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Entropy of infinite systems and transformations

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Abstract

The Kolmogorov-Sinai entropy is a far reaching dynamical generalization of Shannon entropy of information systems. This entropy works perfectly for probability measure preserving (p.m.p.) transformations. However, it is not useful when there is no finite invariant measure. There are certain successful extensions of the notion of entropy to infinite measure spaces, or transformations with infinite invariant measures. The three main extensions are Parry, Krengel, and Poisson entropies. In this survey, we shortly overview the history of entropy, discuss the pioneering notions of Shannon and later contributions of Kolmogorov and Sinai, and discuss in somewhat more details the extensions to infinite systems. We compare and contrast these entropies with each other and with the entropy on finite systems.

Keywords: Infinite invariant measure, Kolmogorov-Sinai entropy, Parry entropy, Krengel entropy, Poisson entropy, Pinsker factor.
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1. A short history of entropy

1.1. Engineering and Physics

The early history of engineering had witnessed several major challenges, the greatest of which, with no doubt, was the issue of lost energy. This was a very serious concern in the early history of engineering, since the original manmade engines were converting less than two percent of the input energy into useful work output. This was the case for such primitive engines as Thomas Savery's steam device (1698), Thomas Newcomen's steam engine (1712), and Nicolas Joseph Cugnot's steam tricycle (1769). In all cases, a great deal of useful energy was lost due to dissipation or friction, mainly

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because of inefficiency of design and production. It took about two more centuries for physicists to find a way of solving the puzzle of lost energy, and the solution was nothing but the *entropy*.

In 1803, Lazare Carnot was the first who discussed the issue of efficiency of fundamental machines (in his work entitled *Fundamental Principles of Equilibrium and Movement*). He also discussed the conservation of (mechanical) energy, anticipating the second law of thermodynamics. One major motivation for him was the idea of impossibility of perpetual motion. His work was continued by his son, Sadi Carnot (in *Motive Power of Fire*). The major contribution of his work was the observation that an ideal engine, converting caloric heat into work, could be reinstated by reversing the motion of the cycle (thermodynamic reversibility). He put one step further to conclude that energy is lost even in an idealized reversible (heat) engine.

In 1854, Rudolf Clausius was the first to discuss the concept of the thermodynamic systems and interior work. He was also the first to observe that in each system, for an irreversible process, (a small amount of) heat is dissipated across the boundary. Clausius further develop the idea of lost energy, and also coined the term entropy in 1865 (though he was also using the term *equivalence value*, referring to the mechanical equivalent of heat). Clausius gave a formula (improved later by himself) to calculate the change of equivalence value (entropy) for the passage of heat from one temperature to another (through the fluid).

Building on the work of Rudolf Clausius and Hermann von Helmholtz, the notion of entropy (and its calculation) was further developed in the works of James Clerk Maxwell (1871), Willard Gibbs (1876), Ludwig Boltzmann (1877), Max Planck (1903) and Erwin Schrödinger (1926).

1.2. Information Theory and Dynamics

In 1948, Claude Shannon began to study the notion of lost information (in his work, *A Mathematical Theory of Communication*) as an information theoretic analog of the thermodynamical notion of lost energy (though he seems not to be aware of the earlier work in thermodynamics). Shannon also defined a (general) notion of information entropy (a name suggested to him later by John von Neumann). The Shannon formula of entropy reads as

$$H = -K \sum_{i=1}^{k} p(i) \log p(i),$$

where K is a positive constant, which amounts to a choice of a unit of measurement. Later in 1957, E. T. Jaynes observed that the statistical thermodynamic entropy is a particular case of the Shannon information entropy.

A parallel *complexity theory* of information was built by Andrey N. Kolmogorov in 1950's (and independently by R.J. Solomonoff and G. Chaitin). While Shannon theory concerns about information of communication, Kolmogorov goes beyond this to capture the information in individual objects as well. Kolmogorov also asked the possibility of measuring structural similarities between dynamical systems, a question answered by Yakov Sinai, leading to the notion of entropy of a dynamical systems, known today as the Kolmogorov-Sinai entropy, a very well established notion with a wide range of applications [15, 20]. This is the right tool for estimating complexity, since by the *Birkhoff Ergodic Theorem*, most properties of the system can be reconstructed from a single orbit with probability one.

