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A Unique Common Fixed Point Theorem for Six Maps in G-metric Spaces

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Abstract

In this paper we obtain a unique common fixed point theorem for six weakly compatible mappings in G-metric spaces.

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1. Introduction

Dhage [1, 2, 3, 4]et al. introduced the concept of D-metric spaces as generalization of ordinary metric functions and went on to presentseveral fixed point results for single and multivalued mappings. Mustafa and Sims [12] and Naidu et al. [8, 9, 10] demonstrated that most of the claims concerning the fundamental topological structure of D-metric space are incorrect,alternatively, Mustafa and Sims introduced in [13] more appropriate notion of generalized metric space which called G-metric spaces, and obtained some topological properties. Later Zead Mustafa , Hamed Obiedat and Fadi Awawdeh [13], Mustafa , Shatanawi and Bataineh [15], Mustafa and Sims [16], Shatanawi [11] and Renu Chugh, Tamanna Kadian, Anju Rani and B.E.Rhoades [7] et al. obtained some fixed point theorems for a single map in G-metric spaces. In this paper, we obtain a unique common fixed point theorem for six weakly compatible mappings in G-metric spaces and obtain some theorems of [11] as corollaries to our theorem. First, we present some known definitions and propositions in G-metric spaces.

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Definition 1.1. *[13]. Let X be a nonempty set and let $G : X \times X \times X \to R^+$ be a function satisfying the following properties :*
(G_1): *$G(x, y, z) = 0$ if $x = y = z$,*
(G_2): *$0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,*
(G_3): *$G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,*
(G_4): *$G(x, y, z) = G(x, z, y) = G(y, z, x) = ...$,symmetry in all three variables,*
(G_5): *$G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.*
Then the function G is called a generalized metric or a G-metric on X and the pair (X, G) is called a G-metric space.

Definition 1.2. *[13]. Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. A point $x \in X$ is said to be limit of $\{x_n\}$ iff $\lim_{n,m\to\infty} G(x, x_n, x_m) = 0$. In this case, the sequence $\{x_n\}$ is said to be G-convergent to x.*

Definition 1.3. *[13]. Let (X, G) be a G-metric space and $\{x_n\}$ be a sequence in X. $\{x_n\}$ is called G-Cauchy iff $\lim_{n,\ m,\ l\to\infty} G(x_l, x_n, x_m) = 0$. (X, G) is called G-complete if every G-Cauchy sequence in (X, G) is G-convergent in (X, G).*

Proposition 1.4. *[13] In a G-metric space, (X, G), the following are equivalent.*
(1) *The sequence $\{x_n\}$ is G-Cauchy.*
(2) *For every $\epsilon > 0$, there exists $N \in \mathbf{N}$ such that $G(x_n, x_m, x_m) < \epsilon$, for all $n, m \geq N$.*

Proposition 1.5. *[13].Let (X, G) be a G-metric space. Then the function $G(x, y, z)$ is jointly continuous in all three of its variables.*

Proposition 1.6. *[13]. Let (X, G) be a G-metric space. Then for any $x, y, z, a \in X$, it follows that*
(i) if $G(x, y, z) = 0$ then $x = y = z$,
(ii) $G(x, y, z) \leq G(x, x, y) + G(x, x, z)$,
(iii) $G(x, y, y) \leq 2G(x, x, y)$,
(iv) $G(x, y, z) \leq G(x, a, z) + G(a, y, z)$,
(v) $G(x, y, z) \leq \frac{2}{3}[G(x, a, a) + G(y, a, a) + G(z, a, a)]$.

