Int. J. Nonlinear Anal. Appl. 7 (2016) No. 1, 249-269
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2015.323

On a Hardy-Hilbert-type inequality with a general homogeneous kernel

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(Communicated by M. Eshaghi Gordji)

Abstract

By the method of weight coefficients and techniques of real analysis, a Hardy-Hilbert-type inequality with a general homogeneous kernel and a best possible constant factor is given. The equivalent forms, the operator expressions with the norm, the reverses and some particular examples are also considered.

Keywords: Hardy-Hilbert-type inequality; weight coefficient; equivalent form; reverse; operator.
2010 MSC: Primary 26D15; Secondary 47A07.

1. Introduction and preliminaries

Suppose that $p > 1, \frac{1}{p} + \frac{1}{q} = 1, f(x), g(y) \geq 0, f \in L^p(\mathbf{R}_+), g \in L^q(\mathbf{R}_+)$,

$$||f||_p = \left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}} > 0,$$

$||g||_q > 0$. We have the following Hardy-Hilbert integral inequality with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ (cf. [1]):

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy < \frac{\pi}{\sin(\pi/p)} ||f||_p ||g||_q. \tag{1.1}$$

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Received: June 2015 *Revised:* November 2015

Assuming that $a_m, b_n \geq 0, a = \{a_m\}_{m=1}^{\infty} \in l^p$, $b = \{b_n\}_{n=1}^{\infty} \in l^q$,

$$||a||_p = (\sum_{m=1}^{\infty} a_m^p)^{\frac{1}{p}} > 0, ||b||_q > 0,$$

we have the following Hardy-Hilbert inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$(cf. [1]):

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}||a||_p||b||_q. \tag{1.2}$$

The inequalities (1.1) and (1.2) have proven to be of importance in Analysis and its applications (cf. [1], [2], [3], [4], [5], [6]).

If $k(x,y)$ is a finite positive homogeneous function of degree -1 in $\mathbf{R}_+^2$ where

$$k = \int_0^{\infty} k(u,1)u^{-1/p}du = \int_0^{\infty} k_{\lambda}(1,u)u^{-1/q}du \in \mathbf{R}_+,$$

$k(x,y)x^{-1/p}(k(x,y)y^{-1/q})$ is strictly decreasing with respect to $x > 0$ $(y > 0)$, $\mu_i, \upsilon_j > 0$ $(i, j \in \mathbf{N} = \{1, 2, \cdots\})$, and

$$U_m := \sum_{i=1}^{m} \mu_i, V_n := \sum_{j=1}^{n} \upsilon_j (m, n \in \mathbf{N}), \tag{1.3}$$

then for

$$0 < \sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}} < \infty, \quad 0 < \sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}} < \infty,$$

we have the following Hardy-Hilbert-type inequality (cf. [1], Theorem 320, replacing $\mu_m^{1/q}a_m$ and $\upsilon_n^{1/p}b_n$ by a_m and b_n) :

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k(U_m, V_n)a_m b_n < k \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{1.4}$$

In particular, for $k(x,y) = \frac{1}{x+y}$, we obtain (cf. [1], Theorem 321, replacing $\mu_m^{1/q}a_m$ and $\upsilon_n^{1/p}b_n$ by a_m and b_n):

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{U_m + V_n} < \frac{\pi}{\sin(\frac{\pi}{p})} \left(\sum_{m=1}^{\infty} \frac{a_m^p}{\mu_m^{p-1}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^{\infty} \frac{b_n^q}{\upsilon_n^{q-1}}\right)^{\frac{1}{q}}. \tag{1.5}$$

For $\mu_i = \upsilon_j = 1 (i, j \in \mathbf{N})$, (1.5) reduces to (1.2). We still consider (1.5) as a Hardy-Hilbert-type inequality.

Note. The authors of [1] did not prove that the inequalities (1.4) and (1.5) are valid with the best possible constant factors.

In 1998, by introducing an independent parameter $\lambda \in (0, 1]$, Yang [7] gave an extension of (1.1) for $p = q = 2$ with the kernel $\frac{1}{(x+y)^{\lambda}}$. Following [7], Yang [5] gave some extensions of (1.1) and (1.2) as follows:

If $\lambda_1, \lambda_2 \in \mathbf{R} = (-\infty, \infty), \lambda_1 + \lambda_2 = \lambda, k_{\lambda}(x,y)$ is a non-negative homogeneous function of degree $-\lambda$, with $k(\lambda_1) = \int_0^{\infty} k_{\lambda}(t,1)t^{\lambda_1 - 1}dt \in \mathbf{R}_+ = (0, \infty), \phi(x) = x^{p(1-\lambda_1)-1}, \psi(y) = y^{q(1-\lambda_2)-1}, f(x), g(y) \geq 0,$

$$f \in L_{p,\phi}(\mathbf{R}_+) = \left\{ f; ||f||_{p,\phi} := (\int_0^{\infty} \phi(x)|f(x)|^p dx)^{\frac{1}{p}} < \infty \right\},$$

$g \in L_{q,\psi}(\mathbf{R}_+), ||f||_{p,\phi}, ||g||_{q,\psi} > 0$, then

$$\int_0^\infty \int_0^\infty k_\lambda(x,y)f(x)g(y)dxdy < k(\lambda_1)||f||_{p,\phi}||g||_{q,\psi}, \tag{1.6}$$

where the constant factor $k(\lambda_1)$ is the best possible.

Moreover, if $k_\lambda(x,y)$ is finite and $k_\lambda(x,y)x^{\lambda_1-1}(k_\lambda(x,y)y^{\lambda_2-1})$ is strictly decreasing with respect to $x > 0$ $(y > 0)$, then for $a_{m,}b_n \geq 0$,

$$a \in l_{p,\phi} = \left\{ a; ||a||_{p,\phi} := (\sum_{n=1}^\infty \phi(n)|a_n|^p)^{\frac{1}{p}} < \infty \right\},$$

$b = \{b_n\}_{n=1}^\infty \in l_{q,\psi}$, $||a||_{p,\phi}, ||b||_{q,\psi} > 0$, we have

$$\sum_{m=1}^\infty \sum_{n=1}^\infty k_\lambda(m,n)a_m b_n < k(\lambda_1)||a||_{p,\phi}||b||_{q,\psi}, \tag{1.7}$$

where the constant factor $k(\lambda_1)$ is still the best possible. We shall call (1.6) as Yang-Hilbert-type integral inequalities, and (1.7) as discrete Yang-Hilbert-type inequality.

Clearly, for

$$\lambda = 1,\ k_1(x,y) = \frac{1}{x+y},\ \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p},$$

the inequality (1.6) reduces to (1.1), while (1.7) reduces to (1.2). Some other results including multiple and multidimensional Hilbert-type inequalities are provided by [9]- [28].

In 2015, by adding a few conditions, Yang [29] gave an extension of (1.5) as follows:

$$\sum_{m=1}^\infty \sum_{n=1}^\infty \frac{a_m b_n}{(U_m+V_n)^\lambda} < B(\lambda_1,\lambda_2) \left(\sum_{m=1}^\infty \frac{U_m^{p(1-\lambda_1)-1} a_m^p}{\mu_m^{p-1}} \right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{V_n^{q(1-\lambda_2)-1} b_n^q}{\upsilon_n^{q-1}} \right)^{\frac{1}{q}},$$

where, the constant factor $B(\lambda_1,\lambda_2)$ is the best possible. For $\lambda = 1, \lambda_1 = \frac{1}{q}$, $\lambda_2 = \frac{1}{p}$, (1.8) reduces to (1.5).

In this paper, by the method of weight coefficients and techniques of real analysis a Hardy-Hilbert-type inequality with a general homogeneous kernel and a best possible constant factor is given, which is an extension of (1.4), (1.8) and (1.7). The equivalent forms, the operator expression with the norm, the reverses and some particular examples are also considered.

2. Some lemmas

In the following, we shall consider that $\mu_i, \upsilon_j > 0$ $(i,j \in \mathbf{N}), U_m$ and V_n are defined as in (1.3), $\lambda_1,\lambda_2 \in \mathbf{R}$, $\lambda_1+\lambda_2 = \lambda, p \neq 0,1$, $\frac{1}{p}+\frac{1}{q} = 1$, $a_m, b_n \geq 0 (m,n \in \mathbf{N})$,

$$||a||_{p,\Phi_\lambda} = (\sum_{m=1}^\infty \Phi_\lambda(m)a_m^p)^{\frac{1}{p}},\quad ||b||_{q,\Psi_\lambda} = (\sum_{n=1}^\infty \Psi_\lambda(n)b_n^q)^{\frac{1}{q}},$$

where

$$\Phi_\lambda(m) := \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}, \Psi_\lambda(n) := \frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}} \quad (m,n \in \mathbf{N}).$$

Lemma 2.1. *Let $h(t)$ be a non-negative measurable function in $\mathbf{R}_+$, $a \in \mathbf{R}$, and suppose that there exists a constant $\delta_0 > 0$, such that for any $\delta \in [0, \delta_0)$,*

$$k(a \pm \delta) := \int_0^\infty h(t) t^{(a\pm\delta)-1} dt \in \mathbf{R}.$$

Then we have

$$k(a \pm \delta) = k(a) + o(1)(\delta \to 0^+). \tag{2.1}$$

Proof . For any $\delta \in [0, \frac{\delta_0}{2})$, it follows that

$$h(t) t^{(a\pm\delta)-1} \le g(t) := \begin{cases} h(t) t^{(a-\frac{\delta_0}{2})-1}, t \in (0,1], \\ h(t) t^{(a+\frac{\delta_0}{2})-1}, t \in (1,\infty). \end{cases}$$

