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An existence result for n^{th}-order nonlinear fractional differential equations

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Abstract

In this paper, we investigate the existence of solutions of some three–point boundary value problems for n^{th}–order nonlinear fractional differential equations with higher boundary conditions by using a fixed point theorem on cones.

Keywords: Caputo fractional derivative, Three-point boundary value problem, Fixed point theorem on cones.
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1. Introduction

Fractional calculus is a generalization of ordinary differentiation and integration to arbitrary non–integer order. Fractional differential equations arise in many engineering and scientific disciplines, such as physics, chemistry, biology, economics, control theory, signal and image processing, biophysics, blood flow phenomena, aerodynamics, fitting of experimental data, etc. (see, for example, [1, 3, 5, 7, 10] and references therein).

Different kind of fixed point theorems are widely used as fundamental tools in order to prove the existence of positive solutions of boundary value problems associated to some differential equations, difference equations, and dynamic equations on scales (see, for example, [2, 3, 6, 8, 9, 11, 12] and the papers cited below).

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**This paper is dedicated to the memory of our colleague, Dr. Ali Benlabbes, who sadly passed away.

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In [2], Bai and Lü studied the existence and multiplicity of positive solutions of nonlinear fractional differential equations with boundary conditions, of the form:

$$\begin{cases} D_{0^+}^{\alpha} u + f(t, u(t)) = 0, \quad 1 < \alpha \leq 2, \quad 0 < t < 1 \\ u(0) = u(1) = 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann–Liouville differential operator of fractional derivative of order α.

In [12], Zhang considered the existence and multiplicity of positive solutions of nonlinear fractional differential equations with boundary conditions, of the form:

$$\begin{cases} {}^cD_{0^+}^{\alpha} u + f(t, u(t)) = 0, \quad 1 < \alpha \leq 2, \quad 0 < t < 1 \\ u(0) + u'(0) = 0, \quad u(1) + u'(1) = 0, \end{cases}$$

where ${}^cD_{0^+}^{\alpha}$ is the Caputo differential operator of fractional derivative of order α.

In [3], Benchohra, Henderson, Ntouyas and Ouahab used the Banach fixed point Theorem and the nonlinear alternative of Leray–Schauder principle to investigate the existence of solutions for fractional order functional and neutral functional differential equations with infinite delay, of the form:

$$\begin{cases} D_{0^+}^{\alpha} u + f(t, u(t)) = 0, \quad 0 < \alpha < 1, \quad 0 \leq t < 1 \\ u(t) = \phi(t), \quad t \leq 0, \end{cases}$$

where $D_{0^+}^{\alpha}$ is the Riemann–Liouville differential operator of fractional derivative of order α,$f : [0, 1) \times B \longrightarrow \mathbb{R}$ is a given function, $\phi \in B$ with $\phi(0) = 0$, and B is the so–called phase space.

Thus, motivated by the results mentioned above, this paper deals with the existence of solutions for the following n^{th}–point boundary value problem for n^{th}–order nonlinear fractional differential equations with higher boundary conditions

$$\begin{cases} {}^cD_{0^+}^{\alpha} u(t) + f(t, u(t)) = 0, \qquad n-1 < \alpha \leq n, \quad 0 \leq t \leq 1 \\ u^{(n-1)}(1) = u^{(n-2)}(p) = u^{(i-1)}(0) = 0, \qquad 1 \leq i \leq n-2, \end{cases} \tag{1.1}$$

where $n > 4$ is an integer, $p \in \left(\frac{1}{2}, 1\right)$ is a constant and ${}^cD_{0^+}^{\alpha}$ is the Caputo differential operator of fractional derivative of order α and $f : [0, 1] \times (0, +\infty) \longrightarrow (0, +\infty)$ is a continuous function.

The rest of the paper is organized as follows. In Section 2, we present some preliminaries. Section 3 is devoted to our main result.

2. Preliminaries

In this section, we recall some definitions and facts which will be used throughout the paper.

Definition 2.1. (Kilbas *et al.*, [4]) The Caputo fractional derivative of order $\alpha > 0$ of a function $u : [0, \infty) \to \mathbb{R}$ is given by

$${}^cD_{0^+}^{\alpha} u(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} u^{(n)}(s) ds,$$

where $n = [\alpha] + 1$ and $[\cdot]$ denotes the integer part.

