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Some notes on "Common fixed point of two R-weakly commuting mappings in b-metric spaces"

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Abstract

Very recently, Kumam *et al.* [J Funct Spaces, Volume 2015, Article ID: 350840] obtained some interesting common fixed point results for two mappings satisfying a generalized contractive condition in b–metric spaces without the assumption of the continuity of the b–metric, but unfortunately, there exists a gap in the proof of the main result. In this note, we point out and fill such gap by making some remarks and offering a new proof for the result. It should be mentioned that our proofs for some key assertions of the main result are new and somewhat different from the original ones. In addition, we also present a result to check the continuity of the b–metrics which is found effective and workable when applied to some examples.

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1. Introduction and Preliminaries

Since Czerwik [3] introduced the concept of b–metric space (sometimes called metric type space or metric–type space, see Jovanović, Kadelburg and Radenović [5], for instance) many authors have focused on the studying the fixed point theory for single–valued and multivalued operators in b–metric spaces (see also [1, 2, 3, 4, 5, 6, 7, 8, 9]). Most of the authors have used in their works b–metric spaces under the assumption that the b–metric is continuous. Very recently, Kumam *et*

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al. [7] presented some interesting common fixed point results for two mappings under a generalized contractive condition in b–metric spaces, where the b–metric is not necessarily continuous. But unfortunately, there exists a gap in the proof of the main result. The aim of this note is to describe the gap by making some remarks and fill the gap by offering a new proof for the result. Our proofs for the key assertions of the main result are new and somewhat different from the original ones. In addition, we discuss the continuity of the b–metrics and present a sufficient condition under which the b–metrics are continuous. We find that this result is effective and workable by applying it to some examples.

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{R}_+$ and $\mathbb{R}$ the sets of positive integers, nonnegative real numbers, and real numbers, respectively.

Consistent with [3], the following definition and results will be needed in the sequel.

Definition 1.1. (Czerwik, [3]) Let X be a nonempty set and let $b \geq 1$ be a given real number. A function $d : X \times X \to \mathbb{R}_+$ is a b–metric on X if, for all $x, y, z \in X$, the following conditions are satisfied:

1. $d(x,\, y) \geq 0$ and $d(x,\, y) = 0$ if and only if $x = y$;
2. $d(x,\, y) = d(y,\, x)$;
3. $d(x,\, y) \leq b(d(x,\, z) + d(z,\, x))$.

Then $(X,\, d)$ is called a b–metric space with coefficient b.

In general, a b–metric need not be continuous, see Remark 2.1 in [2], and Example 3.9 in [9]. However, there are a number of examples in which the b–metrics are actually continuous.

Now we give a sufficient condition under which the b–metric is continuous in a b–metric space.

Proposition 1.2. *Let (X, D) be a b–metric space. If there exist a metric d on X and a continuous function $\varphi : [0, +\infty) \to [0, +\infty)$ such that $D(x, y) = \varphi(d(x, y))$ for all $x, y \in X$, then D is continuous on $X \times X$.*

Proof . Since $\varphi : [0, +\infty] \to [0, +\infty]$ is continuous, taking into account that the metric d on X is continuous, we get that the b–metric $D(x, y)$ with $D(x, y) = \varphi(d(x, y))$ is a function satisfying $\lim_{n\to\infty} D(x_n, y_n) = D(x, y)$ whenever $\lim_{n\to\infty}(x_n, y_n) = (x, y)$. Thus by Corollary 3.8 in [9], we conclude that D is continuous at each $(x, y) \in X \times X$. □

The following can be used to show that a function $D(x, y)$ defined on $X \times X$, where X is a metric space, is a b–metric.

Proposition 1.3. *Let (X, d) be a metric space. Suppose that there exists a convex and nondecreasing function $\varphi : [0, +\infty) \to [0, +\infty)$ with $\varphi(2) \geq 2$ and $\varphi(0) = 0$ such that $\varphi(\frac{x}{2}) \geq \frac{\varphi(x)}{\varphi(2)}$ for all $x \in (0, +\infty)$. Then $D(x, y) = \varphi(d(x, y))$ is a b–metric with $b = \frac{\varphi(2)}{2}$ and so (X, D) is a b–metric space.*

