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# Common Fixed Point of Generalized $(\psi - \varphi)$-Weak Contraction Mappings

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**Abstract**

Let $(X, d)$ be a complete metric space and let $f, g : X \longrightarrow X$ be two mappings which satisfy a $(\psi - \varphi)$-weak contraction condition or generalized $(\psi - \varphi)$-weak contraction condition. Then $f$ and $g$ have a unique common fixed point. Our results extend previous results given by Ćirić (1971), Rhoades (2001), Branciari (2002), Rhoades (2003), Abbas and Ali Khan (2009), Zhang and Song (2009) and Moradi at. el. (2011).

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## 1. Introduction

Let $(X, d)$ be a metric space. A mapping $f : X \longrightarrow X$ is said to be $\varphi-$weak contraction if there exists a map $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(fx, fy) \leq d(x, y) - \varphi(d(x, y)), \tag{1.1}$$

for all $x, y \in X$.
Also a mapping $f : X \longrightarrow X$ is said to be generalized $\varphi-$weak contraction if there exists a map $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$ such that

$$d(fx, fy) \leq N(x, y) - \varphi(N(x, y)), \tag{1.2}$$

for all $x, y \in X$, where

$$N(x, y) := \max\Big\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\Big\}.$$

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The concept of the $\varphi-$weak contraction was defined by Alber and Guerre-Delabriere [2] in 1997, and the generalized $\varphi-$weak contraction was defined by Zhang and Song [15] in 2009. In the following theorem Rhoades [12] extended the Banach Contraction Principle to $\psi-$weak contraction mappings.

**Theorem 1.1.** *Let $(X, d)$ be a complete metric space, and let $T : X \longrightarrow X$ be a mapping such that*

$$d(Tx, Ty) \leq d(x, y) - \varphi(d(x, y)), \tag{1.3}$$

*for every $x, y \in X$ (i.e. $\varphi$-weak contractive), where $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ is a continuous and nondecreasing function with $\varphi(0) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Then $T$ has a unique fixed point.*

In 2009 Zhang and Song [15] extended the Rhoades's Theorem. After that Moradi at. el. [9] generalized the Zhang and Song's result. Their result also, extended Ćirić's theorem [5].
Sessa [14] defined the concept of weakly commuting to obtain common fixed point for pairs of mappings. Jungck generalized this idea, first to compatible mapping [6] and then to weakly compatible mappings [7].

**Definition 1.2.** *Two mapping $f, g : X \longrightarrow X$ are said to be weakly compatible if they commute at their coincidence points (a point $x$ is said to be a coincidence point of $f$ and $g$ if and only if $fx = gx$).*

Recently, Abass and Ali Khan [1] proved the following theorem on the existence of a common fixed point for two mappings. Their result extended the Rhoades's Theorem.

**Theorem 1.3.** *Let $f, g$ be two self maps of a metric space $(X, d)$ satisfying,*

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \varphi(d(gx, gy)), \tag{1.4}$$

*for all $x, y \in X$, where $\psi, \varphi : [0, +\infty) \longrightarrow [0, +\infty)$ are two continuous and nondecreasing functions with $\psi(0) = \varphi(0) = 0$ and $\psi(t) > 0$ and $\varphi(t) > 0$ for all $t > 0$. If range of $g$ contains the range of $f$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

Many authors have studied fixed point and common fixed point for weak and generalized weak contraction mappings. Among many others, see, for example [3-8-10-13], and the references therein. The aim of this paper is to present common fixed point theorems for weakly compatible mappings satisfying a $(\psi-\varphi)$-weak and generalized $(\psi-\varphi)$-weak contractive condition. Our results substantially extend the previous results given by Ćirić [5], Rhoades [12], Branciari [4], Rhoades [11], Abbas and Ali Khan [1], Zhang and Song [15] and Moradi at. el. [9].