Remark 2.2. For $\alpha = n \in \mathbb{N}$, the Caputo fractional derivative of order n of a function coincides with the conventional n–th derivative of this function.

Proposition 2.3. (Kilbas *et al.*, [4]) Let $\alpha > 0$ and $0 \le a < b$. Suppose that u is a function of class $C^m(]a,b[,\mathbb{R})$ and $[\alpha] = n-1$. One have

1. ${}^{C}D_a^{\alpha}\left[I_a^{\alpha}u\right] = u$
2. If ${}^{C}D_a^{\alpha}u = 0$ then $u(t) = \sum\limits_{j=0}^{m-1} c_j(t-a)^j$
3. $I_a^{\alpha}\left[{}^{C}D_a^{\alpha}u\right](t) = u(t) + \sum\limits_{j=0}^{m-1} \frac{(x-a)^j}{j!}u^{(j)}(a)$
4. If $0 \le \alpha, \beta \le 1$ with $\alpha + \beta \le 1$ and u of class C^1, then

$$({}^{C}D_a^{\alpha} \circ {}^{C}D_a^{\beta})u = {}^{C}D_a^{\alpha+\beta} = ({}^{C}D_a^{\beta} \circ^{C} D_a^{\alpha})u.$$

Lemma 2.4. Let $\alpha > 0$. The general solution of the fractional differential equation ${}^{c}D_{0+}^{\alpha}u\,(t) = 0$ is given by

$$u(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1},$$

where $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$, and $[\alpha] = n-1$.

From Lemma 2.4, one can easily deduce the following property.

Lemma 2.5. Let $\alpha > 0$ and $0 \le a < b$. Suppose that u is a function of class $C^m(]a,b[,\mathbb{R})$ and $[\alpha] = n-1$. We have

$$I_{0+}^{\alpha}\,{}^{c}D_{0+}^{\alpha}u\,(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^{n-1}$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \ldots, n-1$.

The hereafter fixed point theorem on cones is the fundamental tool on which the proof of our main theorem is based.

Recall that an operator on a Banach space is called completely continuous if it is continuous and maps bounded sets into relatively compact sets.

Theorem 2.6. (Kilbas *et al.*, [4]) Let $\mathcal{E}$ be a Banach space and let $\mathcal{K} \subset \mathcal{E}$ be a cone in $\mathcal{E}$. Assume that Ω_1 and Ω_2 are open subsets of $\mathcal{E}$ with $0 \in \Omega_1$ and $\overline{\Omega}_1 \subset \Omega_2$. Let $T : \mathcal{K} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right) \longrightarrow \mathcal{K}$ be a completely continuous operator. In addition, suppose either

(B1) $\|Tu\| \le \|u\|$, $\forall u \in \mathcal{K} \cap \partial\Omega_1$ and $\|Tu\| \ge \|u\|$, $\forall u \in \mathcal{K} \cap \partial\Omega_2$, or

(B2) $\|Tu\| \le \|u\|$, $\forall u \in \mathcal{K} \cap \partial\Omega_2$ and $\|Tu\| \ge \|u\|$, $\forall u \in \mathcal{K} \cap \partial\Omega_1$, holds.

Then T has a fixed point in $\mathcal{K} \cap \left(\overline{\Omega}_2 \backslash \Omega_1\right)$.

3. Main result

Before to prove our existence result for the three–point boundary value problem (1.1), we first determine the Green's function associated to this problem. Denote by $\mathcal{C}(J,\mathbb{R})$ the Banach space of all continuous functions from $J = [0,1]$ into $\mathbb{R}$ with the norm $\|u\| := \sup\limits_{t \in J}|u(t)|$, $u \in \mathcal{C}(J,\mathbb{R})$.