Proof . Recall that φ is convex in $[0, +\infty)$, if it satisfies

$$\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$$

for all $x, y \in [0, +\infty)$ and $\lambda \in [0, 1]$. So for any $a, c \in (0, +\infty)$, we have

$$\varphi(a+c) \cdot \frac{1}{\varphi(2)} \leq \varphi(\frac{a+c}{2}) \leq \frac{1}{2}(\varphi(a) + \varphi(c)),$$

since $\varphi(\frac{x}{2}) \geq \frac{\varphi(x)}{\varphi(2)}$ for any $x \in (0,+\infty)$. So we get

$$\varphi(a+c) \leq \frac{\varphi(2)}{2}(\varphi(a)+\varphi(c)). \tag{1.1}$$

By the triangle inequality of d we see that

$$d(x,y) \leq d(x,z)+d(z,y) \tag{1.2}$$

for all $x, y, z \in X$. Let $D(x,y) = \varphi(d(x,y))$. Then by (1.1), (1.2) and the fact that φ is nondecreasing we have

$$\begin{aligned} D(x,y) = \varphi(d(x,y)) &\leq \varphi(d(x,z)+d(z,y)) \\ &\leq \frac{\varphi(2)}{2}(\varphi(d(x,z))+\varphi(d(z,y))) \\ &= b(D(x,z)+D(z,y)), \end{aligned}$$

where $b = \frac{\varphi(2)}{2} \geq 1$ since $\varphi(2) \geq 2$. That is, $D(x,y)$ is a b-metric with $b = \frac{\varphi(2)}{2}$ and so (X,D) is a b-metric space. □

Example 1.4. Suppose $X = \mathbb{R}$ and $s > 1$. Consider the usual Euclidean metric $d : X \times X \to \mathbb{R}_+$ defined by

$$d(x,y) = |x-y|, \ \forall x, y \in X.$$

Then by Proposition 1.3, we see that $D(x,y) = |x-y|^s$ is a b–metric on $\mathbb{R}$ with $b = 2^{s-1}$ (noting that here we can take $\varphi(t) = t^s$), but it is not a metric on $\mathbb{R}$, since the triangle inequality does not hold. In fact, taking $x = 2n+1$, $y = 1$, $z = n+1$ where n is positive real number, we see (noting that $s > 1$)

$$|x-y|^s = (2n)^s > 2n^s = |x-z|^s + |z-y|^s,$$

that is, $D(x,y) > D(x,z)+D(z,y)$, which implies that the triangle inequality does not hold for D. In addition, by Proposition 1.2, we see that the b–metric D is continuous since here $\varphi(t)$ is continuous.

By Proposition 1.2 and Proposition 1.3, we immediately obtain the following result.

Proposition 1.5. Let (X,d) be a metric space. Suppose that there exists a convex, nondecreasing and continuous function $\varphi : [0,+\infty) \to [0,+\infty)$ with $\varphi(2) \geq 2$ and $\varphi(0) = 0$ such that $\varphi(\frac{x}{2}) \geq \frac{\varphi(x)}{\varphi(2)}$ for all $x \in (0,+\infty)$. Then $D(x,y) = \varphi(d(x,y))$ is a b–metric with $b = \frac{\varphi(2)}{2}$ and so (X,D) is a b-metric space. In addition, the b–metric $D(x,y)$ is continuous at each $(x,y) \in X \times X$.

Remark 1.6. *By using Proposition* 1.5 *we may check that the functions* $\rho(x,y) = (d(x,y))^s$ *from Example 2 in [7],* $D(x,y) = (\sum_{n=1}^{\infty} |x_n - y_n|^p)^{\frac{1}{p}}$ *from Example 1.1 in [2] and* $\rho(x,y) = (\int_0^1 |x(t) - y(t)|^p dt)^{\frac{1}{p}}$ *from Example 1.2 in [2] are all b–metrics and each of these b–metrics is also continuous at each* $(x,y) \in X \times X$ *where* X *is the corresponding b-metric space. Take Example 1.2 in [2], for example. In fact, let* $X = L_p[0,1]$ $(0 < p < 1)$*. Define* $d(x,y)$ *by*

$$d(x,y) = \int_0^1 |x(t)-y(t)|^p \, dt, \quad \forall x, y \in L_p[0,1].$$

It can be easily seen that d is a metric on $L_p[0,1]$. Obviously, there is a continuous function $\varphi(t) = t^{\frac{1}{p}}$ such that

$$\rho(x,y) = \varphi(d(x,y)), \quad \forall x, y \in L_p[0,1].$$

Using Proposition 1.5 *(or Proposition* 1.2 *and Proposition* 1.3*) we conclude that the function ρ is a b-metric. Moreover, ρ is also continuous at any $(x,y) \in X \times X$.*

For the definitions of basic concepts concerning the convergence of sequences in a b–metric space such as convergent sequence, Cauchy sequence, complete space, etc., we refer to Boriceanu, Bota and Petrusel [2].

Very recently, Kumam *et al.* introduced commutativity for two mappings in b–metric spaces, which generalized the classical Banach contraction condition, as follows.

Let f, $g : X \to X$ be mappings where (X,d) is a b–metric space. Then we say that f and g are weakly commuting provided that

$$d(fgx, gfx) \leq d(fx, gx)$$

for any $x \in X$. We say that f and g are R–weakly commuting provided that there is a number $R > 0$ such that

$$d(fgx, gfx) \leq Rd(fx, gx)$$

for any $x \in X$.