Since we find

$$\begin{aligned} 0 &\le \int_0^\infty g(t)dt = \int_0^1 h(t) t^{(a-\frac{\delta_0}{2})-1} dt + \int_1^\infty h(t) t^{(a+\frac{\delta_0}{2})-1} dt \\ &\le \int_0^\infty h(t) t^{(a-\frac{\delta_0}{2})-1} dt + \int_0^\infty h(t) t^{(a+\frac{\delta_0}{2})-1} dt \\ &= k(a - \frac{\delta_0}{2}) + k(\alpha + \frac{\delta_0}{2}) \in \mathbf{R}, \end{aligned}$$

then for any $\delta \in (0, \frac{\delta_0}{2})$, by Lebesgue control convergence theorem (cf. [31]), it follows that

$$k(a \pm \delta) = \int_0^\infty h(t) t^{(a\pm\delta)-1} dt = \int_0^\infty h(t) t^{a-1} dt + o(1)(\delta \to 0^+),$$

and then (2.1) follows. □

Lemma 2.2. *If $g(t)$ (> 0) is decreasing in $\mathbf{R}_+$ and strictly decreasing in $[n_0, \infty) \subset \mathbf{R}_+ (n_0 \in \mathbf{N})$, satisfying $\int_0^\infty g(t)dt \in \mathbf{R}_+$, then we have*

$$\int_1^\infty g(t)dt < \sum_{n=1}^\infty g(n) < \int_0^\infty g(t)dt. \tag{2.2}$$

Proof . By the assumption we have

$$\int_n^{n+1} g(t)dt \le g(n) \le \int_{n-1}^n g(t)dt (n = 1, \cdots, n_0),$$

$$\int_{n_0+1}^{n_0+2} g(t)dt < g(n_0+1) < \int_{n_0}^{n_0+1} g(t)dt, \tag{2.3}$$

and thus it follows that

$$0 < \int_1^{n_0+2} g(t)dt < \sum_{n=1}^{n_0+1} g(n) < \sum_{n=1}^{n_0+1} \int_{n-1}^n g(t)dt = \int_0^{n_0+1} g(t)dt.$$

Similarly, we have

$$0 < \int_{n_0+2}^\infty g(t)dt \le \sum_{n=n_0+2}^\infty g(n) \le \int_{n_0+1}^\infty g(t)dt < \infty.$$

Hence, adding by parts the above two inequalities, we obtain (2.2). □

Lemma 2.3. *Let $k_\lambda(x,y)$ be a finite positive homogeneous function of degree $-\lambda$ in $\mathbf{R}_+^2$, satisfying*

$$k_\lambda(ux,uy)=u^{-\lambda}k_\lambda(x,y),$$

for any $u,x,y>0$. Let also

$$k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}\quad (k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}})$$

be strictly decreasing with respect to $x>0$ ($y>0$),

$$k(\lambda_1):=\int_0^\infty k_\lambda(u,1)u^{\lambda_1-1}du=\int_0^\infty k_\lambda(1,u)u^{\lambda_2-1}du\in\mathbf{R}_+,\tag{2.4}$$

and define the weight coefficients as follows:

$$\omega(\lambda_2,m)\ :=\sum_{n=1}^\infty k_\lambda(U_m,V_n)\frac{U_m^{\lambda_1}\upsilon_n}{V_n^{1-\lambda_2}},m\in\mathbf{N},\tag{2.5}$$

$$\varpi(\lambda_1,n)\ :=\sum_{m=1}^\infty k_\lambda(U_m,V_n)\frac{V_n^{\lambda_2}\mu_m}{U_m^{1-\lambda_1}},n\in\mathbf{N}.\tag{2.6}$$

Then, we have the following inequalities:

$$\omega(\lambda_2,m)\ <\ k(\lambda_1)(m\in\mathbf{N}),\tag{2.7}$$

$$\varpi(\lambda_1,n)\ <\ k(\lambda_1)(n\in\mathbf{N}).\tag{2.8}$$

Proof . We set $\mu(t):=\mu_m,t\in(m-1,m](m\in\mathbf{N});\upsilon(t):=\upsilon_n,t\in(n-1,n](n\in\mathbf{N})$,

$$U(x):=\int_0^x\mu(t)dt(x\geq 0),V(y):=\int_0^y\upsilon(t)dt\ (y\geq 0).\tag{2.9}$$

Then by (1.3), it follows that

$$U(m)=U_m,V(n)=V_n\ (m,n\in\mathbf{N}).$$

For $x\in(m-1,m)$, it follows that

$$U'(x)=\mu(x)=\mu_m\ (m\in\mathbf{N});$$

for $y\in(n-1,n)$, it follows that

$$V'(y)=\upsilon(y)=\upsilon_n(n\in\mathbf{N}).$$

Since $V(y)$ is strictly increasing in $(n-1,n]$, in view of (2.3) and (2.2), we derive that

$$\omega(\lambda_2,m)\ =\ \sum_{n=1}^\infty\int_{n-1}^n k_\lambda(U_m,V_n)\frac{U_m^{\lambda_1}}{V_n^{1-\lambda_2}}V'(y)dy<\sum_{n=1}^\infty\int_{n-1}^n k_\lambda(U_m,V(y))\frac{U_m^{\lambda_1}V'(y)}{V^{1-\lambda_2}(y)}dy.$$

Setting $t=\frac{V(y)}{U_m}$, we obtain $V'(y)dy=U_m dt$ and

$$\begin{aligned}\omega(\lambda_2,m)\ &<\ \sum_{n=1}^\infty\int_{\frac{V(n-1)}{U_m}}^{\frac{V(n)}{U_m}}k_\lambda(1,t)t^{\lambda_2-1}dt=\int_0^{\frac{V(\infty)}{U_m}}k_\lambda(1,t)t^{\lambda_2-1}dt\\ &\leq\ \int_0^\infty k_\lambda(1,t)t^{\lambda_2-1}dt=k(\lambda_1).\end{aligned}$$

Since $U(x)$ is strictly increasing in $(m-1, m]$, we similarly obtain that

$$\begin{aligned}\varpi(\lambda_1, n) &= \sum_{m=1}^{\infty}\int_{m-1}^{m} k_\lambda(U_m, V_n)\frac{V_n^{\lambda_2}U'(x)}{U_m^{1-\lambda_1}}dx \\ &< \sum_{m=1}^{\infty}\int_{m-1}^{m} k_\lambda(U(x), V_n)\frac{V_n^{\lambda_2}U'(x)}{U^{1-\lambda_1}(x)}dx \\ &\overset{t=U(x)/V_n}{=} \sum_{m=1}^{\infty}\int_{\frac{U(m-1)}{V_n}}^{\frac{U(m)}{V_n}} k_\lambda(t,1)t^{\lambda_1-1}dt \\ &= \int_0^{\frac{U(\infty)}{V_n}} k_\lambda(t,1)t^{\lambda_1-1}dt \le k(\lambda_1).\end{aligned}$$

Hence, we get inequalities (2.7) and (2.8). □

Lemma 2.4. *Adopting the assumptions of Lemma* 2.3*, if there exist constants* $\tau_1, \tau_2 \in \mathbf{R}$*, and* $L > 0$*, such that* $\tau_1 < \lambda_1 < \tau_2$*,*

$$k_\lambda(u,1) \le \frac{L}{u^{\tau_1}}\ (u \in (0,1)),\quad k_\lambda(u,1) \le \frac{L}{u^{\tau_2}}\ (u \in [1,\infty)),$$

there exist $m_0, n_0 \in \mathbf{N}$*, such that*

$$\mu_m \ge \mu_{m+1}\ (m \in \{m_0, m_0+1, \cdots\}),\ \ \upsilon_n \ge \upsilon_{n+1}(n \in \{n_0, n_0+1, \cdots\}),\ \text{and } U(\infty) = V(\infty) = \infty,$$

then, we have
(i)

$$k(\lambda_1)(1-\theta(\lambda_2, m)) < \omega(\lambda_2, m)(m \in \mathbf{N}), \tag{2.10}$$
$$k(\lambda_1)(1-\vartheta(\lambda_1, n)) < \varpi(\lambda_1, n)(n \in \mathbf{N}), \tag{2.11}$$

where,

$$\begin{aligned}\theta(\lambda_2, m) &:= \frac{1}{k(\lambda_1)}\int_0^{\frac{V_{n_0}}{U_m}} k_\lambda(1,t)t^{\lambda_2-1}dt \\ &= O(\frac{1}{U_m^{\alpha}}) \in (0,1)(\alpha = \tau_2 - \lambda_1 > 0), \\ \vartheta(\lambda_1, n) &:= \frac{1}{k(\lambda_1)}\int_0^{\frac{U_{m_0}}{V_n}} k_\lambda(t,1)t^{\lambda_1-1}dt \\ &= O(\frac{1}{V_n^{\beta}}) \in (0,1)(\beta = \lambda_1 - \tau_1 > 0);\end{aligned}$$

(ii)

$$\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+b}} = \frac{1}{b}\left(\frac{1}{U_{m_0}^b} + bO_1(1)\right), \tag{2.12}$$
$$\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+b}} = \frac{1}{b}\left(\frac{1}{V_{n_0}^b} + bO_2(1)\right), \tag{2.13}$$

for any $b > 0$.

Proof . Since $\upsilon_n \geq \upsilon_{n+1} (n \geq n_0)$ and $V(\infty) = \infty$, by Lemma 2.2, we get

$$\begin{aligned}
\omega(\lambda_2, m) &\geq \sum_{n=n_0}^{\infty} k_\lambda(U_m, V_n) \frac{U_m^{\lambda_1} \upsilon_{n+1}}{V_n^{1-\lambda_2}} = \sum_{n=n_0}^{\infty} \int_n^{n+1} k_\lambda(U_m, V_n) \frac{U_m^{\lambda_1} V'(y)}{V_n^{1-\lambda_2}} dy \\
&> \sum_{n=n_0}^{\infty} \int_n^{n+1} k_\lambda(U_m, V(y)) \frac{U_m^{\lambda_1} V'(y)}{V^{1-\lambda_2}(y)} dy = \sum_{n=n_0}^{\infty} \int_{\frac{V(n)}{U_m}}^{\frac{V(n+1)}{U_m}} k_\lambda(1, t) t^{\lambda_2 - 1} dt \\
&= \int_{\frac{V_{n_0}}{U_m}}^{\infty} k_\lambda(1, t) t^{\lambda_2 - 1} dt = k(\lambda_1)(1 - \theta(\lambda_2, m)),
\end{aligned}$$

where, we derive that

$$\theta(\lambda_2, m) = \frac{1}{k(\lambda_1)} \int_0^{\frac{V_{n_0}}{U_m}} k_\lambda(1, t) t^{\lambda_2 - 1} dt = \frac{1}{k(\lambda_1)} \int_{\frac{U_m}{V_{n_0}}}^{\infty} k_\lambda(v, 1) v^{\lambda_1 - 1} dv \in (0, 1).$$

For $\alpha = \tau_2 - \lambda_1 > 0,\ U_m > V_{n_0}$, we obtain

$$0 < \theta(\lambda_2, m) \leq \frac{L}{k(\lambda_1)} \int_{\frac{U_m}{V_{n_0}}}^{\infty} v^{-\alpha - 1} dv = \frac{L}{\alpha k(\lambda_1)} (\frac{V_{n_0}}{U_m})^\alpha,$$

namely, $\theta(\lambda_2, m) = O(\frac{1}{U_m^\alpha})$. Hence we have (2.10). Similarly, we obtain (2.11).