Proposition 1.7. *[13].Let (X, G) be a G-metric space. Then for a sequence $\{x_n\} \subseteq X$ and a point $x \in X$, the following are equivalent*
(i) $\{x_n\}$ is G-convergent to x,
(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,
(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,
(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

Definition 1.8. *[13] Let (X, G) and (X', G') be two G-metric spaces, and let $f : (X, G) \to (X', G')$ be a function, then f is said to be G-*continuous at a point *$a \in X$ if and only if, given $\epsilon > 0$, there exists $\delta > 0$ such that $x, y \in X$; and $G(a, x, y) < \delta$ implies $G'(f(a), f(x), f(y)) < \epsilon$. A function f is G-continuous at X if and only if it is G-continuous at all $a \in X$.*

Proposition 1.9. *[13] Let (X, G), and (X', G') be two G-metric spaces. Then a function $f : X \longrightarrow X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x; that is, whenever (x_n) is G-convergent to x we have $(f(x_n))$ is G-convergent to $f(x)$.*

Definition 1.10. *[5] A pair of self mappings is called weakly compatible if they commute at their coincidence points.*

2. Main Results

Following to Matkowski [6], let Φ denote the set of all continuous nondecreasing functions $\phi : [0,\infty) \to [0,\infty)$ such that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$. It is clear that $\phi(t) < t$ for all $t > 0$ and $\phi(0) = 0$.

Theorem 2.1. *Let (X, G) be a G-metric space and $S, T, R, f, g, h : X \to X$ be satisfying*
(i) $S(X) \subseteq g(X), T(X) \subseteq h(X)$ and $R(X) \subseteq f(X)$,
(ii) one of $f(X), g(X)$ and $h(X)$ is a complete subspace of X,
(iii) the pairs $(S, f), (T, g)$ and (R, h) are weakly compatible, and

$$(iv)\ G(Sx, Ty, Rz) \leq \phi\left(\max\left\{\begin{array}{c} G(fx, gy, hz), \\ \frac{1}{3}[G(fx, Sx, Ty) + G(gy, Ty, Rz) + G(hz, Rz, Sx)], \\ \frac{1}{4}[G(fx, Ty, hz) + G(Sx, gy, hz) + G(fx, gy, Rz)] \end{array}\right\}\right)$$

for all $x, y, z \in X$, where $\phi \in \Phi$.
Then either one of the pairs $(S, f), (T, g)$ and (R, h) has a coincidence point or the maps S, T, R, f, g and h have a unique common fixed point in X.

Proof . Choose $x_0 \in X$. By (i), there exist $x_1, x_2, x_3 \in X$ such that $Sx_0 = gx_1 = y_0, say$, $Tx_1 = hx_2 = y_1, say$ and $Rx_2 = fx_3 = y_2, say$.
Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that
$y_{3n} = Sx_{3n} = gx_{3n+1}$, $y_{3n+1} = Tx_{3n+1} = hx_{3n+2}$, and $y_{3n+2} = Rx_{3n+2} = fx_{3n+3}$, where $n = 0, 1, 2, ...$
If $y_{3n} = y_{3n+1}$ then x_{3n+1} is a coincidence point of g and T.
If $y_{3n+1} = y_{3n+2}$ then x_{3n+2} is a coincidence point of h and R.
If $y_{3n+2} = y_{3n+3}$ then x_{3n+3} is a coincidence point of f and S.
Now assume that $y_n \neq y_{n+1}$ for all n.
Denote $d_n = G(y_n, y_{n+1}, y_{n+2})$.
Putting $x = x_{3n}, y = x_{3n+1}, z = x_{3n+2}$ in (iv), we get

$$d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2}) =$$
$$G(Sx_{3n}, Tx_{3n+1}, Rx_{3n+2})$$
$$\leq \phi\left(\max\left\{\begin{array}{c} G(fx_{3n}, gx_{3n+1}, hx_{3n+2}), \frac{1}{3}[G(fx_{3n}, Sx_{3n}, Tx_{3n+1}) + \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) + G(hx_{3n+2}, Rx_{3n+2}, Sx_{3n})], \\ \frac{1}{4}[G(fx_{3n}, Tx_{3n+1}, hx_{3n+2}) + G(Sx_{3n}, gx_{3n+1}, hx_{3n+2}) \\ +G(fx_{3n}, gx_{3n+1}, Rx_{3n+2})] \end{array}\right\}\right)$$
$$= \phi\left(\max\left\{\begin{array}{c} G(y_{3n-1}, y_{3n}, y_{3n+1}), \frac{1}{3}[G(y_{3n-1}, y_{3n}, y_{3n+1}) + \\ G(y_{3n}, y_{3n+1}, y_{3n+2}) + G(y_{3n+1}, y_{3n+2}, y_{3n})], \\ \frac{1}{4}[G(y_{3n-1}, y_{3n+1}, y_{3n+1}) + G(y_{3n}, y_{3n}, y_{3n+1}) \\ +G(y_{3n-1}, y_{3n}, y_{3n+2})] \end{array}\right\}\right)$$
$$\leq \phi\left(\max\left\{\begin{array}{c} d_{3n-1}, \frac{1}{3}[d_{3n-1} + d_{3n} + d_{3n}], \\ \frac{1}{4}[d_{3n-1} + d_{3n} + (d_{3n-1} + d_{3n})] \end{array}\right\}\right) \quad (1)$$

