline{\Omega}\to K$ compact and such that $Fix(f)\cap\partial\Omega=\emptyset$. Then we can define an integer $i_K(f,\Omega)$ which has the following properties.

(a) $i_X(f,\Omega)=1$ for $f(\overline{\Omega})\in\Omega$.

(b) Let $f : \Omega \to K$ be a continuous function and assume that $Fix(f)$ is a compact subset of Ω. Let Ω_1 and Ω_2 be disjoint open subsets of Ω such that $Fix(f) \subset \Omega_1 \cup \Omega_2$. Then we obtain $i_K(f, \Omega) = i_K(f, \Omega_1) + i_K(f, \Omega_2)$.

(c) Let G be an open subset of $K \times [0, 1]$ and $F : G \to K$ be a continuous map. Assume that $Fix(F)$ is a compact subset of G. If $G_t = \{x : (x, t) \in G\}$ and $F_t = F(\cdot, t)$, then we have $i_K(F_0, G_0) = i_K(F_1, G_1)$.

(d) If $K_0 \subset K$ is a retract of K and $F(\overline{\Omega}) \subset K_0$, then $i_K(F, \Omega) = i_{K_0}(F, \Omega \cap K_0)$.

In order to follow the main results of this paper easily, now we state the fixed point theorems which we applied to prove main theorems.

Lemma 2.11. (Guo and Lakshmikantham [9], Lan [21]) Suppose P is a cone in a real Banach space B and let D be an open, bounded subset of B with $D_p := D \cap P \neq \emptyset$ and $\overline{D_p} \neq P$. Assume that $A : \overline{D_p} \to P$ is a compact map such that $y \neq Ay$ for $y \in \partial D_p$. The following results hold.

(i) If $\|Ay\| \leq \|y\|$ for $y \in \partial D_p$, then $i_p(A, D_p) = 1$.

(ii) If there exists $b \in P\backslash\{0\}$ such that $y \neq Ay+\lambda b$ for all $y \in \partial D_p$ and all $\lambda > 0$, then $i_p(A, D_p) = 0$.

(iii) Let U be open in P such that $\overline{U_p} \subset D_p$. If $i_p(A, D_p) = 1$ and $i_p(A, U_p) = 0$, then A has a fixed point in $D_p\backslash\overline{U_p}$. The same result holds if $i_p(A, D_p) = 0$ and $i_p(A, U_p) = 1$.

Theorem 2.12. (Avery and Henderson [3]) (Avery–Henderson Fixed Point Theorem) Let P be a cone in a real Banach space E. Set

$$P(\lambda, r) = \{u \in P : \lambda(u) < r\}.$$

Assume there exist positive numbers r and M, nonnegative increasing continuous functionals Φ, λ on P, and a nonnegative continuous functional ν on P with $\nu(0) = 0$ such that

$$\lambda(u) \leq \nu(u) \leq \Phi(u) \text{ and } \|u\| \leq M\lambda(u)$$

for all $u \in \overline{P(\lambda, r)}$. Suppose that there exist positive numbers $p < q < r$ such that

$$\nu(\zeta u) \leq \zeta\nu(u), \text{ for all } 0 \leq \zeta \leq 1 \text{ and } u \in \partial P(\nu, q).$$

If $A : \overline{P(\lambda, r)} \to P$ is a completely continuous operator satisfying

(i) $\lambda(Au) > r$ for all $u \in \partial P(\lambda, r)$,

(ii) $\nu(Au) < q$ for all $u \in \partial P(\nu, q)$,

(iii) $P(\Phi, p) \neq \emptyset$ and $\Phi(Au) > p$ for all $u \in \partial P(\Phi, p)$,

then A has at least two fixed points u_1 and u_2 such that

$$p < \Phi(u_1) \text{ with } \nu(u_1) < q \text{ and } q < \nu(u_2) \text{ with } \lambda(u_2) < r.$$