For $b > 0$, we find

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+b}} &= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \sum_{m=m_0+1}^{\infty} \frac{\mu_m}{U_m^{1+b}} \\
&= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \sum_{m=m_0+1}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U_m^{1+b}} dx \\
&< \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \sum_{m=m_0+1}^{\infty} \int_{m-1}^{m} \frac{U'(x)}{U^{1+b}(x)} dx \\
&= \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \int_{m_0}^{\infty} \frac{dU(x)}{U^{1+b}(x)} = \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} + \frac{1}{bU_{m_0}^b} \\
&= \frac{1}{b} \left(\frac{1}{U_{m_0}^b} + b \sum_{m=1}^{m_0} \frac{\mu_m}{U_m^{1+b}} \right),
\end{aligned}$$

$$\begin{aligned}
\sum_{m=1}^{\infty} \frac{\mu_m}{U_m^{1+b}} &\geq \sum_{m=m_0}^{\infty} \frac{\mu_{m+1}}{U_m^{1+b}} = \sum_{m=m_0}^{\infty} \int_m^{m+1} \frac{U'(x)}{U_m^{1+b}} dx \\
&> \sum_{m=m_0}^{\infty} \int_m^{m+1} \frac{U'(x) dx}{U^{1+b}(x)} = \int_{m_0}^{\infty} \frac{dU(x)}{U^{1+b}(x)} = \frac{1}{bU_{m_0}^b}.
\end{aligned}$$

Hence we obtain (2.12). Similarlyy, we get (2.13). □

Note. For example,

$$\mu_m = \frac{1}{m^\sigma},\ \upsilon_n = \frac{1}{n^\sigma}\ (0 \leq \sigma \leq 1; m, n \in \mathbf{N})$$

satisfy the conditions of Lemma 2.4 (for $m_0 = n_0 = 1$).

3. Main results

Theorem 3.1. *Let $k_\lambda(x,y)$ be a finite positive homogeneous function of degree $-\lambda$ in $\mathbf{R}_+^2$,*

$$k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}(k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}})$$

be strictly decreasing with respect to $x>0$ $(y>0)$, and $k(\lambda_1)$ $(\in \mathbf{R}_+)$ as defined by (2.4). Then for $p>1$, and $||a||_{p,\Phi_\lambda}, ||b||_{q,\Psi_\lambda} \in \mathbf{R}_+$, we have the following equivalent inequalities:

$$I := \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)a_m b_n < k(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\Psi_\lambda}, \tag{3.1}$$

$$J := \left\{\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)a_m\right]^p\right\}^{\frac{1}{p}} < k(\lambda_1)||a||_{p,\Phi_\lambda}. \tag{3.2}$$

Proof . By Hölder's weighted inequality (cf. [30]) and (2.6), we have

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)a_m\right]^p = \left[\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)\left(\frac{U_m^{\frac{1-\lambda_1}{q}}a_m}{V_n^{\frac{1-\lambda_2}{p}}\mu_m^{\frac{1}{q}}}\right)\left(\frac{V_n^{\frac{1-\lambda_2}{p}}\mu_m^{\frac{1}{q}}}{U_m^{\frac{1-\lambda_1}{q}}}\right)\right]^p \\
\leq\ & \sum_{m=1}^{\infty} k_\lambda(U_m,V_n)\left(\frac{U_m^{(1-\lambda_1)p/q}}{V_n^{1-\lambda_2}\mu_m^{p/q}}a_m^p\right)\times\left[\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)\frac{V_n^{(1-\lambda_2)(q-1)}\mu_m}{U_m^{1-\lambda_1}}\right]^{p-1} \\
=\ & \frac{V_n^{1-p\lambda_2}}{(\varpi(\lambda_1,n))^{1-p}\upsilon_n}\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)\frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}}a_m^p.
\end{aligned} \tag{3.3}$$

In view of (2.8) and (2.5), we find

$$\begin{aligned}
J \ &\leq\ (k(\lambda_1))^{\frac{1}{q}}\left[\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)\frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}}a_m^p\right]^{\frac{1}{p}} \\
&=\ (k(\lambda_1))^{\frac{1}{q}}\left[\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(U_m,V_n)\frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}}a_m^p\right]^{\frac{1}{p}} \\
&=\ (k(\lambda_1))^{\frac{1}{q}}\left[\sum_{m=1}^{\infty}\omega(\lambda_2,m)\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}a_m^p\right]^{\frac{1}{p}}.
\end{aligned} \tag{3.4}$$

Then by (2.7), we derive (3.2).

By Hölder's inequality (cf. [30]), we get

$$I = \sum_{n=1}^{\infty}\left[\frac{\upsilon_n^{\frac{1}{p}}}{V_n^{\frac{1}{p}-\lambda_2}}\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)a_m\right]\left(\frac{V_n^{\frac{1}{p}-\lambda_2}}{\upsilon_n^{\frac{1}{p}}}b_n\right) \leq J||b||_{q,\Psi_\lambda}. \tag{3.5}$$

Then by (3.2), we derive (3.1).

On the other hand, assuming that (3.1) is valid, we set

$$b_n := \frac{\upsilon_n}{V_n^{1-p\lambda_2}}\left[\sum_{m=1}^{\infty} k_\lambda(U_m, V_n)a_m\right]^{p-1}, n \in \mathbf{N}.$$

Then we obtain $J^p = ||b||_{q,\Psi_\lambda}^q$. If $J = 0$, then (3.2) is trivially valid; if $J = \infty$, then by (3.4) and (2.7), it is impossible. Suppose that $0 < J < \infty$. By (3.1), it follows that

$$\begin{aligned} ||b||_{q,\Psi_\lambda}^q &= J^p = I < k(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\Psi_\lambda}, \\ ||b||_{q,\Psi_\lambda}^{q-1} &= J < k(\lambda_1)||a||_{p,\Phi_\lambda}, \end{aligned}$$

and then (3.2) follows, which is equivalent to (3.1). □

Theorem 3.2. *Adopting the assumptions of Theorem 3.1, if there exist constants* $\tau_1, \tau_2 \in \mathbf{R}$ *and* $\delta_0, L > 0$, *such that*

$$\tau_1 < \lambda_1 < \tau_2, k(\widetilde{\lambda}_1) \in \mathbf{R}_+ \ (\widetilde{\lambda}_1 \in (\lambda_1 - \delta_0, \lambda_1 + \delta_0),$$
$$k_\lambda(u, 1) \le \frac{L}{u^{\tau_1}}(u \in (0, 1)), k_\lambda(u, 1) \le \frac{L}{u^{\tau_2}}(u \in [1, \infty)),$$

and if there exist $m_0, n_0 \in \mathbf{N}$, *such that*

$$\mu_m \ge \mu_{m+1}(m \in \{m_0, m_0 + 1, \ldots\}),\ \upsilon_n \ge \upsilon_{n+1}\ (n \in \{n_0, n_0 + 1, \ldots\}),$$

and $U(\infty) = V(\infty) = \infty$, *then the constant factor* $k(\lambda_1)$ *in* (3.1) *and* (3.2) *is the best possible.*

Proof . For $\varepsilon \in (0, p\delta_0)$, we set

$$\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p}, \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}, \quad \text{and} \quad \widetilde{a} = \{\widetilde{a}_m\}_{m=1}^\infty, \widetilde{b} = \{\widetilde{b}_n\}_{n=1}^\infty,$$

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1 - 1}\mu_m = U_m^{\lambda_1 - \frac{\varepsilon}{p} - 1}\mu_m, \widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - \varepsilon - 1}\upsilon_n = V_n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\upsilon_n. \tag{3.6}$$

Then

$$k_\lambda(x, y)\frac{1}{x^{1-\widetilde{\lambda}_1}} \ (= k_\lambda(x, y)\frac{1}{x^{1-\lambda_1}}\frac{1}{x^{\varepsilon/p}})$$

remains strictly decreasing with respect to $x > 0$ and by (2.11), we have

$$k(\widetilde{\lambda}_1)(1 - \vartheta(\widetilde{\lambda}_1, n)) < \varpi(\widetilde{\lambda}_1, n), \vartheta(\widetilde{\lambda}_1, n) = O(\frac{1}{V_n^{\widetilde{\beta}}}) \in (0, 1), \tag{3.7}$$

where, $\lim_{\varepsilon \to 0^+} \widetilde{\beta} = \lim_{\varepsilon \to 0^+}(\widetilde{\lambda}_1 - \tau_1) = \lambda_1 - \tau_1 > 0$.

By (2.12) and (2.13), we find

$$\begin{aligned} ||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\Psi_\lambda} &= \left(\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\ &= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon O_2(1)\right)^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned}
\widetilde{I} \ : \ &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)\widetilde{a}_m\widetilde{b}_n \\
&= \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} k_\lambda(U_m,V_n)\frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}}\right)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \sum_{n=1}^{\infty}\varpi(\widetilde{\lambda}_1,n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \geq k(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}(1-\vartheta(\widetilde{\lambda}_1,n))\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= k(\widetilde{\lambda}_1)\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{\varepsilon+1}} - \sum_{n=1}^{\infty}O(\frac{\upsilon_n}{V_n^{(\frac{\varepsilon}{q}+\widetilde{\beta})+1}})\right) \\
&= \frac{1}{\varepsilon}k(\widetilde{\lambda}_1)\left[\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon(O_2(1)-\widetilde{O}(1))\right].
\end{aligned}$$

If there exists a positive constant $K \leq k(\lambda_1)$, such that (3.1) is satisfied when replacing $k(\lambda_1)$ by K, then in particular we have $\varepsilon\widetilde{I} < \varepsilon K||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\Psi_\lambda}$, namely

$$k(\lambda_1-\frac{\varepsilon}{p})\left[\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon(O_2(1)-\widetilde{O}(1))\right] < K\left(\frac{1}{U_{m_0}^{\varepsilon}}+\varepsilon O_1(1)\right)^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}}+\varepsilon O_2(1)\right)^{\frac{1}{q}}.$$

In view of Lemma 2.1, it follows that $k(\lambda_1) \leq K(\varepsilon \to 0^+)$. Hence, $K = k(\lambda_1)$ is the best possible constant factor of (3.1).