If $d_{3n} \geq d_{3n-1}$ then from (1),we have $d_{3n} \leq \phi(d_{3n}) < d_{3n}$. It is a contradiction. Hence $d_{3n} \leq d_{3n-1}$.
Now from (1), $d_{3n} \leq \phi(d_{3n-1})$.
Similarly, by putting $x = x_{3n+3}, y = x_{3n+1}, z = x_{3n+2}$ and $x = x_{3n+3}$,
$y = x_{3n+4}, z = x_{3n+2}$ in (iv), we get

$$d_{3n+1} \leq \phi(d_{3n}) \quad (2)$$

and

$$d_{3n+2} \leq \phi(d_{3n+1}) \quad (3)$$

respectively. Thus from (1),(2) and (3), we have

$$\begin{aligned} G(y_n, y_{n+1}, y_{n+2}) &\le \phi(G(y_{n-1}, y_n, y_{n+1}) \\ &\le \phi^2(G(y_{n-2}, y_{n-1}, y_n) \\ &. \\ &. \\ &. \\ &\le \phi^n(G(y_0, y_1, y_2)) \qquad (4) \end{aligned}$$

From (G_3) and (4), we have
$G(y_n, y_n, y_{n+1}) \le G(y_n, y_{n+1}, y_{n+2}) \le \phi^n(G(y_0, y_1, y_2))$.
Now for $m > n$, from (G_5) and (4), we have

$$\begin{aligned} G(y_n, y_n, y_m) &\le G(y_n, y_n, y_{n+1}) + G(y_{n+1}, y_{n+1}, y_{n+2}) + ... + G(y_{m-1}, y_{m-1}, y_m) \\ &\le \phi^n(G(y_0, y_1, y_2)) + \phi^{n+1}(G(y_0, y_1, y_2)) + ... + \phi^{m-1}(G(y_0, y_1, y_2)) \\ &\to 0 \quad as \quad n \to \infty, \text{ since } \phi^n(t) \to 0 \quad as \quad n \to \infty \quad for \quad all \quad t > 0 \ . \end{aligned}$$

Hence $\{y_n\}$ is G-Cauchy. Suppose $f(X)$ is G-complete.
Then there exist $p, t \in X$ such that $y_{3n+2} \to p = ft$. Since $\{y_n\}$ is G-Cauchy, it follows that $y_{3n} \to p$ and $y_{3n+1} \to p$ as $n \to \infty$.

$$\begin{aligned} &G(St, Tx_{3n+1}, Rx_{3n+2}) \\ &\le \phi\left(\max\left\{\begin{array}{c} G(ft, gx_{3n+1}, hx_{3n+2}), \frac{1}{3}[G(ft, St, Tx_{3n+1})+ \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) + G(hx_{3n+2}, Rx_{3n+2}, St)], \\ \frac{1}{4}[G(ft, Tx_{3n+1}, hx_{3n+2}) + G(St, gx_{3n+1}, hx_{3n+2}) \\ +G(ft, gx_{3n+1}, Rx_{3n+2})] \end{array}\right\}\right) \end{aligned}$$

Letting $n \to \infty$, we get

$$G(St, p, p) \le \phi\left(\max\left\{\begin{array}{c} 0, \frac{1}{3}[G(p, St, p) + 0 + G(p, p, St)], \\ \frac{1}{4}[0 + G(St, p, p) + 0)] \end{array}\right\}\right).$$