## 2. Preliminaries

In this work, $(X, d)$ denotes a complete metric space.
Let $G$ denotes the class of all nondecreasing and continuous mapping $\psi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\psi(0) = 0$ and $\psi(t) > 0$ for all $t > 0$. Also let $D$ denotes the class of all nondecreasing and lower semi-continuous from the right mapping $\varphi : [0, +\infty) \longrightarrow [0, +\infty)$ with $\varphi(t) = 0$ and $\varphi(t) > 0$ for all $t > 0$. Obviously, $G \subset D$.

**Definition 2.1.** *Let $(X, d)$ be a metric space and let $f, g : X \longrightarrow X$ be two self mappings. The pair $(f, g)$ are said to be $(\psi - \varphi)-$weak contraction, if there exist $\psi \in G$ and $\varphi \in D$ such that*

$$\psi(d(fx, fy)) \leq \psi(d(gx, gy)) - \varphi(d(gx, gy)), \tag{2.1}$$

*for all $x, y \in X$.*

**Definition 2.2.** *Let $(X, d)$ be a metric space and let $f, g : X \longrightarrow X$ be two self mappings. The pair $(f, g)$ are said to be generalized $(\psi - \varphi)-$weak contraction, if there exist $\psi \in G$ and $\varphi \in D$ such that*

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{2.2}$$

*for all $x, y \in X$, where*

$$M(x, y) := \max\Big\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2}\Big\}.$$

## 3. Main Results

In this section we aim to present our main result. At first we generalized the Abbas and Ali Khan's theorem for generalized $(\psi - \varphi)-$weak contraction mappings. After that, by a similar method, we can extend the Abbas and Ali Khan's theorem for $(\psi - \varphi)-$weak contraction mappings. These generalizations also extend Ćirić, Rhoades, Branciari, Zhang and Song and Moradi at. el.'s theorems .

**Theorem 3.1.** *Let $(X, d)$ be a metric space and let $f, g : X \longrightarrow X$ be two self mappings, which satisfying,*

$$\psi(d(fx, fy)) \leq \psi(M(x, y)) - \varphi(M(x, y)), \tag{3.1}$$

*for all $x, y \in X$ (i.e. generalized $(\psi - \varphi)-$weak contraction), where $\psi \in G$ and $\varphi \in D$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$ (there exists $x \in X$ such that $fx = gx$). Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

**Proof .** Let $x_0 \in X$. Using $f(X) \subseteq g(X)$ there exist tow sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ such that $y_n = fx_n = gx_{n+1}$ for all $n \geq 0$.
At first we prove that $f$ and $g$ have a unique point of coincidence.
Unicity of the point of coincidence follows from (3.1). Suppose, for any $n$, $y_n \neq y_{n+1}$, since, otherwise, $f$ and $g$ have a point of coincidence. We break the argument into four steps.
**Step 1.** $\lim_{n\to\infty} d(y_{n+1}, y_n) = 0$.
**proof.** Using (3.1),

$$\begin{aligned} \psi(d(y_{n+1}, y_n)) &= \psi(d(fx_{n+1}, fx_n)) \\ &\leq \psi(M(x_{n+1}, x_n)) - \varphi(M(x_{n+1}, x_n)), \end{aligned} \tag{3.2}$$

where,

$$\begin{aligned} d(y_n, y_{n-1}) &\leq M(x_{n+1}, x_n) = \max\{d(y_n, y_{n-1}), d(y_n, y_{n+1}), d(y_{n-1}, y_n), \\ &\quad \frac{1}{2}[d(y_n, y_n) + d(y_{n-1}, y_{n+1})]\} \\ &\leq \max\{d(y_n, y_{n-1}), d(y_n, y_{n+1})\} = d(y_n, y_{n-1}). \qquad (by\ 3.2) \end{aligned} \tag{3.3}$$