The constant factor $k(\lambda_1)$ in (3.2) is still the best possible. Otherwise, by (3.5) we would reach the contradiction that the constant factor in (3.1) is not the best possible. □

Remark 3.3. For $\lambda = 1, \lambda_1 = \frac{1}{q}, \lambda_2 = \frac{1}{p}$, (3.1) reduces to (1.4); for $k_\lambda(x,y) = \frac{1}{(x+y)^\lambda}$, (3.1) reduces to (1.8); for $\mu_i = \upsilon_j = 1 (i,j \in \mathbf{N})$, (3.1) reduces to (1.7). Hence, the inequality (3.1) is an extension of (1.4), (1.8) and (1.7) with the best possible constant factor $k(\lambda_1)$.

4. Operator expressions and examples

For $p > 1$, we find $\Psi_\lambda^{1-p}(n) = \frac{\upsilon_n}{V_n^{1-p\lambda_2}}$ and define the following normed spaces:

$$\begin{aligned}
l_{p,\Phi_\lambda} \ : \ &= \{a = \{a_m\}_{m=1}^{\infty}; ||a||_{p,\Phi_\lambda} < \infty\}, \\
l_{q,\Psi_\lambda} \ : \ &= \{b = \{b_n\}_{n=1}^{\infty}; ||b||_{q,\Psi_\lambda} < \infty\}, \\
l_{p,\Psi_\lambda^{1-p}} \ : \ &= \{c = \{c_n\}_{n=1}^{\infty}; ||c||_{p,\Psi_\lambda^{1-p}} < \infty\}.
\end{aligned}$$

Assuming that $a = \{a_m\}_{m=1}^{\infty} \in l_{p,\Phi_\lambda}$, and setting

$$c = \{c_n\}_{n=1}^{\infty}, c_n := \sum_{m=1}^{\infty} k_\lambda(U_m,V_n)a_m, n \in \mathbf{N},$$

we can rewrite (3.2) as follows:

$$||c||_{p,\Psi_\lambda^{1-p}} < k(\lambda_1)||a||_{p,\Phi_\lambda} < \infty.$$

Namely, $c \in l_{p,\Psi_\lambda^{1-p}}$.

Definition 4.1. Define a Hardy-Hilbert-type operator $T : l_{p,\Phi_\lambda} \to l_{p,\Psi_\lambda^{1-p}}$ as follows: For any $a = \{a_m\}_{m=1}^\infty \in l_{p,\Phi_\lambda}$, there exists a unique representation $Ta = c \in l_{p,\Psi_\lambda^{1-p}}$. Additionally, we define the formal inner product of Ta and $b = \{b_n\}_{n=1}^\infty (\in l_{q,\Psi_\lambda})$ as follows:

$$(Ta, b) := \sum_{n=1}^\infty \left(\sum_{m=1}^\infty k_\lambda(U_m, V_n) a_m \right) b_n. \tag{4.1}$$

Then, we can rewrite (3.1) and (3.2) in the following manner:

$$(Ta, b) \; < \; k(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\Psi_\lambda}, \tag{4.2}$$

$$||Ta||_{p,\Psi_\lambda^{1-p}} \; < \; k(\lambda_1)||a||_{p,\Phi_\lambda}. \tag{4.3}$$

We define the norm of the operator T as follows:

$$||T|| := \sup_{a(\neq\theta)\in l_{p,\Phi_\lambda}} \frac{||Ta||_{p,\Psi_\lambda^{1-p}}}{||a||_{p,\Phi_\lambda}}. \tag{4.4}$$

Then by (4.3), we obtain that $||T|| \leq k(\lambda_1)$. Since by Theorem 3.2, the constant factor in (4.3) is the best possible, we have

$$||T|| = k(\lambda_1). \tag{4.5}$$

Example 4.2. For $s \in \mathbf{N}$, $0 < c_1 \leq \cdots \leq c_s < \infty$, $\lambda_1, \lambda_2 > -\alpha$, $\lambda_1 + \lambda_2 = \lambda$, we set

$$k_\lambda(x, y) := \prod_{k=1}^s \frac{(\min\{x, c_k y\})^{\frac{\alpha}{s}}}{(\max\{x, c_k y\})^{\frac{\lambda+\alpha}{s}}} ((x, y) \in \mathbf{R}_+^2 = \mathbf{R}_+ \times \mathbf{R}_+).$$

(a) We find

$$\begin{aligned}
k_s(\lambda_1) \;:\; &= \int_0^\infty k_\lambda(1, u) t^{\lambda_2 - 1} du = \int_0^\infty k_\lambda(t, 1) t^{\lambda_1 - 1} dt \\
&= \int_0^\infty \prod_{k=1}^s \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1 - 1} dt \\
&= \int_0^{c_1} \prod_{k=1}^s \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}} t^{\lambda_1 - 1}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} dt + \int_{c_s}^\infty \prod_{k=1}^s \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}} t^{\lambda_1 - 1}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} dt \\
&\quad + \sum_{i=1}^{s-1} \int_{c_i}^{c_{i+1}} \prod_{k=1}^s \frac{(\min\{t, c_k\})^{\frac{\alpha}{s}} t^{\lambda_1 - 1}}{(\max\{t, c_k\})^{\frac{\lambda+\alpha}{s}}} dt \\
&= \prod_{k=1}^s \frac{1}{c_k^{(\lambda+\alpha)/s}} \int_0^{c_1} t^{\lambda_1 + \alpha - 1} dt + \prod_{k=1}^s c_k^{\alpha/s} \int_{c_s}^\infty t^{-\lambda_2 - \alpha - 1} dt \\
&\quad + \sum_{i=1}^{s-1} \int_{c_i}^{c_{i+1}} \prod_{k=1}^i \frac{c_k^{\frac{\alpha}{s}}}{t^{\frac{\lambda+\alpha}{s}}} \prod_{k=i+1}^s \frac{t^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda+\alpha}{s}}} t^{\lambda_1 - 1} dt \\
&= \frac{c_1^{\lambda_1 + \alpha}}{\lambda_1 + \alpha} \frac{1}{\prod_{k=1}^s c_k^{\frac{\lambda+\alpha}{s}}} + \frac{1}{(\lambda_2 + \alpha) c_s^{\lambda_2 + \alpha}} \prod_{k=1}^s c_k^{\frac{\alpha}{s}} \\
&\quad + \sum_{i=1}^{s-1} \frac{\prod_{k=1}^i c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^s c_k^{\frac{\lambda+\alpha}{s}}} \int_{c_i}^{c_{i+1}} t^{\lambda_1 - \frac{i\lambda}{s} + (1 - \frac{2i}{s})\alpha - 1} dt.
\end{aligned}$$

If $\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha \neq 0$, then

$$\int_{c_i}^{c_{i+1}} t^{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha - 1} dt = \frac{c_{i+1}^{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha} - c_i^{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha}}{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha};$$

if there exists a $i_0 \in \{1, \cdots, s-1\}$, such that $\lambda_1 - \frac{i_0\lambda}{s} + (1-\frac{2i_0}{s})\alpha = 0$, then we find

$$\int_{c_{i_0}}^{c_{i_0+1}} t^{\lambda_1 - \frac{i_0\lambda}{s} + (1-\frac{2i_0}{s})\alpha - 1} dt = \ln(\frac{c_{i_0+1}}{c_{i_0}}) = \lim_{i \to i_0} \int_{c_i}^{c_{i+1}} t^{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha - 1} dt,$$

and we still indicate $\ln(\frac{c_{i_0+1}}{c_{i_0}})$ by the following formal expression:

$$\frac{c_{i_0+1}^{\lambda_1 - \frac{i_0\lambda}{s} + (1-\frac{2i_0}{s})\alpha} - c_{i_0}^{\lambda_1 - \frac{i_0\lambda}{s} + (1-\frac{2i_0}{s})\alpha}}{\lambda_1 - \frac{i_0\lambda}{s} + (1-\frac{2i_0}{s})\alpha}.$$

Hence, we may reset

$$\begin{aligned} k_s(\lambda_1) &= \frac{c_1^{\lambda_1+\alpha}}{\lambda_1+\alpha} \frac{1}{\prod_{k=1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} + \frac{1}{(\lambda_2+\alpha)c_s^{\lambda_2+\alpha}} \prod_{k=1}^{s} c_k^{\frac{\alpha}{s}} \\ &+ \sum_{i=1}^{s-1} \left[\frac{c_{i+1}^{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha} - c_i^{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha}}{\lambda_1 - \frac{i\lambda}{s} + (1-\frac{2i}{s})\alpha} \frac{\prod_{k=1}^{i} c_k^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s} c_k^{\frac{\lambda+\alpha}{s}}} \right] \in \mathbf{R}_+. \end{aligned} \tag{4.6}$$

In particular,
(i) for $s = 1$ (or $c_s = \cdots = c_1$), we have

$$k_\lambda(x,y) = \frac{(\min\{x, c_1 y\})^\alpha}{(\max\{x, c_1 y\})^{\lambda+\alpha}}$$

and

$$k_1(\lambda_1) = \frac{\lambda + 2\alpha}{(\lambda_1+\alpha)(\lambda_2+\alpha)} \frac{1}{c_1^{\lambda_2}}; \tag{4.7}$$