$G(St, p, p) \le \phi(G(St, p, p))$, since ϕ is nondecreasing.
Hence $St = p$. Thus $p = ft = St$.
Since the pair (S, f) is weakly compatible, we have $fp = Sp$.
Putting $x = p, y = x_{3n+1}, z = x_{3n+2}$ in (iv), we get

$$\begin{aligned} &G(Sp, Tx_{3n+1}, Rx_{3n+2}) \\ &\le \phi\left(\max\left\{\begin{array}{c} G(fp, gx_{3n+1}, hx_{3n+2}), \frac{1}{3}[G(fp, Sp, Tx_{3n+1})+ \\ G(gx_{3n+1}, Tx_{3n+1}, Rx_{3n+2}) + G(hx_{3n+2}, Rx_{3n+2}, Sp)], \\ \frac{1}{4}[G(fp, Tx_{3n+1}, hx_{3n+2}) + G(Sp, gx_{3n+1}, hx_{3n+2}) \\ +G(fp, gx_{3n+1}, Rx_{3n+2})] \end{array}\right\}\right) \end{aligned}$$

Letting $n \to \infty$, we have

$$G(Sp, p, p) \le \phi\left(\max\left\{\begin{array}{c} G(Sp, p, p), \frac{1}{3}[G(Sp, Sp, p) + 0 + G(p, p, Sp)], \\ \frac{1}{4}[G(Sp, p, p) + G(Sp, p, p) + G(Sp, p, p)] \end{array}\right\}\right)$$

Since $G(Sp, Sp, p) \le 2G(Sp, p, p)$, we have $G(Sp, p, p) \le \phi(G(Sp, p, p))$
Thus $Sp = p$. Hence $fp = Sp = p$. (5)

Since $p = Sp \in g(X)$, there exists $v \in X$ such that $p = gv$.
Putting $x = p, y = v, z = x_{3n+2}$ in (iv), we get

$$G(Sp, Tv, Rx_{3n+2}) \le \phi\left(\max\left\{\begin{array}{c} G(fp, gv, hx_{3n+2}), \frac{1}{3}[G(fp, Sp, Tv)+ \\ G(gv, Tv, Rx_{3n+2}) + G(hx_{3n+2}, Rx_{3n+2}, Sp)], \\ \frac{1}{4}[G(fp, Tv, hx_{3n+2}) + G(Sp, gv, hx_{3n+2}) \\ +G(fp, gv, Rx_{3n+2})] \end{array}\right\}\right)$$

Letting $n \to \infty$, we deduce that

$$G(p, Tv, p) \le \phi\left(\max\left\{\begin{array}{c} 0, \frac{1}{3}[G(p, p, Tv) + G(p, Tv, p) + 0], \\ \frac{1}{4}[G(p, Tv, p) + 0 + 0] \end{array}\right\}\right)$$

$$\le \phi(G(p, Tv, p)) \text{ , since } \phi \text{ is nondecreasing.}$$

Thus $Tv = p$, so that $p = Tv = gv$.
Since the pair (T, g) is weakly compatible, we have $Tp = gp$.

$$G(Sp, Tp, Rx_{3n+2}) \le \phi\left(\max\left\{\begin{array}{c} G(fp, gp, hx_{3n+2}), \frac{1}{3}[G(fp, Sp, Tp)+ \\ G(gp, Tp, Rx_{3n+2}) + G(hx_{3n+2}, Rx_{3n+2}, Sp)], \\ \frac{1}{4}[G(fp, Tp, hx_{3n+2}) + G(Sp, gp, hx_{3n+2}) \\ +G(fp, gp, Rx_{3n+2})] \end{array}\right\}\right)$$

Letting $n \to \infty$, we have

$$G(p, Tp, p) \le \phi\left(\max\left\{\begin{array}{c} G(p, Tp, p), \frac{1}{3}[G(p, p, Tp) + G(Tp, Tp, p) + 0], \\ \frac{1}{4}[G(p, Tp, p) + G(p, Tp, p) + G(p, Tp, p)] \end{array}\right\}\right),$$