Lemma 3.1. (Podlubny, [7]) Let $n-1<\alpha\le n$ and $h\in\mathcal{C}(J,\mathbb{R})$. A function u is a solution of the initial value problem

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t)=h(t), & t\in J\\ u(0)=u_0 \end{cases}$$

if and only if it is a solution of the fractional integral equation

$$u(t)=u_0+\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}h(s)\,ds, \quad t\in J.$$

Lemma 3.2. *Let $n-1<\alpha\le n$ and $h\in\mathcal{C}(J,\mathbb{R})$ with $h(t)\ge 0$, for $t\in J$. The following n^{th}–point boundary value problem*

$$\begin{cases} {}^cD^{\alpha}_{0^+}u(t)=h(t), & t\in J\\ u^{(n-1)}(1)=u^{(n-2)}(p)=u^{(i-1)}(0)=0, & 1\le i\le n-2 \end{cases} \tag{3.1}$$

has the unique solution

$$u(t)=\int_0^1 G(t,s)h(s)ds+\int_0^1 \overline{G}_p(t,s)h(s)ds, \quad t\in J \tag{3.2}$$

where

$$-G(t,s)=\begin{cases} \dfrac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}, & t\le s\\ \dfrac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}-\dfrac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}, & t\ge s \end{cases}$$

and

$$\overline{G}_p(t,s)=\begin{cases} \dfrac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} \\ \qquad -\dfrac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+2)}(p-s)^{\alpha-n+1}, & p\ge s\\ \dfrac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}, & s\ge p. \end{cases}$$

Proof . By applying lemmas 2.5 and 3.1, equation ${}^cD^{\alpha}_{0^+}u(t)=h(t)$ is equivalent to the following integral equation

$$u(t)=c_0+c_1t+c_2t^2+\cdots+c_{n-1}t^{n-1}+I^{\alpha}_{0^+}h(t) \tag{3.3}$$

for some arbitrary constants $c_i\in\mathbb{R}$, $i=1,2,\ldots,n-1$. From the n^{th}–point boundary conditions, we deduce the exact values $c_0=c_1=\cdots=c_{n-3}=0$ in (3.3) so that

$$u(t)=c_{n-2}t^{n-2}+c_{n-1}t^{n-1}+I^{\alpha}_{0^+}h(t), \qquad t\in J \tag{3.4}$$

with

$$c_{n-1}=-\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}\int_0^1(1-s)^{\alpha-n}h(s)ds.$$

Indeed, for $t\in J$

$$u^{(n-1)}(t)=(n-1)!c_{n-1}+I^{\alpha-n+1}_{0^+}h(t)$$

implies that

$$u^{(n-1)}(1) = (n-1)!c_{n-1} + I_{0^+}^{\alpha-n+1}h(1) = 0,$$

so that

$$c_{n-1} = -\frac{1}{\Gamma(n)} I_{0^+}^{\alpha-n+1}h(1), \qquad \text{with } \Gamma(n) = (n-1)!.$$

Also we have

$$u^{(n-2)}(p) = 0 \Leftrightarrow (n-2)!c_{n-2} + (n-1)!c_{n-1}p + I_{0^+}^{\alpha-n+2}h(p) = 0$$

from which we deduce that

$$\begin{aligned}
c_{n-2} &= -\frac{(n-1)}{(n-2)}pc_{n-1} - \frac{I_{0^+}^{\alpha-n+2}h(p)}{(n-2)!} \\
&= \frac{(n-1)\,p}{\Gamma(n)\,\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s)ds \\
&\quad - \frac{1}{\Gamma(n-1)\,\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s)ds \\
&= \frac{p}{\Gamma(n-1)\,\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s)ds \\
&\quad - \frac{1}{\Gamma(n-1)\,\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s)ds.
\end{aligned}$$

After replacing c_{n-2} and c_{n-1} in (3.4), we obtain

$$\begin{aligned}
u(t) &= \frac{pt^{n-2}}{\Gamma(n-1)\,\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s)ds \\
&\quad - \frac{t^{n-2}}{\Gamma(n-1)\,\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s)ds \\
&\quad - \frac{t^{n-1}}{\Gamma(n)\,\Gamma(\alpha-n+1)} \int_0^1 (1-s)^{\alpha-n} h(s)ds + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds \\
&= \frac{pt^{n-2}}{\Gamma(n-1)\,\Gamma(\alpha-n+1)} \int_0^p (1-s)^{\alpha-n} h(s)ds \\
&\quad - \frac{t^{n-2}}{\Gamma(n-1)\,\Gamma(\alpha-n+2)} \int_0^p (p-s)^{\alpha-n+1} h(s)ds \\
&\quad + \frac{pt^{n-2}}{\Gamma(n-1)\,\Gamma(\alpha-n+1)} \int_p^1 (1-s)^{\alpha-n} h(s)ds \\
&\quad - \frac{t^{n-1}}{\Gamma(n)\,\Gamma(\alpha-n+1)} \int_0^t (1-s)^{\alpha-n} h(s)ds \\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} h(s)ds - \frac{t^{n-1}}{\Gamma(n)\,\Gamma(\alpha-n+1)} \int_t^1 (1-s)^{\alpha-n} h(s)ds.
\end{aligned}$$