(ii) for $s = 2$, we have

$$k_\lambda(x,y) = \frac{(\min\{x, c_1 y\}\min\{x, c_2 y\})^{\alpha/2}}{(\max\{x, c_1 y\}\max\{x, c_2 y\})^{(\lambda+\alpha)/2}}$$

and

$$k_2(\lambda_1) = \left(\frac{c_1}{c_2}\right)^{\frac{\alpha}{2}} \left[\frac{c_1^{\lambda_1 - \frac{\lambda}{2}}}{(\lambda_1+\alpha)c_2^{\lambda/2}} + \frac{1}{(\lambda_2+\alpha)c_2^{\lambda_2}} + \frac{c_2^{\lambda_1-\frac{\lambda}{2}} - c_1^{\lambda_1-\frac{\lambda}{2}}}{(\lambda_1 - \frac{\lambda}{2})c_2^{\lambda/2}} \right]; \tag{4.8}$$

(iii) for $\alpha = 0$, we have

$$\lambda_1, \lambda_2 > 0, \quad k_\lambda(x,y) = \frac{1}{\prod_{k=1}^{s}(\max\{x, c_k y\})^{\lambda/s}}$$

and

$$k_s(\lambda_1) = \widetilde{k}_s(\lambda_1) := \frac{c_1^{\lambda_1}}{\lambda_1} \frac{1}{\prod_{k=1}^{s} c_k^{\frac{\lambda}{s}}} + \frac{1}{\lambda_2 c_s^{\lambda_2}} + \sum_{i=1}^{s-1} \frac{c_{i+1}^{\lambda_1 - \frac{i}{s}\lambda} - c_i^{\lambda_1 - \frac{i}{s}\lambda}}{\lambda_1 - \frac{i}{s}\lambda} \frac{1}{\prod_{k=i+1}^{s} c_k^{\lambda/s}}; \tag{4.9}$$

(iv) for $\alpha = -\lambda$, we have

$$\lambda < \lambda_1, \lambda_2 < 0, \quad k_\lambda(x,y) = \frac{1}{\prod_{k=1}^{s}(\min\{x, c_k y\})^{\lambda/s}}$$

and

$$\begin{aligned} k_s(\lambda_1) &= \widehat{k}_s(\lambda_1) := \frac{c_1^{-\lambda_2}}{(-\lambda_2)} + \frac{1}{(-\lambda_1)c_s^{-\lambda_1}} \prod_{k=1}^{s} c_k^{-\lambda/s} \\ &+ \sum_{i=1}^{s-1} \left(\frac{c_{i+1}^{\lambda_1 - \frac{s-i}{s}\lambda} - c_i^{\lambda_1 - \frac{s-i}{s}\lambda}}{\lambda_1 - \frac{s-i}{s}\lambda} \prod_{k=1}^{i} c_k^{-\lambda/s} \right); \end{aligned} \tag{4.10}$$

(v) for $\lambda = 0$, we have $\lambda_2 = -\lambda_1, |\lambda_1| < \alpha(\alpha > 0)$,

$$k_0(x,y) = \prod_{k=1}^{s} \left(\frac{\min\{x, c_k y\}}{\max\{x, c_k y\}} \right)^{\alpha/s},$$

and

$$\begin{aligned} k_s(\lambda_1) &= k_s^{(0)}(\lambda_1) := \frac{c_1^{\lambda_1+\alpha}}{a+\lambda_1} \frac{1}{\prod_{k=1}^{s} c_k^{\alpha/\widetilde{s}}} + \frac{c_s^{\lambda_1-\alpha}}{a-\lambda_1} \prod_{k=1}^{s} c_k^{\alpha/s} \\ &+ \sum_{i=1}^{s-1} \left[\frac{c_{i+1}^{\lambda_1+(1-\frac{2i}{s})\alpha} - c_i^{\lambda_1+(1-\frac{2i}{s})\alpha}}{\lambda_1 + (1-\frac{2i}{s})\alpha} \frac{\prod_{k=1}^{i} c_k^{\alpha/s}}{\prod_{k=i+1}^{s} c_k^{\alpha/s}} \right]. \end{aligned} \tag{4.11}$$

We set $\delta_0 = \min\{\lambda_1 + \alpha, \lambda_2 + \alpha\}$. For $\widetilde{\lambda}_1 \in (\lambda_1 - \delta_0, \lambda_1 + \delta_0), \widetilde{\lambda}_1 + \widetilde{\lambda}_2 = \lambda$, we obtain that

$$\widetilde{\lambda}_1 + \alpha > \lambda_1 - \delta_0 + \alpha \geq 0, \; \widetilde{\lambda}_2 + \alpha = \lambda - \widetilde{\lambda}_1 + \alpha > \lambda - (\lambda_1 + \delta_0) + \alpha = \lambda_2 + \alpha - \delta_0 \geq 0,$$

and $k_s(\widetilde{\lambda}_1) \in \mathbf{R}_+$.

(b) Since we find

$$\begin{aligned} k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}} &= \frac{1}{y^{1-\lambda_2}} \prod_{k=1}^{s} \frac{(\min\{c_k^{-1}x, y\})^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda}{s}} (\max\{c_k^{-1}x, y\})^{\frac{\lambda+\alpha}{s}}} \\ &= \begin{cases} \frac{1}{y^{1-\lambda_2-\alpha}} \prod_{k=1}^{s} \frac{1}{c_k^{\frac{\lambda}{s}} (c_k^{-1}x)^{\frac{\lambda+\alpha}{s}}}, 0 < y \leq c_s^{-1}x, \\ \frac{1}{y^{1+\lambda_1+\alpha-\frac{i}{s}(\lambda+2\alpha)}} \frac{\prod_{k=i+1}^{s}(c_k^{-1}x)^{\frac{\alpha}{s}}}{\prod_{k=1}^{s} c_k^{\frac{\lambda}{s}} \prod_{k=1}^{i}(c_k^{-1}x)^{\frac{\lambda+\alpha}{s}}}, c_{i+1}^{-1}x < y \leq c_i^{-1}x \\ (i = 1, \cdots, s-1), \\ \frac{1}{y^{1+\lambda_1+\alpha}} \prod_{k=1}^{s} \frac{(c_k^{-1}x)^{\frac{\alpha}{s}}}{c_k^{\frac{\lambda}{s}} (y)^{\frac{\lambda+\alpha}{s}}}, c_1^{-1}x < y < \infty, \end{cases} \end{aligned}$$

then for $\lambda_2 < 1 - \alpha$ ($\lambda_1 > -\alpha$), we get that $k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}$ is strict decreasing with respect to $y > 0$. Similarly, since

$$k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}} = \frac{1}{x^{1-\lambda_1}} \prod_{k=1}^{s} \frac{(\min\{x, c_k y\})^{\frac{\alpha}{s}}}{(\max\{x, c_k y\})^{\frac{\lambda+\alpha}{s}}}$$

$$= \begin{cases} \frac{1}{x^{1-\lambda_1-\alpha}} \prod_{k=1}^{s} \frac{1}{(c_k y)^{\frac{\lambda+\alpha}{s}}}, 0 < x \le c_1 y, \\ \frac{1}{x^{1-\lambda_1-\alpha+\frac{i}{s}(\lambda+2\alpha)}} \frac{\prod_{k=1}^{i}(c_k y)^{\frac{\alpha}{s}}}{\prod_{k=i+1}^{s}(c_k y)^{\frac{\lambda+\alpha}{s}}}, c_i y < x \le c_{i+1} y \\ (i = 1, \cdots, s-1), \\ \frac{1}{x^{1+\lambda_2+\alpha}} \prod_{k=1}^{s}(c_k y)^{\frac{\alpha}{s}}, c_s y < x < \infty, \end{cases}$$

then for $\lambda_1 < 1-\alpha$ ($\lambda_2 > -\alpha$), we have that $k_\lambda(x,y)\frac{1}{x^{1-\lambda_1}}$ is strict decreasing with respect to $x > 0$.

(c) There exist $\tau_1 \in (-\alpha, \lambda_1)$ and $\tau_2 \in (\lambda_1, \lambda+\alpha)$, such that

$$\lim_{u\to 0^+} u^{\tau_1} k_\lambda(u,1) = \lim_{u\to 0^+} u^{\tau_1} \frac{u^\alpha}{\prod_{k=1}^{s} c_k^{(\lambda+\alpha)/s}} = 0,$$

and

$$\lim_{u\to\infty} u^{\tau_2} k_\lambda(u,1) = \lim_{u\to\infty} \frac{\prod_{k=1}^{s} c_k^{\alpha/s}}{u^{\lambda+\alpha-\tau_2}} = 0.$$

It follows that there exists $L > 0$, such that

$$k_\lambda(u,1) \le \frac{L}{u^{\tau_1}} (u \in (0,1)), k_\lambda(u,1) \le \frac{L}{u^{\tau_2}} (u \in [1,\infty)).$$

In view of (a), (b) and (c), by (4.5), we have

$$||T|| = k_s(\lambda_1). \tag{4.12}$$

Example 4.3. For $-1 < \alpha \le 1, 0 < \lambda_1, \lambda_2 < 1$, $\lambda_1 + \lambda_2 = \lambda$, we set

$$k_\lambda(x,y) := \frac{1}{x^\lambda + y^\lambda + \alpha|x^\lambda - y^\lambda|} \ ((x,y) \in \mathbf{R}_+^2).$$

(a) We find

$$\begin{aligned} 0 &< k_\alpha(\lambda_1) := \int_0^\infty k_\lambda(1,t) t^{\lambda_2-1} dt = \int_0^\infty k_\lambda(t,1) t^{\lambda_1-1} dt \\ &= \int_0^\infty \frac{t^{\lambda_1-1} dt}{t^\lambda + 1 + \alpha|t^\lambda - 1|} = \int_0^1 \frac{t^{\lambda_1-1} + t^{\lambda_2-1}}{1+\alpha+(1-\alpha)t^\lambda} dt \\ &\le \int_0^1 \frac{t^{\lambda_1-1} + t^{\lambda_2-1}}{1+\alpha} dt = \frac{1}{1+\alpha}(\frac{1}{\lambda_1} + \frac{1}{\lambda_2}) < \infty, \end{aligned} \tag{4.13}$$

namely, $k_\alpha(\lambda_1) \in \mathbf{R}_+$.