Since $G(Tp, Tp, p) \le 2G(Tp, p, p)$,we have, $G(p, Tp, p) \le \phi(G(p, Tp, p))$.
Thus $Tp = p$. Hence $gp = Tp = p$. (6)
Since $p = Tp \in h(X)$, there exists $w \in X$ such that $p = hw$.
Putting $x = p, y = p, z = w$ in (iv), we get

$$G(Sp, Tp, Rw) \le \phi\left(\max\left\{\begin{array}{c} G(fp, gp, hw), \frac{1}{3}[G(fp, Sp, Tp)+ \\ G(gp, Tp, Rw) + G(hw, Rw, Sp)], \\ \frac{1}{4}[G(fp, Tp, hw) + G(Sp, gp, hw) \\ +G(fp, gp, Rw)] \end{array}\right\}\right)$$

$$G(p, p, Rw) \le \phi\left(\max\left\{\begin{array}{c} 0, \frac{1}{3}[0 + G(p, p, Rw) + G(p, Rw, p)], \\ \frac{1}{4}[0 + 0 + G(p, p, Rw)] \end{array}\right\}\right)$$

$$\le \phi(G(p, p, Rw)) \text{ , since } \phi \text{ is nondecreasing.}$$

Thus $Rw = p$,so that $p = hw = Rw$.
Since the pair (R, h) is weakly compatible, we have $Rp = hp$.
Putting $x = p, y = p, z = p$ in (iv), we get

$$G(p, p, Rp) = G(Sp, Tp, Rp) \le \phi\left(\max\left\{\begin{array}{c} G(fp, gp, Rp), \frac{1}{3}[0+ \\ G(p, p, Rp) + G(Rp, Rp, p)], \\ \frac{1}{4}[G(p, p, Rp) + G(p, p, Rp) \\ +G(p, p, Rp)] \end{array}\right\}\right)$$

Since $G(Rp, Rp, p) \leq 2G(p, p, Rp)$, we have $G(p, p, Rp) \leq \phi(G(p, p, Rp))$.
Thus $Rp = p$ so that $Rp = hp = p$. (7)

From (5),(6) and (7), it follows that p is a common fixed point of S, T, R, f, g and h.

Uniqueness of common fixed point follows easily from (iv). Similarly, we can prove the theorem when $g(X)$ or $h(X)$ is a complete subspace of X □

Corollary 2.2. *Let (X, G) be a G-metric space and $S, T, R, f, g, h : X \to X$ be satisfying*
(i) $S(X) \subseteq g(X), T(X) \subseteq h(X)$ and $R(X) \subseteq f(X)$,
(ii) one of $f(X), g(X)$ and $h(X)$ is a complete subspace of X,
(iii) the pairs $(S, f), (T, g)$ and (R, h) are weakly compatible and
(iv)$G(Sx, Ty, Rz) \leq \phi(G(fx, gy, hz))$
for all $x, y, z \in X$, where $\phi \in \Phi$.
Then the maps S, T, R, f, g and h have a unique common fixed point in X.

Corollary 2.3. *Let (X, G) be a complete G-metric space and $S, T, R : X \to X$ be satisfying $G(Sx, Ty, Rz) \leq \phi(G(x, y, z))$ for all $x, y, z \in X$, where $\phi \in \Phi$.*
Then the maps S, T and R have a unique common fixed point,say, $p \in X$ and S, T and R are G-continuous at p.

Proof . *There exists $p \in X$ such that p is the unique common fixed point of S, T and R as in Theorem 2.1.*
Let $\{y_n\}$ be any sequence in X which G-converges to p.
Then
$G(Sy_n, Sp, Sp) = G(Sy_n, Tp, Rp) \leq \phi(G(y_n, p, p)) \to 0$ *as* $n \to \infty$.
Hence S is G-continuous at p.
Similarly, we can show that T and R are also G-continuous at p. □

Remark 2.4. *Theorem* 3.1, *Corollaries* 3.2 *to* 3.5 *of [11] follows from Corollary* 2.3 *with $S = T = R$.*

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