Theorem 2.13. (Legget and Williams [22]) (Leggett–Williams Fixed Point Theorem) Let P be a cone in the real Banach space E. Set

$$P_r := \{x \in P : \|x\| < r\}$$

$$P(\psi, a, b) := \{x \in P : a \leq \psi(x), \|x\| \leq b\}.$$

Suppose $A : \overline{P_r} \to \overline{P_r}$ be a completely continuous operator and ψ be a nonnegative continuous concave functional on P with $\psi(u) \leq \|u\|$ for all $u \in \overline{P_r}$. If there exists $0 < p < q < l \leq r$ such that the following condition hold,

(i) $\{u \in P(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ and $\psi(Au) > q$ for all $u \in P(\psi, q, l)$;
(ii) $\|Au\| < p$ for $\|u\| \leq p$;
(iii) $\psi(Au) > q$ for $u \in P(\psi, q, r)$ with $\|Au\| > l$,

then A has at least three fixed points u_1, u_2 and u_3 in $\overline{P_r}$ satisfying

$$\|u_1\| < p,\ \psi(u_2) > q,\ p < \|u_3\| \text{ with } \psi(u_3) < q.$$

3. Main Results

For convenience, we define

$$L := \int_0^1 s(1-s)^{\alpha-1} ds \tag{3.1}$$

and we take arbitrary constant η such that $0 < \eta < 1$. Now, we will give the sufficient conditions to have at least one positive solution for the BVP (1.1) by using Lemma 2.11. For the cone P given in (2.9) and any positive real number r, define the convex set

$$P_r := \{u \in P : \|u\| < r\} \tag{3.2}$$

and the set

$$\Omega_r := \{u \in P : \min_{t\in[\eta,1]} u(t) < \frac{m}{M} r\} \tag{3.3}$$

where m and M are as in (2.8).

Theorem 3.1. *Let there exist number $r > 0$ such that the function f satisfies the following conditions:*

(i) $f(t, u) \leq \frac{u}{ML}$ and $u \neq Au$ for $(t, u) \in [0, 1] \times [0, r]$;
(ii) $f(t, u) \geq \frac{r}{M\eta^{\alpha-1}L}$ and $u \neq Au$ for $(t, u) \in [\eta, 1] \times [\frac{m}{M} r, r]$.

Then the BVP (1.1) has at least one positive solution u in $P_r \backslash \overline{\Omega_r}$.

Proof . If $u \in \partial P_r$, then we obtain $0 \leq u(t) \leq r$ for $t \in [0, 1]$. Thus, we have

$$Au(t) = \int_0^1 G(t, s)f(s, u(s))ds \leq M \int_0^1 s(1-s)^{\alpha-1} f(s, u(s))ds \leq \frac{1}{L} \int_0^1 s(1-s)^{\alpha-1} u(s)ds \leq \|u\|,$$

by using hypothesis (i), Lemma 2.7 and (3.1). It follows that $\|Au\| \leq \|u\|$ for $u \in \partial P_r$. By Lemma 2.11 (i), we get $i_P(A, P_r) = 1$. Let $u \in \partial\Omega_r$. Since $\overline{\Omega_r} \subset P_r$ and $u \in \partial\Omega_r$, we have $\frac{m}{M} r \leq u(t) \leq r$ for $t \in [\eta, 1]$. If $b(t) \equiv 1$ for $t \in [0, 1]$, then $b \in \partial P_1$. Assume that there exist $u_0 \in \partial\Omega_r$ and $\lambda_0 > 0$ such that $u_0 = Au_0 + \lambda_0 b$. Then for $t \in [\eta, 1]$ we have

$$\begin{aligned} u_0(t) = Au_0(t) + \lambda_0 b(t) &= \int_0^1 G(t, s)f(s, u_0(s))ds + \lambda_0 \\ &\geq \int_0^1 \min_{t\in[\eta,1]} G(t, s)f(s, u_0(s))ds + \lambda_0 \geq \int_0^1 \min_{t\in[\eta,1]} mt^{\alpha-1} s(1-s)^{\alpha-1} f(s, u_0(s))ds + \lambda_0 \\ &= m\eta^{\alpha-1} \int_0^1 s(1-s)^{\alpha-1} f(s, u_0(s))ds + \lambda_0 \geq \frac{m}{M} r + \lambda_0, \end{aligned}$$