This completes the proof. □

Hereafter, we give some properties of functions $G(t,s)$ and $\overline{G}_p(t,s)$ in Lemma 3.2, and of the solution of the n^{th}–point boundary value problem (3.1), which we will use later.

Lemma 3.3. *For all* $(t,s) \in [\tau,1] \times J$*, with* $\tau \geq 0$*, and* $p \in (\frac{1}{2},1)$*, we have:*

(E_1) $\overline{G}_{\frac{1}{2}}(t,s) \leq \overline{G}_p(t,s) \leq \overline{G}_1(t,s)$.

(E_2) $0 \leq \overline{G}_p(t,s) \leq \dfrac{pt^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}$.

(E_3) $\tau^{\alpha-1}\overline{G}_p(1,s) \leq \min_{\tau\leq t\leq 1}\overline{G}_p(t,s) = \tau^{n-2}\overline{G}_p(1,s)$.

(E_4) $\gamma G(1,s) \leq G(t,s) \leq G(1,s)$*, where* $\gamma = \left(1 - \dfrac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha)}\right)\tau^{\alpha-1}$.

Proof . • *Property* (E_1). We have

$$\frac{\partial \overline{G}_p(t,s)}{\partial p} = \begin{cases} \dfrac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} \\ \qquad -\dfrac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+2)}(p-s)^{\alpha-n}, & p \geq s \\ \dfrac{t^{n-2}}{\Gamma(n-1)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}, & p \leq s. \end{cases}$$

Taking into account that $p \in (\frac{1}{2},1)$, it is easy to see that $\frac{\partial \overline{G}_p(t,s)}{\partial p} \geq 0$. Hence $\overline{G}_p(t,s)$ is nondecreasing with respect to p and then

$$\overline{G}_{\frac{1}{2}}(t,s) \leq \overline{G}_p(t,s) \leq \overline{G}_1(t,s).$$

- *Properties* (E_2) *and* (E_3). The proofs are obvious and then omitted.
- *Property* (E_4). For $s \geq t$, taking into account that $t \in [\tau,1]$ with $\tau \geq 0$ and $\alpha \leq n$, we have

$$\gamma \leq \tau^{n-1} \leq \frac{G(t,s)}{G(1,s)} = t^{n-1} \leq 1.$$

For $s \leq t$, we have

$$\begin{aligned}
\frac{G(t,s)}{G(1,s)} &= \frac{\frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}}{G(1,s)} \\
&\geq \min_{\tau\leq t\leq 1} \frac{\frac{t^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}}{G(1,s)} - \max_{\tau\leq t\leq 1}\frac{\frac{1}{\Gamma(\alpha)}(t-s)^{\alpha-1}}{G(1,s)} \\
&\geq \frac{\frac{\tau^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}{G(1,s)} \\
&= \frac{\frac{\tau^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}
\end{aligned}$$

$$\geq \frac{\frac{\tau^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}}$$
$$\geq \tau^{\alpha-1} - \frac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha)}(1-s)^{\alpha-1}$$

so that when $1-s \leq \tau$, we get

$$\frac{G(t,s)}{G(1,s)} \geq \left(1 - \frac{\Gamma(n)\Gamma(\alpha-n+1)}{\Gamma(\alpha)}\right)\tau^{\alpha-1} =: \gamma.$$

Now, taking into account that

$$0 \leq 1 - \frac{s}{t} \leq 1 - s \Rightarrow -(1-\frac{s}{t})^{\alpha-1} \geq -(1-s)^{\alpha-1},$$

we have

$$\frac{G(t,s)}{G(1,s)} = t^{\alpha-1}\left[\frac{\frac{t^{n-\alpha}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}\left(1-\frac{s}{t}\right)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}\right]$$
$$\leq t^{\alpha-1}\left[\frac{\frac{t^{n-\alpha}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}{\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-1} - \frac{1}{\Gamma(\alpha)}(1-s)^{\alpha-1}}\right]$$
$$\leq t^{\alpha-1} \leq 1.$$