(i) For $\alpha = 1$, we obtain

$$\begin{aligned} k_1(\lambda_1) &= \int_0^\infty \frac{t^{\lambda_1-1}}{t^\lambda + 1 + |t^\lambda - 1|} dt = \int_0^\infty \frac{t^{\lambda_1-1}}{2\max\{t^\lambda, 1\}} dt \\ &= \frac{1}{2}\int_0^1 (t^{\lambda_1-1} + t^{\lambda_2-1}) dt = \frac{\lambda}{2\lambda_1\lambda_2}; \end{aligned} \tag{4.14}$$

(ii) for

$$0 < \alpha < 1, \ 0 < \frac{1-\alpha}{1+\alpha} < 1,$$

in view of the Lebesgue term by term integration theorem (cf. [31]), we find

$$
\begin{aligned}
k_\alpha(\lambda_1) &= \frac{1}{1+\alpha}\int_0^1 \frac{t^{\lambda_1-1}+t^{\lambda_2-1}}{1+\frac{1-\alpha}{1+\alpha}t^\lambda}dt \\
&= \frac{1}{1+\alpha}\int_0^1 (t^{\lambda_1-1}+t^{\lambda_2-1})\sum_{k=0}^{\infty}(-1)^k\left(\frac{1-\alpha}{1+\alpha}\right)^k t^{\lambda k}dt \\
&= \frac{1}{1+\alpha}\int_0^1 (t^{\lambda_1-1}+t^{\lambda_2-1})\sum_{k=0}^{\infty}\left(\frac{1-\alpha}{1+\alpha}\right)^{2k} t^{2\lambda k}\left(1-\frac{1-\alpha}{1+\alpha}t^\lambda\right)dt \\
&= \frac{1}{1+\alpha}\sum_{k=0}^{\infty}\left(\frac{1-\alpha}{1+\alpha}\right)^{2k}\int_0^1 (t^{\lambda_1-1}+t^{\lambda_2-1})t^{2\lambda k}\left(1-\frac{1-\alpha}{1+\alpha}t^\lambda\right)dt \\
&= \frac{1}{1+\alpha}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1-\alpha}{1+\alpha}\right)^k\int_0^1 (t^{\lambda_1-1}+t^{\lambda_2-1})t^{\lambda k}dt \\
&= \frac{1}{1+\alpha}\sum_{k=0}^{\infty}(-1)^k\left(\frac{1-\alpha}{1+\alpha}\right)^k\left(\frac{1}{\lambda k+\lambda_1}+\frac{1}{\lambda k+\lambda_2}\right); \qquad (4.15)
\end{aligned}
$$

(iii) for $\alpha = 0$, we obtain

$$k_0(\lambda_1) = \int_0^\infty \frac{t^{\lambda_1-1}dt}{t^\lambda+1} = \frac{1}{\lambda}\int_0^\infty \frac{v^{(\lambda_1/\lambda)-1}}{v+1}dv = \frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}; \qquad (4.16)$$

(iv) for

$$-1<\alpha<0,\ 0<\frac{1+\alpha}{1-\alpha}<1,$$

by the Lebesgue term by term integration theorem (cf. [31]), we find

$$
\begin{aligned}
k_\alpha(\lambda_1) &= \frac{1}{1+\alpha}\int_0^1 \frac{t^{\lambda_1-1}+t^{\lambda_2-1}}{1+\frac{1-a}{1+a}t^\lambda}dt \\
&\overset{v=\frac{1+\alpha}{1-\alpha}t^{-\lambda}}{=} \frac{1}{\lambda(1+\alpha)}\int_{\frac{1+\alpha}{1-\alpha}}^\infty \frac{1}{v+1}\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}v^{\frac{-\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}v^{\frac{-\lambda_2}{\lambda}}\right]dv \\
&= \frac{1}{\lambda(1+\alpha)}(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}\int_0^\infty\frac{1}{v+1}v^{(1-\frac{\lambda_1}{\lambda})-1}dv+\frac{1}{\lambda(1+\alpha)}(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}\int_0^\infty\frac{1}{v+1}v^{(1-\frac{\lambda_2}{\lambda})-1}dv \\
&\quad -\frac{1}{\lambda(1+\alpha)}\int_0^{\frac{1+\alpha}{1-\alpha}}\frac{1}{v+1}\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}v^{\frac{-\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}v^{\frac{-\lambda_2}{\lambda}}\right]dv \\
&= \frac{1}{\lambda(1+\alpha)}\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}\frac{\pi}{\sin(\frac{\pi\lambda_2}{\lambda})}\right] \\
&\quad -\frac{1}{\lambda(1+\alpha)}\int_0^{\frac{1+\alpha}{1-\alpha}}\sum_{k=0}^{\infty}(-1)^k v^k \\
&\quad \times\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}v^{\frac{-\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}v^{\frac{-\lambda_2}{\lambda}}\right]dv
\end{aligned}
$$

$$= \frac{1}{\lambda(1+\alpha)}\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}$$
$$-\frac{1}{\lambda(1+\alpha)}\int_0^{\frac{1+\alpha}{1-\alpha}}\sum_{k=0}^{\infty}(-1)^k v^k\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}v^{\frac{-\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}v^{\frac{-\lambda_2}{\lambda}}\right]dv$$

$$= \frac{1}{\lambda(1+\alpha)}\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}$$
$$-\frac{1}{\lambda(1+\alpha)}\int_0^{\frac{1+\alpha}{1-\alpha}}\sum_{k=0}^{\infty}(v^{2k}-v^{2k+1})$$
$$\times\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}v^{\frac{-\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}v^{\frac{-\lambda_2}{\lambda}}\right]dv$$
$$= \frac{1}{\lambda(1+a)}\left[(\frac{1+a}{1-a})^{\frac{\lambda_1}{\lambda}}+(\frac{1+a}{1-a})^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}$$
$$-\frac{1}{\lambda(1+a)}\sum_{k=0}^{\infty}\int_0^{\frac{1+a}{1-a}}(v^{2k}-v^{2k+1})$$
$$\times\left[(\frac{1+a}{1-a})^{\frac{\lambda_1}{\lambda}}v^{\frac{-\lambda_1}{\lambda}}+(\frac{1+a}{1-a})^{\frac{\lambda_2}{\lambda}}v^{\frac{-\lambda_2}{\lambda}}\right]dv$$
$$= \frac{1}{\lambda(1+\alpha)}\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\sin(\frac{\pi\lambda_1}{\lambda})}$$
$$-\frac{1}{\lambda(1+\alpha)}\sum_{k=0}^{\infty}\int_0^{\frac{1+\alpha}{1-\alpha}}(-1)^k v^k\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}v^{\frac{-\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}v^{\frac{-\lambda_2}{\lambda}}\right]dv$$
$$= \frac{1}{1+\alpha}\left[(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_1}{\lambda}}+(\frac{1+\alpha}{1-\alpha})^{\frac{\lambda_2}{\lambda}}\right]\frac{\pi}{\lambda\sin(\frac{\pi\lambda_1}{\lambda})}$$
$$-\frac{1}{1+\alpha}\sum_{k=0}^{\infty}(-1)^k(\frac{1+\alpha}{1-\alpha})^{k+1}\left(\frac{1}{\lambda k+\lambda_2}+\frac{1}{\lambda k+\lambda_1}\right); \tag{4.17}$$

(v) for $\lambda_1=\lambda_2=\frac{\lambda}{2}, 0<\lambda<2, -1<\alpha<1$, we find

$$k_\alpha(\frac{\lambda}{2}) = 2\int_0^1\frac{t^{(\lambda/2)-1}}{1+\alpha+(1-\alpha)t^\lambda}dt$$
$$\overset{u=(\frac{1-\alpha}{1+\alpha}t^\lambda)^{\frac{1}{2}}}{=} \frac{4}{\lambda(1+\alpha)}(\frac{1+\alpha}{1-\alpha})^{\frac{1}{2}}\int_0^{(\frac{1-\alpha}{1+\alpha})^{\frac{1}{2}}}\frac{1}{1+u^2}du$$
$$= \frac{4}{\lambda(1+\alpha)}(\frac{1+\alpha}{1-\alpha})^{\frac{1}{2}}\arctan(\frac{1-\alpha}{1+\alpha})^{\frac{1}{2}}. \tag{4.18}$$

We set $\delta_0=\min\{1-\lambda_1,1-\lambda_2\}$. For $\widetilde{\lambda}_1\in(\lambda_1-\delta_0,\lambda_1+\delta_0),\widetilde{\lambda}_1+\widetilde{\lambda}_2=\lambda$, we find $\widetilde{\lambda}_1-1<\lambda_1+\delta_0-1\le 0,\widetilde{\lambda}_2-1=\lambda-\widetilde{\lambda}_1-1<\lambda-(\lambda_1-\delta_0)-1=\lambda_2-1+\delta_0\le 0$ and then $k_\alpha(\widetilde{\lambda}_1)\in\mathbf{R}_+$.

(b) For fixed $x>0$, in view of $-1<\alpha\le 1, 0<\lambda_2<1$, we obtain that

$$k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}=\begin{cases}\frac{1}{(1+\alpha)x^\lambda+(1-\alpha)y^\lambda}\frac{1}{y^{1-\lambda_2}}, 0<y<x\\ \frac{1}{(1-\alpha)x^\lambda+(1+\alpha)y^\lambda}\frac{1}{y^{1-\lambda_2}}, y\ge x\end{cases},$$

is strictly decreasing with respect to $y > 0$. Similarly, for fixed $y > 0$, $k_\lambda(x, y)\frac{1}{x^{1-\lambda_1}}$ is strict decreasing with respect to $x > 0$.