Note that the commutativity for two mappings defined above is crucial in the studying of the theory of common fixed point.

2. The main results

In this section, we will emphasize the insufficiency of the proof of the main result of Kumam *et al.* [7] and correct the weaknesses appearing in [7]. The main result (Theorem 12) of [7] was presented as follows.

Theorem 2.1. (Kumam *et al.* [7]) Let (X,d) be a complete b–metric space with $b \geq 1$. Suppose that $f, g : X \to X$ are R–weakly commuting mappings satisfying the following three conditions:

(a) $f(X) \subset g(X)$;

(b) f or g is continuous;

(c) $d(fx, fy) \leq \gamma\left(\frac{1}{b^4} d(gx, gy)\right)$ for all $x, y \in X$, where $\gamma : [0,\infty) \to [0,\infty)$ is a continuous and nondecreasing function such that $\gamma(a) < a$ for each $a > 0$ and $\gamma(0) = 0$. Then f and g have a unique common fixed point.

This result and its proof are interesting. However, there is some insufficiency or gap in the proof of Theorem 2.1 appearing in Kumam *et al.* [7]. We first present some remarks.

Remark 2.2. In Theorem 2.1, it follows from (c) that if g is continuous then f is also continuous. Hence, in (b): "f or g is continuous" may be replaced by "f is continuous".

Remark 2.3. In the proof of Theorem 2.1 from [7], in order to prove the existence of common fixed point of f and g, the authors of [7] first discuss the Jungck sequence $\{y_n\}$ where

$$y_n = fx_n = gx_{n+1} \quad (n \in \mathbb{N})$$

and prove that it is a Cauchy sequence in $f(X)$. To achieve this end they use the contradiction method. However, there is another method to show this. In fact, by using (c) and Lemma 3.1 from

[8] we know that $\{y_n\}$ is a Cauchy sequence in $f(X)$. This direct method is quite different from the above-mentioned contradiction method. In addition, the authors get the inequality

$$\varepsilon \leq \lim_{k\to\infty} d_k \leq b\varepsilon$$

from (17) in [7], but they have not shown that the limit of the sequence $\{d_k\}$ exists, so it is unreasonable to assert this.

Remark 2.4. In the proof of Theorem 2.1 from [7], since $\{y_n\}$ is a Cauchy sequence in $f(X)$, it is easily seen that $fx_n \to z \in X$ because X is complete. So it follows that $ffx_n \to fz \in X$ since f is continuous. Moreover, a key assertion, namely, $gfx_n \to fz \in X$ must be proved. In order to do this, the authors of [3] utilize the R-weakly commuting condition and get

$$d\left(fgx_n, gfx_n\right) \leq Rd\left(fx_n, gx_n\right) \quad (n \in \mathbb{N}). \tag{2.1}$$

Based on (2.1), by Lemma 1.2 from [7] the authors take the upper limit as $n \to \infty$ and obtain

$$\begin{aligned} \frac{1}{b^2} d\left(fz, \limsup_{n\to\infty} gfx_n\right) &\leq \limsup_{n\to\infty} d\left(fgx_n, gfx_n\right) \\ &\leq \limsup_{n\to\infty} d\left(fx_n, gx_n\right) \\ &\leq Rb^2 d(z,z) \\ &= 0 \end{aligned} \tag{2.2}$$

which means that

$$\frac{1}{b^2} d\left(fz, \limsup_{n\to\infty} gfx_n\right) = 0. \tag{2.3}$$

Similarly, they also get

$$\frac{1}{b^2} d\left(fz, \liminf_{n\to\infty} gfx_n\right) = 0. \tag{2.4}$$

By (2.3) together with (2.4) they obtain

$$\lim_{n\to\infty} gfx_n = fz.$$

Note that there is something wrong with (2.2), (2.3) and (2.4) since the notation "$\limsup_{n\to\infty} gfx_n$" and "$\liminf_{n\to\infty} gfx_n$" are not well defined. Indeed, the authors never give the definition of the upper limit or upper lower limit of a sequence in a b–metric space. This may be the crucial gap in the proof of the main theorem in [7].