with the assumption (ii), Lemma 2.8 and (3.1). Since this implies that $\frac{m}{M}r \geq \frac{m}{M}r + \lambda_0$, we obtain a contradiction. Hence, we have $u_0 \neq Au_0 + \lambda_0 b$ for $u_0 \in \partial\Omega_r$ and $\lambda_0 > 0$, so by Lemma 2.11 (ii), we get $i_P(A, \Omega_r) = 0$. From Lemma 2.11 (iii), A has a fixed point in $P_r \backslash \overline{\Omega_r}$. So, the BVP (1.1) has at least one positive solution u in $P_r \backslash \overline{\Omega_r}$. □

To prove the existence of at least two positive solutions of the BVP (1.1), we apply the Avery–Henderson fixed point theorem.

Theorem 3.2. *Suppose that there exist numbers* $0 < p < q < r$ *such that the function* f *satisfies the following conditions:*

(i) $f(t,u) > \frac{r}{m\eta^{\alpha-1}L}$ *for* $(t,u) \in [\eta,1] \times [r, \frac{M}{m\eta^{\alpha-1}}r]$*;*
(ii) $f(t,u) < \frac{q}{ML}$ *for* $(t,u) \in [0,1] \times [0,q]$*;*
(iii) $f(t,u) > \frac{p}{mL}$ *for* $(t,u) \in [0,1] \times [0,p]$*,*

where m*,* M *and* L *are as in* (2.8) *and* (3.1)*, respectively. Then the BVP* (1.1) *has at least two positive solutions* u_1 *and* u_2 *satisfying*

$$p < \|u_1\| \text{ with } \|u_1\| < q \text{ and } q < \|u_2\| \text{ with } \min_{t\in[\eta,1]} u_2(t) < r.$$

Proof . Define the cone P as in (2.9). From Lemma 2.9, $AP \subset P$ and A is completely continuous operator. Let nonnegative increasing continuous functionals ϕ, θ and ψ defined on the cone P by

$$\phi(u) = \min_{t\in[\eta,1]} u(t), \ \theta(u) = \psi(u) = \max_{t\in[0,1]} u(t) = \|u\|.$$

For each $u \in P$, we have $\phi(u) \leq \theta(u) \leq \psi(u)$ and

$$\phi(u) = \min_{t\in[\eta,1]} u(t) \geq \min_{t\in[\eta,1]} \frac{m}{M} t^{\alpha-1}\|u\| = \frac{m}{M}\eta^{\alpha-1}\|u\|.$$

Also, $\theta(0) = 0$ and for all $u \in P$, $\lambda \in [0,1]$ we have $\theta(\lambda u) = \lambda\theta(u)$. We now verify that other conditions of Avery–Henderson fixed point theorem are satisfied. If $u \in \partial P(\phi, r)$, we have $r \leq u(t) \leq \frac{M}{m\eta^{\alpha-1}}r$ for $t \in [\eta,1]$. Then from assumption (i) and Lemma 2.8, we have

$$\begin{aligned}\phi(Au) &= \min_{t\in[\eta,1]} \int_0^1 G(t,s)f(s,u(s))ds \\ &\geq \min_{t\in[\eta,1]} mt^{\alpha-1} \int_0^1 s(1-s)^{\alpha-1} f(s,u(s))ds > r.\end{aligned}$$

Thus, condition (i) of Avery–Henderson fixed point theorem holds. If $u \in \partial P(\theta, q)$, we have $0 \leq u(t) \leq q$ for $t \in [0,1]$. Then, we get

$$\begin{aligned}\theta(Au) &= \max_{t\in[0,1]} \int_0^1 G(t,s)f(s,u(s))ds \\ &\leq M \int_0^1 s(1-s)^{\alpha-1} f(s,u(s))ds < q\end{aligned}$$

by hypothesis (ii) and Lemma 2.7. Hence, condition (ii) of Avery–Henderson fixed point theorem holds. Since $0 \in P$ and $p > 0$, $P(\psi, p) \neq \emptyset$. If $u \in \partial P(\psi, p)$, we have $0 \leq u(t) \leq p$ for $t \in [0, 1]$. Then from assumption (iii), we obtain

$$\begin{aligned}\psi(Au) &= \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))ds \\ &\geq \max_{t\in[0,1]} mt^{\alpha-1} \int_0^1 s(1-s)^{\alpha-1} f(s,u(s))ds \\ &= m\int_0^1 s(1-s)^{\alpha-1} f(s,u(s))ds > p.\end{aligned}$$