This completes the proof. □

Lemma 3.4. *Let $\tau \leq 1$, $p \in (\frac{1}{2}, 1)$ and $h \in \mathcal{C}(J, R)$ with $h(t) \geq 0$, for $t \in J$. The unique solution of the n^{th}–point boundary value problem (3.1) is such that*

$$\min_{t\in[0,\tau]} u(t) \geq \gamma \|u\|, \tag{3.5}$$

where $\gamma = \tau^{\alpha-1}$.

Proof . Let $t \in [\tau, 1]$. Taking into account that $\gamma \leq 1$, by (3.2) and Lemma 3.3, we have

$$u(t) = \int_0^1 G(t,s)h(s)ds + \int_0^1 \overline{G}_p(t,s)h(s)ds$$
$$\geq \gamma \int_0^1 G(1,s)h(s)ds + \int_0^1 \overline{G}_p(t,s)h(s)ds$$
$$\geq \gamma \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \overline{G}_p(1,s)h(s)ds\right]$$

so that

$$\min_{\tau\leq t\leq 1} u(t) \geq \gamma \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \overline{G}_p(1,s)h(s)ds\right]. \tag{3.6}$$

On the other hand, we have

$$u(t) = \int_0^1 G(t,s)h(s)ds + \int_0^1 \overline{G}_p(t,s)h(s)ds$$
$$\leq \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \overline{G}_p(1,s)h(s)ds\right]$$

from which we deduce that

$$\|u\| \leq \left[\int_0^1 G(1,s)h(s)ds + \int_0^1 \overline{G}_p(1,s)h(s)ds\right]. \tag{3.7}$$

Finally, by (3.6) and (3.7) we get the desired inequality (3.5). □

Now, we will apply Theorem 2.6 to prove the existence of solutions for the n^{th}–point boundary value problem (1.1). First, we recall that a function $u \in \mathcal{C}(J,\mathbb{R})$ is a solution of (1.1), if u satisfies equation

$${}^cD_0^\alpha u\,(t) = f(t,u(t)), \quad t \in J$$

and conditions

$$u^{(n-1)}(1) = u^{(n-2)}(p) = u^{(i-1)}\,(0) = 0, \quad 1 \leq i \leq n-2.$$

In the other hand, by Lemma 3.2, we see that u is a solution of (1.1) if and only if

$$u(t) = \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds, \quad t \in J.$$

Let us consider the Banach space $\mathcal{E} = \{u \in \mathcal{C}(J,\mathbb{R}) : \ u(t) \geq 0, \ t \in J\}$ equipped with the norm $\|u\| = \max\{u(t) : \ t \in J\}$. We define on $\mathcal{E}$ the integral operator $T : \mathcal{E} \longrightarrow \mathcal{E}$ given by

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds, \quad u \in \mathcal{E}, \ t \in J. \tag{3.8}$$

It is not difficult to see that T is well defined and fixed points of T are solutions of problem (1.1).

Let $\mathcal{K}$ be the cone in $\mathcal{E}$ given by $\mathcal{K} = \left\{u \in \mathcal{E} : \min\limits_{t\in[0,\tau]} u(t) \geq \gamma\,\|u\|\right\}$. The following result gives some properties of the restriction of T to $\mathcal{K}$.

Lemma 3.5. *Let $f : J \times (0,+\infty) \longrightarrow (0,+\infty)$ be continuous. The operator $T : \mathcal{E} \longrightarrow \mathcal{E}$ given by (3.8) is such that*

(1) *$T(\mathcal{K}) \subset \mathcal{K}$.*

(2) *$T_{\mathcal{K}} := T_{|\mathcal{K}} : \mathcal{K} \longrightarrow \mathcal{K}$ is completely continuous.*

Proof . • *Property* (1). Let $u \in \mathcal{K}$ and $t \in J$. We have

$$Tu(t) = \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds$$
$$\leq \gamma \int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds$$
$$\leq \int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds,$$

which implies that

$$\|Tu\| \leq \int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 G_1(p,s)f(s,u(s))ds. \tag{3.9}$$

On the other hand, we have

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds \\ &\geq \int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds \\ &\geq \gamma\left[\int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u(s))ds\right]. \end{aligned} \tag{3.10}$$

From inequalities (3.9) and (3.10) we deduce that $\min\limits_{t\in[0,\tau]} Tu(t) \geq \gamma \|Tu\|$ and consequently we have $T(\mathcal{K}) \subset \mathcal{K}$.