(c) There exist $\tau_1 \in (0, \lambda_1)$ and $\tau_2 \in (\lambda_1, \lambda)$, such that

$$\lim_{u\to 0^+} u^{\tau_1} k_\lambda(u, 1) = \lim_{u\to 0^+} \frac{u^{\tau_1}}{u^\lambda + 1 + \alpha|u^\lambda - 1|} = 0,$$

and

$$\lim_{u\to\infty} u^{\tau_2} k_\lambda(u, 1) = \lim_{u\to\infty} \frac{u^{\tau_2}}{u^\lambda + 1 + \alpha|u^\lambda - 1|} = 0.$$

It follows that there exists $L > 0$, such that

$$k_\lambda(u, 1) \le \frac{L}{u^{\tau_1}} \ (u \in (0, 1)), k_\lambda(u, 1) \le \frac{L}{u^{\tau_2}} \ (u \in [1, \infty)).$$

In view of (a), (b) and (c), by (4.5), we have

$$||T|| = k_\alpha(\lambda_1) = \int_0^1 \frac{t^{\lambda_1-1} + t^{\lambda_2-1}}{1 + \alpha + (1-\alpha)t^\lambda} dt. \tag{4.19}$$

5. Some reverses

In the following, we also set

$$\begin{aligned} \widetilde{\Phi}_\lambda(m) \ &: = (1 - \theta(\lambda_2, m))\frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}}, \\ \widetilde{\Psi}_\lambda(n) \ &: = (1 - \vartheta(\lambda_1, n))\frac{V_n^{q(1-\lambda_2)-1}}{\upsilon_n^{q-1}} (m, n \in \mathbf{N}). \end{aligned}$$

For $0 < p < 1$ or $p < 0$, we still use the formal symbols of $||a||_{p,\Phi_\lambda}$, $||b||_{q,\Psi_\lambda}$, $||a||_{p,\widetilde{\Phi}_\lambda}$ and $||b||_{q,\widetilde{\Psi}_\lambda}$.

Theorem 5.1. *Adopting the assumptions of Theorem 3.2, for $0 < p < 1$, $||a||_{p,\Phi_\lambda}, ||b||_{q,\Psi_\lambda} \in \mathbf{R}_+$, we have the following equivalent inequalities with the best possible constant factor $k(\lambda_1)$:*

$$I = \sum_{n=1}^\infty \sum_{m=1}^\infty k_\lambda(U_m, V_n) a_m b_n > k(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda} ||b||_{q,\Psi_\lambda}, \tag{5.1}$$

$$J = \left\{ \sum_{n=1}^\infty \frac{\upsilon_n}{V_n^{1-p\lambda_2}} \left[\sum_{m=1}^\infty k_\lambda(U_m, V_n) a_m \right]^p \right\}^{\frac{1}{p}} > k(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda}. \tag{5.2}$$

Proof . By the reverse Hölder inequality (cf. [30]) and (2.6), we obtain the reverses of (3.3). By (2.8) and (2.5), we derive the reverse of (3.4). Then by (2.10), we have (5.2). By the reverse Hölder inequality, we get the reverse of (3.5). Then by (5.2), we obtain (5.1).

On the other hand, assuming that (5.1) is valid, we set b_n as in Theorem **??**. Then we find $J^p = ||b||_{q,\Psi_\lambda}^q$. If $J = \infty$, then (5.2) is trivially valid; if $J = 0$, then by the reverse of (3.4) and (2.10), it is impossible. Suppose that $0 < J < \infty$. By (5.1), it follows that

$$\begin{aligned} ||b||_{q,\Psi_\lambda}^q \ &= \ J^p = I > k(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda} ||b||_{q,\Psi_\lambda}, \\ ||b||_{q,\Psi_\lambda}^{q-1} \ &= \ J > k(\lambda_1)||a||_{p,\widetilde{\Phi}_\lambda}, \end{aligned}$$

and then (5.2) follows, which is equivalent to (5.1).

For $\varepsilon \in (0, p\delta_0)$, we set $\widetilde{\lambda}_1, \widetilde{\lambda}_2, \widetilde{a}_m$ and $\widetilde{b}_n$ as (3.6). Then it follows that

$$\varpi(\widetilde{\lambda}_1, n) < k(\widetilde{\lambda}_1)(n \in \mathbf{N}).$$

By (2.12), (2.13) and (3.7), we find

$$\begin{aligned}
||a||_{p,\widetilde{\Phi}_\lambda}||b||_{q,\Psi_\lambda} &= \left[\sum_{m=1}^{\infty}(1-\theta(\lambda_2, m))\frac{\mu_m}{U_m^{1+\varepsilon}}\right]^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \left(\sum_{m=1}^{\infty}\frac{\mu_m}{U_m^{1+\varepsilon}} - \sum_{m=1}^{\infty}O(\frac{\mu_m}{U_m^{1+\alpha+\varepsilon}})\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right)^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon}\left[\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon(O_1(1) - O_3(1))\right]^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon O_2(1)\right)^{\frac{1}{q}},
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &= \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda(U_m, V_n)\widetilde{a}_m\widetilde{b}_n = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty}k_\lambda(U_m, V_n)\frac{V_n^{\widetilde{\lambda}_2}\mu_m}{U_m^{1-\widetilde{\lambda}_1}}\right)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \\
&= \sum_{n=1}^{\infty}\varpi(\widetilde{\lambda}_1, n)\frac{\upsilon_n}{V_n^{\varepsilon+1}} \le k(\widetilde{\lambda}_1)\sum_{n=1}^{\infty}\frac{\upsilon_n}{V_n^{\varepsilon+1}} = \frac{1}{\varepsilon}k(\widetilde{\lambda}_1)\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon O_2(1)\right).
\end{aligned}$$

If there exists a constant $K \ge k(\lambda_1)$, such that (5.1) is valid when replacing $k(\lambda_1)$ by K, then in particular we have $\varepsilon\widetilde{I} > \varepsilon K||\widetilde{a}||_{p,\widetilde{\Phi}_\lambda}||\widetilde{b}||_{q,\Psi_\lambda}$, namely

$$k(\lambda_1 - \frac{\varepsilon}{p})\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon O_2(1)\right) > K\left[\frac{1}{U_{m_0}^{\varepsilon}} + \varepsilon(O_1(1) - O_3(1))\right]^{\frac{1}{p}}\left(\frac{1}{V_{n_0}^{\varepsilon}} + \varepsilon O_2(1)\right)^{\frac{1}{q}}.$$

In view of Lemma 2.1, it follows that $k(\lambda_1) \ge K(\varepsilon \to 0^+)$. Hence, $K = k(\lambda_1)$ is the best possible constant factor of (5.1).

The constant factor $k(\lambda_1)$ in (5.2) is still the best possible. Otherwise, we would reach a contradiction by the reverse of (3.5) that the constant factor in (5.1) is not the best possible. □

Theorem 5.2. *Adopting the assumptions of Theorem 3.2, if $p < 0$, then we have the following equivalent inequalities with the best possible constant factor $k(\lambda_1)$:*

$$I = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty}k_\lambda(U_m, V_n)a_m b_n > k(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\widetilde{\Psi}_\lambda}, \tag{5.3}$$

$$\begin{aligned}
J_1 &:= \left\{\sum_{n=1}^{\infty}\frac{V_n^{p\lambda_2-1}\upsilon_n}{(1-\vartheta(\lambda_1, n))^{p-1}}\left[\sum_{m=1}^{\infty}k_\lambda(U_m, V_n)a_m\right]^p\right\}^{\frac{1}{p}} \\
&> k(\lambda_1)||a||_{p,\Phi_\lambda}. \qquad (5.4)
\end{aligned}$$

Proof . By the reverse weighted Hölder inequality (cf. [30]) and (2.6), since $p < 0$, in view of (2.11), we get

$$\begin{aligned}
&\left[\sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m\right]^p \\
= &\left[\sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m \left(\frac{U_m^{(1-\lambda_1)/q}}{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}} a_m\right)\left(\frac{V_n^{(1-\lambda_2)/p}\mu_m^{1/q}}{U_m^{(1-\lambda_1)/q}}\right)\right]^p \\
\leq &\sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m \frac{U_m^{(1-\lambda_1)p/q}}{V_n^{1-\lambda_2}\mu_m^{p/q}} a_m^p \\
&\times \left[\sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m \frac{V_n^{(1-\lambda_2)(q-1)}\mu_m}{U_m^{1-\lambda_1}}\right]^{p-1} \\
= &\frac{V_n^{1-p\lambda_2}}{(\varpi(\lambda_1, n))^{1-p}} \sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m \frac{U_m^{(1-\lambda_1)(p-1)}}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p \\
\leq &\frac{(k(\lambda_1))^{p-1} V_n^{1-p\lambda_2}}{(1-\vartheta(\lambda_1, n))^{1-p}\upsilon_n} \sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p,
\end{aligned}$$

$$\begin{aligned}
J_1 \geq &(k(\lambda_1))^{\frac{1}{q}} \left\{\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} k_\lambda(U_m, V_n) \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p\right\}^{\frac{1}{p}} \\
= &(k(\lambda_1))^{\frac{1}{q}} \left\{\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} k_\lambda(U_m, V_n) \frac{U_m^{(1-\lambda_1)(p-1)}\upsilon_n}{V_n^{1-\lambda_2}\mu_m^{p-1}} a_m^p\right\}^{\frac{1}{p}} \\
= &(k(\lambda_1))^{\frac{1}{q}} \left\{\sum_{m=1}^{\infty} \omega(\lambda_2, m) \frac{U_m^{p(1-\lambda_1)-1}}{\mu_m^{p-1}} a_m^p\right\}^{\frac{1}{p}}. \qquad (5.5)
\end{aligned}$$

Then by (2.7), we have (5.4).

By the reverse Hölder inequality (cf. [30]), we have

$$I = \sum_{n=1}^{\infty} \frac{V_n^{\lambda_2 - \frac{1}{p}}\upsilon_n^{1/p}}{(1-\vartheta(\lambda_1, n))^{1/q}} \left[\sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m\right] \times \left[(1-\vartheta(\lambda_1, n))^{\frac{1}{q}} \frac{V_n^{\frac{1}{p}-\lambda_2}}{\upsilon_n^{1/p}} b_n\right] \geq J_1 ||b||_{q,\widetilde{\Psi}_\lambda}. \qquad (5.6)$$

Then by (5.4), we have (5.3).

On the other hand, assuming that (5.3) is valid, we set b_n as follows:

$$b_n := \frac{V_n^{p\lambda_2 - 1}\upsilon_n}{(1-\vartheta(\lambda_1, n))^{p-1}} \left[\sum_{m=1}^{\infty} k_\lambda(U_m, V_n) a_m\right]^{p-1}, n \in \mathbf{N}.$$

Then we find $J_1^p = ||b||_{q,\widetilde{\Psi}_\lambda}^q$. If $J_1 = \infty$, then (5.4) is trivially valid; if $J_1 = 0$, then by (5.5) and (2.7), it is impossible. Suppose that $0 < J_1 < \infty$. By (5.3), it follows that

$$\begin{aligned}
||b||_{q,\widetilde{\Psi}_\lambda}^q &= J_1^p = I > k(\lambda_1)||a||_{p,\Phi_\lambda}||b||_{q,\widetilde{\Psi}_\lambda}, \\
||b||_{q,\widetilde{\Psi}_\lambda}^{q-1} &= J_1 > k(\lambda_1)||a||_{p,\Phi_\lambda},
\end{aligned}$$

and then (5.4) follows, which is equivalent to (5.3).