So $M(x_{n+1}, x_n) = d(y_n, y_{n-1})$. Hence, by (3.2)

$$\psi(d(y_{n+1}, y_n)) < \psi(d(y_n, y_{n-1})). \tag{3.4}$$

Since $\psi$ is nondecreasing

$$d(y_{n+1}, y_n) < d(y_n, y_{n-1}), \tag{3.5}$$

for all $n \in \mathbb{N}$. Therefore the sequence $\{d(y_{n+1}, y_n)\}$ is monotone nondecreasing and bounded below. So there exists $r \geq 0$ such that

$$d(y_{n+1}, y_n) \to r^+ \ as \ n \to \infty. \tag{3.6}$$

Since $\psi$ is nondecreasing, from 3.2

$$\psi(r) \leq \psi(d(y_{n+1}, y_n)) \leq \psi(d(y_n, y_{n-1})) - \varphi(d(y_n, y_{n-1})). \tag{3.7}$$

for all $n \in \mathbb{N}$. By $\psi \in G$, $\varphi \in D$ and using 3.6

$$\psi(r) \leq \psi(r) - \varphi(r), \tag{3.8}$$

and so $\psi(r) = 0$. Hence $r = 0$.
**Step 2.** $\{y_n\}$ is Cauchy.
**proof.**If $\{y_n\}$ is not a Cauchy sequence, then there exists an $\varepsilon > 0$ such that for each $k \in \mathbb{N}$ there are $m(k), n(k) \in \mathbb{N}$, with $k < n(k) < m(k)$, such that $d(y_{n(k)}, y_{m(k)}) \geq \varepsilon$, then we chose the sequences $\{m(k)\}$ and $n(k)$ such that for each $k \in \mathbb{N}$, $m(k)$ is minimal in the sense that $d(y_{n(k)}, y_{m(k)}) \geq \varepsilon$ but $d(y_{n(k)}, y_m) < \varepsilon$ for each $k \in \{n(k)+1, n(k)+2, ..., m(k)-1\}$.
From Step 1, for large enough $k$, we have $d(y_{n(k)+1}, y_{n(k)}) < \frac{\varepsilon}{2}$ and $d(y_{m(k)+1}, y_{m(k)}) < \frac{\varepsilon}{2}$. Hence, for large enough $k$, $m(k) - n(k) \geq 2$ and

$$\begin{aligned} \varepsilon < d(y_{n(k)}, y_{m(k)}) &\leq d(y_{n(k)}, y_{m(k)-1}) + d(y_{m(k)-1}, y_{m(k)}) \\ &\leq \varepsilon + d(y_{m(k)-1}, y_{m(k)}). \end{aligned} \tag{3.9}$$

This inequality shows that $d(y_{n(k)}, y_{m(k)}) \to \varepsilon^+$ as $k \to \infty$. Furthermore

$$\begin{aligned} & d(y_{n(k)}, y_{m(k)}) - d(y_{n(k)}, y_{n(k)+1}) - d(y_{m(k)}, y_{m(k)+1}) \\ \leq\ & d(y_{n(k)+1}, y_{m(k)+1}) \\ \leq\ & d(y_{n(k)}, y_{m(k)}) + d(y_{n(k)}, y_{n(k)+1}) + d(y_{m(k)}, y_{m(k)+1}), \end{aligned} \tag{3.10}$$

and this shows that $n \to \infty \lim d(y_{n(k)+1}, y_{r(k)+1}) = \varepsilon$. Also from (3.1)

$$\psi(d(y_{n(k)+1}, y_{m(k)+1})) \leq \psi(M(x_{n(k)+1}, x_{m(k)+1})) - \varphi(M(x_{n(k)+1}, x_{m(k)+1})), \tag{3.11}$$