Since all conditions of Avery–Henderson fixed point theorem are satisfied, the BVP (1.1) has at least two positive solutions u_1 and u_2 such that

$$p < \|u_1\| \text{ with } \|u_1\| < q \text{ and } q < \|u_2\| \text{ with } \min_{t\in[\eta,1]} u_2(t) < r.$$

□

Now we will use the Legget–Williams fixed point theorem to prove the next theorem.

Theorem 3.3. *Suppose that there exist numbers* $0 < p < q < \frac{M}{m\eta^{\alpha-1}} q \leq r$ *such that the function* f *satisfies the following conditions:*

(i) $f(t, u) \leq \frac{r}{ML}$ *for* $(t, u) \in [0, 1] \times [0, r]$*;*
(ii) $f(t, u) > \frac{q}{m\eta^{\alpha-1}L}$ *for* $(t, u) \in [\eta, 1] \times [q, \frac{M}{m\eta^{\alpha-1}} q]$*;*
(iii) $f(t, u) < \frac{p}{ML}$ *for* $(t, u) \in [0, 1] \times [0, p]$*,*

where m*,* M *and* L *are as in* (2.8) *and* (3.1)*, respectively. Then the BVP* (1.1) *has at least three positive solutions* u_1*,* u_2 *and* u_3 *satisfying*

$$\|u_1\| < p,\ \min_{t\in[\eta,1]} u_2(t) > q,\ p < \|u_3\| \text{ with } \min_{t\in[\eta,1]} u_3(t) < q.$$

Proof . Define the nonnegative, continuous, concave functional $\psi : P \to [0, \infty)$ to be $\psi(u) = \min_{t\in[\eta,1]} u(t)$ and the cone P as in (2.9). For all $u \in P$, we have $\psi(u) \leq \|u\|$. If $u \in \overline{P_r}$, then we obtain $0 \leq u(t) \leq r$ for all $t \in [0, 1]$. We get

$$\begin{aligned}\|Au\| &= \max_{t\in[0,1]} \int_0^1 G(t,s) f(s,u(s))ds \\ &\leq M\int_0^1 s(1-s)^{\alpha-1} f(s,u(s))ds \leq r,\end{aligned}$$

by hypothesis (i) and Lemma 2.7. Thus, we have $A : \overline{P_r} \to \overline{P_r}$. From Lemma 2.9, $A : \overline{P_r} \to \overline{P_r}$ is completely continuous. In the same way, we can show that if (iii) holds, then $A(\overline{P_p}) \subset P_p$. Hence, we get $\|Au\| < p$, $\forall u \in \overline{P_p}$ and so condition (ii) of Theorem 2.13 holds. Since $\frac{M}{m\eta^{\alpha-1}} > 1$, we have $\frac{M}{m\eta^{\alpha-1}}q \in P(\psi, q, \frac{M}{m\eta^{\alpha-1}}q)$ and $\psi(\frac{M}{m\eta^{\alpha-1}}q) > q$. Then, we obtain

$$\{u \in P(\psi, q, \frac{M}{m\eta^{\alpha-1}}q) : \psi(u) > q\} \neq \emptyset.$$

On the other hand, for all $u \in P(\psi, q, \frac{M}{m\eta^{\alpha-1}}q)$, we have $q \le u(t) \le \frac{M}{m\eta^{\alpha-1}}q$ for $t \in [\eta, 1]$. Using assumption (ii) and Lemma 2.8, we find

$$\begin{aligned}\psi(Au) &= \min_{t\in[\eta,1]} Au(t) = \min_{t\in[\eta,1]} \int_0^1 G(t,s)f(s,u(s))ds \\ &\ge \min_{t\in[\eta,1]} mt^{\alpha-1} \int_0^1 s(1-s)^{\alpha-1} f(s,u(s))ds > q.\end{aligned}$$