- *Property* (2). We divide the proof into two steps.

Step 1. $T_{\mathcal{K}}$ is continuous.

Let $(u_n)_n$ be a sequence such that $u_n \to u_0$ in $\mathcal{K}$, and let

$$M = 2\left(\frac{1}{\Gamma(n-1)\Gamma(\alpha-n+2)} - \frac{1}{\Gamma(\alpha+1)}\right).$$

Fix $\varepsilon > 0$. For n sufficiently large, we have $\|Tu_n - Tu_0\|$

$$\begin{aligned} &= \left\|\int_0^1 G(t,s)f(s,u_n(s))ds + \int_0^1 \overline{G}_p(1,s)f(s,u_n(s))ds \right.\\ &\qquad \left. - \int_0^1 G(t,s)f(s,u_0(s))ds - \int_0^1 \overline{G}_p(1,s)f(s,u_0(s))ds\right\| \\ &= \left\|\int_0^1 G(t,s)\left(f(s,u_n(s) - f(s,u_0(s))\right)ds + \int_0^1 \overline{G}_p(1,s)\left(f(s,u_n(s) - f(s,u_0(s))\right)ds\right\| \\ &\leq \left\|\gamma\int_0^1 G(1,s)\left(f(s,u_n(s) - f(s,u_0(s))\right)ds + \int_0^1 \overline{G}_1(1,s)\left(f(s,u_n(s) - f(s,u_0(s))\right)ds\right\| \\ &\leq \left\|\int_0^1 G(1,s)\left(f(s,u_n(s) - f(s,u_0(s))\right)ds + \int_0^1 \overline{G}_1(1,s)\left(f(s,u_n(s) - f(s,u_0(s))\right)ds\right\| \\ &\leq \frac{\varepsilon}{M}\left\|\int_0^1 \left[G(1,s)ds + \overline{G}_1(1,s)\right]ds\right\|. \end{aligned}$$

If we replace $G(1,s) + \overline{G}_1(1,s)$ by its value, we obtain

$$\begin{aligned} \|Tu_n - Tu_0\| \leq \frac{\varepsilon}{M}\frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)}\int_0^1 (1-s)^{\alpha-n}ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}ds \\ + t^{n-2}\frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)}\int_0^1 \left[(1-s)^{\alpha-n} - (1-s)^{\alpha-n+1}\right]ds \end{aligned}$$

$$\leq \frac{\varepsilon}{M}\frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)}\int_0^1 (1-s)^{\alpha-n}ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}ds$$
$$+\frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)}\int_0^1 \left[(1-s)^{\alpha-n}-(1-s)^{\alpha-n+1}\right]ds$$
$$\leq \frac{\varepsilon}{M}\frac{2}{\Gamma(n-1)\Gamma(\alpha-n+1)}\int_0^1 (1-s)^{\alpha-n}ds - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}ds$$
$$-\frac{1}{\Gamma(n-1)\Gamma(\alpha-n+1)}\int_0^1 (1-s)^{\alpha-1}ds.$$

After some calculations we get

$$\|Tu_n - Tu_0\| \leq \frac{\varepsilon}{M}.M = \varepsilon,$$

which finishes to prove that T is continuous.