For $\varepsilon \in (0, q\delta_0)$, we set $\widetilde{\lambda}_1 = \lambda_1 + \frac{\varepsilon}{q}, \widetilde{\lambda}_2 = \lambda_2 - \frac{\varepsilon}{q}$, and

$$\widetilde{a}_m := U_m^{\widetilde{\lambda}_1 - 1 - \varepsilon}\mu_m = U_m^{\lambda_1 - \frac{\varepsilon}{p} - 1}\mu_m, \widetilde{b}_n = V_n^{\widetilde{\lambda}_2 - 1}\upsilon_n = V_n^{\lambda_2 - \frac{\varepsilon}{q} - 1}\upsilon_n.$$

Then $k_\lambda(x,y)\frac{1}{y^{1-\widetilde{\lambda}_2}}(= k_\lambda(x,y)\frac{1}{y^{1-\lambda_2}}\frac{1}{x^{\varepsilon/q}})$ $(0 < q < 1)$ is still strictly decreasing with respect to $y > 0$ and

$$\omega(\widetilde{\lambda}_2, m) < k(\widetilde{\lambda}_1). \tag{5.7}$$

By (2.12), (2.13) and (5.7), we have

$$||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\widetilde{\Psi}_\lambda} = \left(\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}} \left[\sum_{n=1}^\infty (1 - \vartheta(\lambda_1, n))\frac{\upsilon_n}{V_n^{1+\varepsilon}}\right]^{\frac{1}{q}}$$

$$= \left(\sum_{m=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}}\right)^{\frac{1}{p}} \left(\sum_{n=1}^\infty \frac{\upsilon_n}{V_n^{1+\varepsilon}} - \sum_{n=1}^\infty O(\frac{\upsilon_n}{V_n^{1+\beta+\varepsilon}})\right)^{\frac{1}{q}}$$

$$= \frac{1}{\varepsilon}\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O_1(1)\right)^{\frac{1}{p}} \left[\frac{1}{V_{n_0}^\varepsilon} + \varepsilon(O_2(1) - O_4(1))\right]^{\frac{1}{q}},$$

$$\widetilde{I} = \sum_{n=1}^\infty \sum_{m=1}^\infty k_\lambda(U_m, V_n)\widetilde{a}_m\widetilde{b}_n = \sum_{m=1}^\infty \left[\sum_{n=1}^\infty k_\lambda(U_m, V_n)\frac{U_m^{\widetilde{\lambda}_1}\upsilon_n}{V_n^{1-\widetilde{\lambda}_2}}\right]\frac{\mu_m}{U_m^{1+\varepsilon}}$$

$$= \sum_{m=1}^\infty \omega(\widetilde{\lambda}_2, m)\frac{\mu_m}{U_m^{1+\varepsilon}} \le k(\widetilde{\lambda}_1)\sum_{n=1}^\infty \frac{\mu_m}{U_m^{1+\varepsilon}} = \frac{1}{\varepsilon}k(\widetilde{\lambda}_1)\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O_1(1)\right).$$

If there exists a constant $K \ge k(\lambda_1)$, such that (5.3) is valid when replacing $k(\lambda_1)$ by K, then in particular, we have $\varepsilon\widetilde{I} > \varepsilon K||\widetilde{a}||_{p,\Phi_\lambda}||\widetilde{b}||_{q,\widetilde{\Psi}_\lambda}$, namely

$$k(\lambda_1 + \frac{1}{q})\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O_1(1)\right) > K\left(\frac{1}{U_{m_0}^\varepsilon} + \varepsilon O_1(1)\right)^{\frac{1}{p}}\left[\frac{1}{V_{n_0}^\varepsilon} + \varepsilon(O_2(1) - O_4(1))\right]^{\frac{1}{q}}.$$

In view of Lemma 2.1, it follows that $k(\lambda_1) \ge K(\varepsilon \to 0^+)$. Hence, $K = k(\lambda_1)$ is the best possible constant factor of (5.3).

The constant factor $k(\lambda_1)$ in (5.4) is still the best possible. Otherwise, by (5.6) we would reach the contradiction that the constant factor in (5.3) is not the best possible. □

Acknowledgments

The authors wish to express their thanks to the referees for their careful reading of the manuscript and for their valuable suggestions.

B. Yang: This work is supported by the National Natural Science Foundation (No. 61370186), and the Science and Technology Planning Project of Guangzhou (No.2014J4100032, No. 201510010203). I would like to express my thanks for their support.

M. Th. Rassias: This work is supported by the SNF grant: SNF PP00P2_138906. I would like to express my gratitude to Professor P. -O. Dehaye and the Swiss National Science Foundation for providing me with financial support to conduct postdoctoral research at the University of Zurich during the academic year 2015-2016.

References

[1] G.H. Hardy, J.E. Littlewood and G. Pólya, *Inequalities*, Cambridge University Press, Cambridge, 1934.

[2] D.S. Mitrinović, J.E. Pečarić and A.M. Fink, *Inequalities involving functions and their integrals and derivatives*, Kluwer Acaremic Publishers, Boston, 1991.

[3] B.C. Yang, *Hilbert-type integral inequalities*, Bentham Science Publishers Ltd., The United Arab Emirates, 2009.

[4] B.C. Yang, *Discrete Hilbert-type inequalities*, Bentham Science Publishers Ltd., The United Arab Emirates, 2011.

[5] B.C. Yang, *The norm of operator and Hilbert-type inequalities*, Science Press, Beijin, 2009.

[6] B.C. Yang, *Two types of multiple half-discrete Hilbert-type inequalities*, Lambert Academic Publishing, 2012.

[7] B.C. Yang, *On Hilbert's integral inequality*, J. Math. Anal. Appl. 220 (1998) 778–785.

[8] B.C. Yang, *On a more accurate multidimensional Hilbert-type inequality with parameters*, Math. Ineq. Appl. 18 (2015) 429–441.

[9] M. Krnić and J.E. Pečarić, *General Hilbert's and Hardy's inequalities*, Math. Ineq. Appl. 8 (2005) 29–51.

[10] B.C. Yang, I. Brnetić, M. Krnić and J. E. Pečarić, *Generalization of Hilbert and Hardy-Hilbert integral inequalities*, Math. Ineq. Appl. 8 (2005) 259–272.

[11] M. Krnić and J.E. Pečarić, *Hilbert's inequalities and their reverses*, Publ. Math. Debrecen 67 (2005) 315–331.

[12] B.C. Yang and Th.M. Rassias, *On the way of weight coefficient and research for Hilbert-type inequalities*, Math. Ineq. Appl. 6 (2003) 625–658.

[13] B.C. Yang and Th.M. Rassias, *On a Hilbert-type integral inequality in the subinterval and its operator expression*, Banach J. Math. Anal. 4 (2010) 100–110.

[14] L. Azar, *On some extensions of Hardy-Hilbert's inequality and applications*, J. Ineq. Appl., 2009, no. 546829.

[15] B. Arpad and O. Choonghong, *Best constant for certain multilinear integral operator*, J. Ineq. Appl., 2006, no. 28582.

[16] J.C. Kuang and L. Debnath, *On Hilbert's type inequalities on the weighted Orlicz spaces*, Pacific J. Appl. Math. 1 (2007) 95–103.

[17] W.Y. Zhong, *The Hilbert-type integral inequality with a homogeneous kernel of Lambda-degree*, J. Ineq. Appl., 2008, no. 917392.

[18] Y. Hong, *On Hardy-Hilbert integral inequalities with some parameters*, J. Ineq. Pure Appl. Math. 6 (2005) 1–10.

[19] W.Y. Zhong and B.C. Yang, *On multiple Hardy-Hilbert's integral inequality with kernel*, J. Ineq. Appl., Vol. 2007, Art.ID 27962, 17 pages, doi: 10.1155/ 2007/27.

[20] I. Perić and P. Vuković, *Multiple Hilbert's type inequalities with a homogeneous kernel*, Banach J. Math. Anal. 5 (2011) 33–43.

[21] B.C. Yang and M. Krnić, *On the Norm of a Mult-dimensional Hilbert-type Operator*, Sarajevo J. Math. 7 (2011) 223–243.

[22] M. Krnić, J.E. Pečarić and P. Vuković, *On some higher-dimensional Hilbert's and Hardy-Hilbert's type integral inequalities with parameters*, Math. Inequal. Appl. 11 (2008) 701–716.

[23] M. Krnić and P. Vuković, *On a multidimensional version of the Hilbert-type inequality*, Anal. Math. 38 (2012) 291–303.

[24] M.Th. Rassias and B.C. Yang, *On half-discrete Hilbert's inequality*, Appl. Math. Comput. 220 (2013) 75–93.

[25] M.Th. Rassias and B.C. Yang, *A multidimensional half-discrete Hilbert-type inequality and the Riemann zeta function*, Appl. Math. Comput. 225 (2013) 263–277.

[26] M.Th. Rassias and B.C. Yang, *On a multidimensional half-discrete Hilbert-type inequality related to the hyperbolic cotangent function*, Appl. Math. Comput. 242 (2014) 800–813.

[27] M.Th. Rassias and B.C. Yang, *On a multidimensional Hilbert-type integral inequality associated to the gamma function*, Appl. Math. Comput. 249 (2014) 408–418.

[28] Y.J. Li and B. He, *On inequalities of Hilbert's type,* Bull. Australian Math. Soc. 76 (2007) 1–13.

[29] B.C. Yang, *An extension of a Hardy-Hilbert-type inequality*, J. Guangdong Univ. Edu. 35 (2015) 1–8.

[30] J.C. Kuang, *Applied inequalities*, Shangdong Science Technic Press, Jinan, China, 2004.

[31] J.C. Kuang, *Real and functional analysis (continuation)*, Higher Education Press, Beijing, China, 2015.