Remark 2.5. *In Eq. (27) from [7], the formula*

$$\frac{1}{b^2} d\left(fz, fz_1\right) \leq \limsup_{n\to\infty} d\left(ffx_n, fz\right)$$

should be replaced by

$$\frac{1}{b^2} d\left(fz, fz_1\right) \leq \limsup_{n\to\infty} d\left(ffx_n, fz_1\right).$$

Now we present a new proof of Theorem 2.1

Proof . Let x_0 be any given point in X. From (a), we can find an $x_1 \in X$ satisfying $fx_0 = gx_1$. In general, we can find the point $x_{n+1} \in X$ satisfying $fx_n = gx_{n+1}$ for any $n \geq 0$. Let $y_n = fx_n$. Now we show that $\{y_n\}$ is a Cauchy sequence in $f(X)$. As is indicated in (11) from [7], the following is obvious:

$$\begin{aligned} d(y_n, y_{n+1}) &= d(fx_n, fx_{n+1}) \\ &\leq \gamma\left(\frac{1}{b^4} d(gx_n, gx_{n+1})\right) \\ &= \gamma\left(\frac{1}{b^4} d(fx_{n-1}, fx_n)\right) \\ &\leq \frac{1}{b^4} d(y_{n-1}, y_n) \\ &= \lambda d(y_{n-1}, y_n), \end{aligned}$$

which implies that

$$d(y_n, y_{n+1}) \leq \lambda d(y_{n-1}, y_n)$$

for all $n \in \mathbb{N}$, where $\lambda = \frac{1}{b^4} \in [0, \frac{1}{b})$. So it follows from Lemma 3.1 in [8] that $\{y_n\}$ is a Cauchy sequence in $f(X)$. Noting that $f(X) \subset X$ and X is complete, we see there exists $z \in X$ such that $\{fx_n\}$ converges to $z \in X$. Also, $\{gx_n\}$ converges to $z \in X$. So it follows that $ffx_n \to fz$, $fgx_n \to fz$ since f is continuous. Further, we have also that $gfx_n \to fz$. In fact,

$$\begin{aligned} d(fz, gfx_n) &\leq b\left[d(fz, fgx_n) + d(fgx_n, gfx_n)\right] \\ &\leq bd(fz, fgx_n) + bRd(fx_n, gx_n) \\ &\leq bd(fz, fgx_n) + Rb^2 d(fx_n, z) + Rb^2 d(z, gx_n) \\ &\to 0 + 0 + 0 = 0, \end{aligned}$$

which implies that $gfx_n \to fz$.

Next, we prove $z = fz$ by contradiction method. Indeed, if it is not true, then by Lemma 1.2 in [1] and (c) we have

$$\begin{aligned} \frac{1}{b^2} d(fz, z) \leq \limsup_{n\to\infty} d(ffx_n, fx_n) &\leq \limsup_{n\to\infty} \gamma\left(\frac{1}{b^4} d(gfx_n, gx_n)\right) \\ &= \gamma\left(\frac{1}{b^4} \limsup_{n\to\infty} d(gfx_n, gx_n)\right) \\ &\leq \gamma\left(\frac{1}{b^4} b^2 d(fz, z)\right) \\ &= \gamma\left(\frac{1}{b^2} d(fz, z)\right) < \frac{1}{b^2} d(fz, z), \end{aligned}$$

a contradiction. Hence, $z = fz$. Since $f(X) \subset g(X)$, there is $z_1 \in X$ such that $z = fz = gz_1$. So we have

$$d(ffx_n, fz_1) \leq \gamma\left(\frac{1}{b^4} d(gfx_n, gz_1)\right) \quad \forall n \in \mathbb{N}.$$

Then, by $\gamma(0) = 0$, it follows from Lemma 1.2 in [1] that

$$\begin{aligned}\frac{1}{b}d\,(fz, z_1) &\leq \limsup_{n\to\infty} d(ffx_n, fz_1)\\ &\leq \gamma\left(\limsup_{n\to\infty}\frac{1}{b^4}d\,(gfx_n, gz_1)\right)\\ &\leq \gamma\left(\frac{1}{b^3}d\,(fz, gz_1)\right)\\ &= 0,\end{aligned}$$

which implies that $fz = fz_1$. So, $z = fz = fz_1 = gz_1$. Hence, by the fact that f and g are R-weakly commuting we see that

$$d(fz, gz) = d(fgz_1, gfz_1) \leq Rd(fz_1, gz_1) = 0,$$

which implies that $fz = gz$. That is, z is a common fixed point of f and g.

Finally, we need to prove the uniqueness of the common fixed point of f and g. This can be done in the same way as in the proof of Theorem 12 from [7], so we omit it for convenience. □

Remark 2.6. Compared to the proof of Theorem 12 from [7], our proof for Theorem 2.1 above gives a new method to show the assertion $\limsup_{n\to\infty} gfx_n = fz$, filling the gap caused by the unwell–defined notations $\limsup_{n\to\infty} gfx_n$ and $\liminf_{n\to\infty} gfx_n$ appearing in (22) and (23) from [7], respectively.

Remark 2.7. In Theorem 2.1, if we replace the condition "(X, d) is a complete b–metric space" by "(X, d) is a b–metric space and $f(X)$ or $g(X)$ is complete", while the rest remains unchanged, then the conclusion also holds true.

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