where

$$\begin{aligned} d(y_{n(k)}, y_{m(k)}) &\leq M(x_{n(k)+1}, x_{m(k)+1}) \\ &= \max\Big\{d(y_{n(k)}, y_{m(k)}), d(y_{n(k)}, y_{n(k)+1}), d(y_{m(k)}, y_{m(k)+1}), \\ &\quad \frac{d(y_{n(k)}, y_{m(k)+1}) + d(y_{m(k)}, y_{n(k)+1})}{2}\Big\} \\ &\leq \max\Big\{d(y_{n(k)}, y_{m(k)}), d(y_{n(k)}, y_{n(k)+1}), d(y_{m(k)}, y_{m(k)+1}), \\ &\quad \frac{2d(y_{n(k)}, y_{m(k)}) + d(y_{m(k)+1}, y_{m(k)}) + d(y_{n(k)+1}, y_{n(k)})}{2}\Big\} \\ &\leq d(y_{n(k)}, y_{m(k)}) + d(y_{m(k)+1}, y_{m(k)}) + d(y_{n(k)+1}, y_{n(k)}). \end{aligned} \tag{3.12}$$

Letting $k \to \infty$ in above inequality, we conclude that $M(x_{n(k)+1}, x_{m(k)+1}) \to \varepsilon^+$ as $k \to \infty$.
Hence from 3.14,

$$\psi(\varepsilon) \leq \psi(\varepsilon) - \varphi(\varepsilon). \tag{3.13}$$

Therefore $\psi(\varepsilon) = 0$ and this is a contradiction.
**Step 3.** $f$ and $g$ have a point of coincidence.
**proof.** Since $(X, d)$ is complete and $\{y_n\}$ is a Cauchy sequence, there exists $z \in X$ such that $n \to \infty \lim y_n = z$. Since $g(X)$ is closed, then $z \in g(X)$. So there exists $u \in X$ such that $g(u) = z$. From (3.1), we have

$$\psi(d(fu, fx_n)) \leq \psi(M(u, x_n)) - \varphi(M(u, x_n)), \tag{3.14}$$

where

$$M(u, x_n) = \max\Big\{d(z, y_{n-1}), d(z, fu), d(y_{n-1}, y_n), \frac{d(z, y_n) + d(y_{n-1}, fu)}{2}\Big\}, \tag{3.15}$$

and this shows that $M(u, x_n) \to d(z, fu)^+$ as $n \to \infty$.
Since $\psi \in G$, $\varphi \in D$ and (3.14) holds, we conclude that

$$\psi(d(fu, z)) \leq \psi(d(z, fu)) - \varphi(d(z, fu)), \tag{3.16}$$

and hence $d(z, fu) = 0$. So $fu = z = gu$. Therefore $z$ is a point of coincidence of $f$ and $g$.
Now, if $f$ and $g$ are weakly compatible, then we prove that $z$ is a common fixed point of $f$ and $g$. Since $fu = gu = z$ and $f$ and $g$ are weakly compatible, then $fz = gz$. Using (3.1)

$$\psi(d(fz, fx_n)) \leq \psi(M(z, x_n)) - \varphi(M(z, x_n)), \tag{3.17}$$

where

$$M(z, x_n) = \max\Big\{d(gz, gx_n), d(gz, fz), d(gx_n, fx_n), \frac{d(gz, fx_n) + d(gx_n, fz)}{2}\Big\}. \tag{3.18}$$

This inequality shows that $n \to \infty \lim M(z, x_n) = d(z, fz)$. We need to show that $z = fz$.
If $z \neq fz$ then for $\varepsilon_0 = \frac{d(z,fz)}{2}$, there exists $N_0 \in \mathbb{N}$ such that for all $n \geq N_0$, $M(z, x_n) \geq \varepsilon_0$. Since $\varphi$ is nondecreasing, from (3.17)

$$\psi(d(fz, fx_n)) \leq \psi(M(z, x_n)) - \varphi(\varepsilon_0), \tag{3.19}$$

for all $n \geq N_0$. Letting $n \to \infty$ in above inequality, we conclude that

$$\psi(d(fz, z)) \leq \psi(d(z, fz)) - \varphi(\varepsilon_0), \tag{3.20}$$

and this shows that $\varphi(\varepsilon_0) = 0$. This is a contradiction.
So $z = fz$, and hence, $gz = fz = z$. Therefore $f$ and $g$ have a common fixed point.
Uniquness of the common fixed point follows from (3.1), and this completes the proof. □

The following corollary extend Ćirić and Rhoades's Theorems.