Step 2. $T_{\mathcal{K}}$ takes bounded sets into relatively compact sets in $\mathcal{K}$. From the Ascoli–Arzela Theorem, it is sufficient to prove that for each bounded subset $\mathcal{B}$ of $\mathcal{K}$, the set $T_{\mathcal{K}}(\mathcal{B})$ is bounded and equicontinuous. Let $\mathcal{B}$ be a bounded set in $\mathcal{K}$. Then there exists a real number $m > 0$ such that $\|u\| \leq m$, for all $u \in \mathcal{B}$. For $u \in \mathcal{B}$, we have

$$\|Tu\| = \left\| \int_0^1 G(t,s)f(s,u(s))ds + \int_0^1 \overline{G}_p(p,s)f(s,u(s))ds \right\|$$
$$\leq \max_{t\in J, 0\leq u(t)\leq m} \mid f(t,u(t) \mid \left\| \int_0^1 G(t,s)ds + \int_0^1 \overline{G}_p(p,s)ds \right\| \leq c_m M$$

where $c_m = \max\{|f(t,u)| : \ t \in J, \ \|u\| \leq m\}$. Hence $T(\mathcal{B})$ is bounded Now, let $u \in \mathcal{B}$ and $t_1, t_2 \in J$, $t_1 < t_2$. We have

$$|Tu(t_1) - Tu(t_2)| = \left| \int_0^1 G(t_1,s)f(s,u(s))ds - \int_0^1 G(t_2,s)f(s,u(s))ds \right|$$
$$\leq c_m \left| \int_0^1 (G(t_1,s)-G(t_2,s))\,ds \right|$$
$$\leq c_m \left| \int_0^{t_1} (G(t_1,s)-G(t_2,s))\,ds + \int_{t_1}^{t_2} (G(t_1,s)-G(t_2,s))\,ds \right.$$
$$\left. + \int_{t_2}^1 (G(t_1,s)-G(t_2,s))\,ds \right|$$
$$\leq c_m \left| \int_0^{t_1} \left[\left(\frac{t_1^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(t_1-s)^{\alpha-1} \right) \right. \right.$$
$$\left. - \left(\frac{t_2^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(t_2-s)^{\alpha-1} \right) \right] ds$$
$$+ \int_{t_1}^{t_2} \left[\left(\frac{t_1^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(t_1-s)^{\alpha-1} \right) \right.$$
$$\left. - \left(\frac{t_2^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n} - \frac{1}{\Gamma(\alpha)}(t_2-s)^{\alpha-1} \right) \right] ds$$

$$+\int_{t_2}^{1}\left[\left(\frac{t_1^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}-\frac{1}{\Gamma(\alpha)}(t_1-s)^{\alpha-1}\right)\right.$$
$$\left.\left.-\left(\frac{t_2^{n-1}}{\Gamma(n)\Gamma(\alpha-n+1)}(1-s)^{\alpha-n}-\frac{1}{\Gamma(\alpha)}(t_2-s)^{\alpha-1}\right)\right]ds\right|.$$

After some standard calculations we get

$$|Tu(t_1)-Tu(t_2)|\leq c_m\left|\frac{1}{\Gamma(n)\Gamma(\alpha-n+1)}\left(t_1^{n-1}-t_2^{n-1}\right)\right.$$
$$\left.+\frac{1}{\Gamma(\alpha+1)}\left((t_1-1)^{\alpha}-(t_2-1)^{\alpha}\right)-\frac{1}{\Gamma(\alpha+1)}\left(t_1^{\alpha}-t_2^{\alpha}\right)\right|.$$

When $t_1\to t_2$ the right–hand side of the above inequality tends to zero. This property holds also for $t_2<t_1$. This finishes to prove that $T(\mathcal{B})$ is equicontinuous. The proof is complete. □

In what follows, we will use the following notations:

$$f_0=\liminf_{\|u\|\to 0}\min_{t\in J}\frac{f(t,u(t))}{u(t)},\quad f_\infty=\limsup_{\|u\|\to+\infty}\max_{t\in J}\frac{f(t,u(t))}{u(t)}$$

where $f:[0,1]\times(0,+\infty)\longrightarrow(0,+\infty)$ is a given function. f is called superlinear if $f_0=0$ and $f_\infty=\infty$.

Finally, our main result reads as follows.