Thus, condition (i) of Theorem 2.13 holds. Finally, we will check that condition (iii) of Theorem 2.13. We suppose that $u \in P(\psi, q, r)$ with $\|Au\| > \frac{M}{m\eta^{\alpha-1}}q$. Then we obtain

$$\begin{aligned}\psi(Au) &= \min_{t\in[\eta,1]} Au(t) \ge \min_{t\in[\eta,1]} \frac{m}{M} t^{\alpha-1}\|Au\| = \frac{m}{M}\eta^{\alpha-1}\|Au\| \\ &> \frac{m}{M}\eta^{\alpha-1}\frac{M}{m\eta^{\alpha-1}}q = q.\end{aligned}$$

Since all conditions of the Legget–Williams fixed point theorem are satisfied. The BVP (1.1) has at least three positive solutions u_1, u_2 and u_3 such that

$$\|u_1\| < p, \ \min_{t\in[\eta,1]} u_2(t) > q, \ p < \|u_3\| \text{ with } \min_{t\in[\eta,1]} u_3(t) < q.$$

□

Example 3.4. Taking $n = 4$, $m = 4$, $\alpha = \frac{7}{2}$, $a_1 = \frac{1}{4} = \xi_1$, $a_2 = \frac{1}{2} = \xi_2$, we consider the following boundary value problem:

$$\begin{cases} D_{0+}^{\frac{7}{2}} u(t) + f(t, u(t)) = 0, \ \ t \in [0,1] \\ u(0) = u'(0) = u''(0) = 0 \\ u(1) = \frac{1}{4}\int_0^{\frac{1}{4}} u(s)ds + \frac{1}{2}\int_0^{\frac{1}{2}} u(s)ds, \end{cases} \tag{3.4}$$

where

$$f(t, u(t)) = \begin{cases} (\frac{10^9}{54} + \frac{1}{2})\frac{u^3}{1+u^3}, & u \ge 0.003 \\ \frac{1}{2}, & u \le 0.003. \end{cases}$$

Then by simple calculations, we can obtain that $K \approx 0.9868$, $L = \frac{1}{14}$, $m \approx 0.004$, $M \approx 1.237$. If we take $p = 0.0001$, $q = 0.003$, $r = 2000$, then all conditions in Theorem 3.2 are verified. Thus the BVP (3.4) has at least two positive solutions u_1 and u_2 satisfying

$$0.001 < \|u_1\| \text{ with } \|u_1\| < 0.003 \text{ and } 0.003 < \|u_2\| \text{ with } \min_{t\in[0.9,1]} u_2(t) < 2000.$$

Example 3.5. Taking $n = 3$, $\alpha = \frac{5}{2}$, $m = 3$, $a_1 = 1$, $\xi_1 = \frac{1}{2}$ and $\eta = 0.9$ we consider the following boundary value problem:

$$\begin{cases} D_{0_+}^{\frac{5}{2}} u(t) + \frac{2000u^2}{1+u^2} = 0, \quad t \in [0,1] \\ u(0) = u'(0) = 0 \\ u(1) = \int_0^{\frac{1}{2}} u(s)ds. \end{cases} \tag{3.5}$$

Then we get $K \approx 0.92928932$, $m \approx 0.05723976$, $M \approx 2.60019693$ and $L = \frac{4}{35}$. If we take $p = 0.00001$, $q = 10$ and $r = 2000$, then $0 < p < q < \frac{M}{m\eta^{\alpha-1}} q \leq r$ and the conditions $(i) - (iii)$ of Theorem 3.3 are satisfied. Thus, the BVP (3.5) has at least three positive solutions u_1, u_2 and u_3 satisfying

$$\|u_1\| < p, \ \min_{t\in[0.9,1]} u_2(t) > q, \ p < \|u_3\| \text{ with } \min_{t\in[0.9,1]} u_3(t) < q.$$

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