2. Entropy of infinite systems

Kolmogorov-Sinai entropy works only for measure preserving transformations on probability spaces.

There have been several attempts to generalize the notion of entropy to measure preserving transformations on infinite (or at least σ-finite) spaces [1], [2]. There are basically three suggested notions of entropy in this more general setting: Krengel [11], [12] Parry [13], [14] and Poisson [17], [16] entropies.

2.1. *generalized notions of entropy*

We briefly review and compare the above mentioned three notions of entropy based on [10]. It is known that these three entropies (and their relative counterparts) coincide for quasi-finite or rank-one transformations, and that these are all linear functionals [10]. Also there are spectral criteria for zero Poisson entropy (implying also zero Parry entropy), which give a dichotomy for an ergodic quasi-finite infinite measure preserving transformation: either it is remotely infinite or there exists a maximum (Pinsker) factor with zero Poisson, Krengel and Parry entropy [10]. We comment more on this in the next subsection.

In the Poisson suspension (X^*, μ^*) of a standard σ-finite space (X, μ), X^* is the space of measures on X, the σ-algebra is generated by the family of the sets $\{\gamma \in X^* : a \leq \gamma(B) \leq b\}$ where B ranges over the σ-algebra of X and $a, b \in [0, \infty]$. Now μ^* is a probability measure, independent on disjoint sets, with

$$\mu^*(\gamma(A) = k) = \frac{e^{-\mu(A)}}{k!}\mu(A)^k.$$

For a measure preserving transformation $T : X \to Y$, we set $T^*\gamma = \gamma \circ T^{-1}$, and call it the Poisson suspension of T. The Poisson entropy of a measure preserving transformation T is now the Kolmogorov-Sinai entropy of its Poisson suspension. This is the same as the Kolmogorov-Sinai entropy for finite spaces [16].

The Krengel entropy of a conservative measure preserving transformation T on a σ-finite space (X, μ) is defined by $\sup \mu(A) h_{\mu_A} T_A$, where A ranges over sets with finite strictly positive measures, $T_A(x) := T^{\phi_A(x)}(x)$, with $\phi_A(x) := \min\{k \geq 1 : T^k(x) \in A\}$. If T is not purely periodic, the sup is attained on any sweep-out (i.e., A with $\cup_{n\geq 0} T^{-n}A = X$) [11]. The Krengel entropy is the same as the Kolmogorov-Sinai entropy by Abramov formula. For σ-finite subalgebras α and β of the underlying σ-algebra, $H_\mu(\alpha) := \int_X I_\mu(\alpha) d\mu$ and $H_\mu(\alpha|\beta) := \int_X I_\mu(\alpha|\beta) d\mu$

The information function of a measurable partition α of a σ-finite space (X, μ) is given by $I_\mu(\alpha)(x) := -\log \mu(\alpha(x))$ (with the convention: $\log 0 = -\infty, \log \infty = 0$), where $\alpha(x)$ is the unique element in α which contains x (we take only those α with such a uniqueness property). Similarly, the (value of) conditional information function $I_\mu(\alpha|\beta)(x)$ is defined to be $I_{\mu(\cdot|\beta(x))}(\alpha)(x)$, when $\mu(\beta(x))$ is finite, and $I_\mu(\alpha \vee \{\beta(x), \beta(x)^c\})(x)$, otherwise. Now the Parry entropy of a measure preserving transformation T on a σ-finite space (X, μ) is defined by $\sup H_\mu(\alpha|T^{-1}\alpha)$, where α runs over all σ-finite subalgebras with $T^{-1}\alpha \subseteq \alpha$.

For a conservative transformation T, X is uniquely partitioned into T-invariant sets X_1 and X_∞ which are union of finite and union of infinite ergodic components of μ. Then T is of type II_∞ (resp. II_1) if $\mu(X_1) = 0$ (resp. $\mu(X_\infty) = 0$). Parry entropy is known (for II_∞ transformations) to be less than Krengel [14, Theorem 10.11] and Poisson [10] entropies. Also we know situations where these entropies are equal [10, 7.2, 9.1].