Theorem 3.6. *Suppose that $f:[0,1]\times(0,+\infty)\longrightarrow(0,+\infty)$ is continuous and superlinear. Then the n^{th}–point boundary value problem (1.1) has at least one solution.*

Proof . Since $f_0=0$, for any $\varepsilon>0$ there exists $\eta(\varepsilon)>0$ such that, for $u\in\mathcal{C}(J)$ and $t\in J$,

$$\|u\|\leq\eta(\varepsilon)\ \Rightarrow\ \frac{f(t,u(t))}{u(t)}\leq\varepsilon\ \Rightarrow\ f(t,u(t))\leq\varepsilon u(t). \tag{3.11}$$

Let $\Omega_1=\{u\in\mathcal{C}(J,\mathbb{R}):\|u\|\leq\eta(\varepsilon)\}$. For any $u\in\mathcal{K}\cap\partial\Omega_1$ and $t\in J$, it follows from (3.8) and Lemma 3.3 that

$$\begin{aligned}Tu(t)&=\int_0^1 G(t,s)f(s,u(s))ds+\int_0^1\overline{G}_p(p,s)f(s,u(s))ds\\&\leq\gamma\int_0^1 G(1,s)f(s,u(s))ds+\int_0^1\overline{G}_p(1,s)f(s,u(s))ds\\&\leq\int_0^1 G(1,s)f(s,u(s))ds+\int_0^1\overline{G}_p(1,s)f(s,u(s))ds.\end{aligned}$$

By use of (3.11) we get

$$\begin{aligned}Tu(t)&\leq\varepsilon\left(\int_0^1 G(1,s)u(s)ds+\int_0^1\overline{G}_1(1,s)u(s)ds\right)\\&\leq\varepsilon\left(\int_0^1 G(1,s)ds+\int_0^1\overline{G}_1(1,s)ds\right)\|u\|.\end{aligned}$$

Now, if we let

$$\varepsilon = \left(\int_0^1 G(1,s)ds + \int_0^1 \overline{G}_1(1,s)ds \right)^{-1}$$

we get

$$\|Tu\| \le \|u\| . \tag{3.12}$$

In the same way, since $f_\infty = \infty$, for any $A > 0$ there exists $B > \eta(\varepsilon) > 0$ such that, for $u \in \mathcal{C}(J)$ and $t \in J$,

$$\|u\| \ge \gamma B \;\Rightarrow\; \frac{f(t,u(t))}{u(t)} > A \;\Rightarrow\; f(t,u(t)) > Au(t). \tag{3.13}$$

Let $\Omega_2 = \{u \in \mathcal{C}(J,\mathbb{R}) : \|u\| \le B\}$. For any $u \in \mathcal{K} \cap \partial\Omega_2$ and $t \in J$, we have $\|u\| = B$ and by Lemma 3.4, we get

$$\min_{t\in[0,\tau]} u(t) \ge \gamma \|u\| = \gamma B.$$

Thus, from (3.8) and Lemma 3.3 we can conclude that

$$Tu(t) \ge \int_0^1 G(1,s)f(s,u(s))ds + \int_0^1 \overline{G}_{\frac{1}{2}}(1,s)f(s,u(s))ds.$$

By use of (3.13) we get

$$\begin{aligned} Tu(t) &\ge A\left(\int_0^1 G(1,s)u(s)ds + \int_0^1 \overline{G}_{\frac{1}{2}}(1,s)u(s)ds \right) \\ &\ge A\gamma \left(\int_0^1 G(1,s)ds + \int_0^1 \overline{G}_{\frac{1}{2}}(1,s)ds \right) \|u\| . \end{aligned}$$

At this stage, if we let

$$A = \left[\gamma \left(\int_0^1 G(1,s)ds + \int_0^1 \overline{G}_{\frac{1}{2}}(1,s)ds \right) \right]^{-1}$$

we get

$$\|Tu\| \ge \|u\| . \tag{3.14}$$

From inequalities (3.12), (3.14) and Theorem 2.6, case (B1), we deduce the existence of a solution for the n^{th}–point boundary value problem (1.1). □

Example 3.7. Let $f : J \times (0,+\infty) \longrightarrow (0,+\infty)$ be the continuous function defined by

$$f(t,x) = t\ln(2e^{x^2} - 1), \quad t \in J,\; x > 0.$$

It is easy to verify that

$$f_0 = \liminf_{\|u\|\to 0} \min_{t\in J} \frac{f(t,u(t))}{u(t)} = 0$$

and

$$f_\infty = \limsup_{\|u\|\to +\infty} \max_{t\in J} \frac{f(t,u(t))}{u(t)} = +\infty,$$

so that f is a superlinear function. Thus, by Theorem 3.6, the n^{th}–point boundary value problem (1.1), associated to f, has at least one solution.

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