**Corollary 3.2.** *Let $(X, d)$ be a metric space and let $f, g : X \longrightarrow X$ be two self mappings, which satisfying,*

$$\int_0^{d(fx,fy)} \eta(t)dt \leq \int_0^{M(x,y)} \eta(t)dt - \int_0^{M(x,y)} \theta(t)dt, \tag{3.21}$$

*for all $x, y \in X$, where $\eta, \theta : [0, +\infty] \longrightarrow [0, +\infty]$ are two Lebesgue integrable mappings which are summable and satisfy $\int_0^{\varepsilon} \eta(t)dt > 0$ and $\int_0^{\varepsilon} \theta(t)dt > 0$ for each $\varepsilon > 0$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

**Proof .** Define $\psi, \varphi : [0, +\infty) \longrightarrow [0, +\infty)$ by $\psi(x) = \int_0^x \eta(t)dt$ and $\varphi(x) = \int_0^x \theta(t)dt$. Obviously $\psi \in G$ and $\varphi \in D$. Hence by Theorem 3.1 $f$ and $g$ have a unique point of coincidence and if $f$ and $g$ are weakly compatible then have a unique common fixed point. □

The following theorem is another main result of this paper.

**Theorem 3.3.** *Let $(X, d)$ be a metric space and let $f, g : X \longrightarrow X$ be two self mappings, which satisfying,*

$$\psi(d(fx, fy)) \le \psi(d(gx, gy)) - \varphi(d(gx, gy)), \tag{3.22}$$

*for all $x, y \in X$ (i.e. $(\psi - \varphi)-$weak contraction), where $\psi \in G$ and $\varphi \in D$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

**Proof .** The proof is similar to the proof of Theorem 3.1, by replacing $M(x, y)$ with $d(gx, gy)$. □
The following corollary extends Branciari's theorems.

**Corollary 3.4.** *Let $(X, d)$ be a metric space and let $f, g : X \longrightarrow X$ be two self mappings, which satisfying,*

$$\int_0^{d(fx,fy)} \eta(t)dt \le \int_0^{d(gx,gy)} \eta(t)dt - \int_0^{d(gx,gy)} \theta(t)dt, \tag{3.23}$$

*for all $x, y \in X$, where $\eta, \theta : [0, +\infty] \longrightarrow [0, +\infty]$ are two Lebesgue integrable mappings which are summable and satisfy $\int_0^\varepsilon \eta(t)dt > 0$ and $\int_0^\varepsilon \theta(t)dt > 0$ for each $\varepsilon > 0$. If $f(X) \subseteq g(X)$ and $g(X)$ is a complete subspace of $X$, then $f$ and $g$ have a unique point of coincidence in $X$. Moreover, if $f$ and $g$ are weakly compatible, then $f$ and $g$ have a unique common fixed point.*

**Proof .** The proof is similar to the proof of Corollary 3.2. □ Now we present an example in the support of Theorem 3.1.

**Example 3.5.** *Let $X = [0, 2]$ be endowed with the Euclidean metric. Let $f, g : X \longrightarrow X$ be defined by $gx = x$ and $fx = 0$ for $x \in [0, \frac{3}{2})$ and $fx = \frac{1}{5}$ for $x \in [\frac{3}{2}, 2]$. Also let $\psi, \varphi : [0, +\infty) \longrightarrow [0, +\infty)$ be defined by $\psi(t) = \frac{t}{2}$ and $\varphi(t) = \frac{t}{3}$. We have*

$$\psi(d(f\frac{3}{2}, f1)) > \psi(d(\frac{3}{2}, 1)) - \varphi(d(\frac{3}{2}, 1)).$$

*So we can not use Abbas and Ali Khan's Theorem. But for all $x, y \in X$*

$$\psi(d(fx, fy)) \le \psi(M(x, y)) - \varphi(M(x, y)).$$

*Hence, from Theorem 3.1, $f$ and $g$ have a common fixed point.*

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