2.2. *Pinsker partition*

A factor of T is a σ-finite subalgebra α satisfying $T^{-1}\alpha = \alpha$. For probability spaces, the Pinsker factor *pin* is the maximum factor with zero entropy. If Pinsker factor of the Poisson suspension is $\mathcal{P}^*$ for some σ-finite σ-algebra $\mathcal{P}$, then $\mathcal{P}$ is both the Poisson-Pinsker and the Parry-Pinsker factor of T. Moreover, if there exists a factor with zero Krengel entropy, then P is also the Krengel-Pinsker factor [10, Proposition 11.1].

When T is ergodic type II_∞ with a zero Poisson entropy factor, then there is a Poisson-Pinsker factor $\mathcal{P}$ such that $\mathcal{P}^*$ is the Pinsker factor of T^* [10, Proposition 11.4]. In this case, when the Parry entropy of T (which is the same as Kolmogorov-Sinai entropy of T^*) is finite and the Pinsker partition of T^* is not the trivial partition ν, then T possesses a Poisson-Pinsker factor $\mathcal{P}$ such that $\mathcal{P}^*$ is the Pinsker partition of T^* [10, Proposition 12.1].

Recall that a transformation T of $(X, \mathfrak{B}, \mu)$ is remotely infinite if there exists a σ-finite subalgebra α such that $T^{-1}\alpha \subseteq \alpha$ and $T^n\alpha \uparrow \mathfrak{B}$ and $T^{-n}\alpha \downarrow \xi$ (mod μ), where ξ has no set of positive finite measure. A subset A of strictly positive finite measure is quasi-finite if $H_\mu(\rho_A) < \infty$, where $\rho_A = \{A \cap T^{-n}A \setminus \cup_{k=1}^{n-1} T^{-k}A\}_{n\geq 1}$ is the first-return-time partition of A. When there is such an A which is also a sweep-out, we say that (X, μ) is quasi-finite. For a quasi-finite system (X, μ) and non remotely infinite, ergodic transformation T, there exists a Poisson-Pinsker factor, which is also a Parry and Krengel-Pinsker factor [10, Corollary 12.7].

Let T be a measure preserving invertible transformation of a σ-finite measure space (X, μ). A sub σ-algebra $\mathfrak{F} \subseteq \mathfrak{B}_X$ is a factor if $T^{-1}\mathfrak{F} = \mathfrak{F}$. and the restriction of μ on $\mathfrak{F}$ is σ-finite. In this case, we have a decomposition $\mu = \int_X \mu_x d\mu|_{\mathfrak{F}}(x)$ with μ_x probability measure for all $x \in X$. Given a countable partition α of X, the conditional entropy $H_\mu(\alpha|\mathfrak{F})$ is defined by

$$H_\mu(\alpha|\mathfrak{F}) = \int_X H_{\mu_x}(\alpha) d\mu(x).$$

One usually works with the set $\mathcal{Z}$ of partitions α with $H_\mu(\alpha|\mathfrak{F}) < \infty$. Two such partitions α, β have distance

$$\rho(\alpha, \beta) := \int_X H_{\mu_x}(\alpha|\beta) + H_{\mu_x}(\beta|\alpha) d\mu(x).$$

We have $\rho(\alpha, \beta) = 0$ if and only if $\alpha \vee \mathfrak{F} = \beta \vee \mathfrak{F}$. In this case, we write $\alpha \sim \beta$. It is known that the quotient $(\mathcal{Z}, \rho)/\sim$ is a Polish space [9].

3. Formal Definitions of Entropies

In this section, we give more formal definition of the entropies for infinite systems and compare them.

3.1. Krengel entropy

The Krengel entropy of a conservative measure preserving transformation $(X, \mathcal{B}, \mu, T)$ is defined as follows [11].

$$h_{\mathrm{Kr}}(X, \mathcal{B}, \mu, T) := \sup_{A \in \mathcal{F}_+} \mu(A)\, h(A, \mathcal{B} \cap A, \mu_A, T_A),$$

where $\mathcal{F}_+$ is the collection of sets in $\mathcal{B}$ with finite positive measure, μ_A is the normalized probability measure on A obtained by restricting μ to $\mathcal{B} \cap A$, and $T_A : A \to A$ is the induced map on A, defined by

$$T_A(x) := T^{\phi_A(x)}(x),$$

where $\phi_A(x) := \min\{k \geq 1 : \; T^k(x) \in A\}$ is the first-return-time map associated to A (c.f., [10]). Krengel proved that if T is not purely periodic,

$$h_{\mathrm{Kr}}(X, \mathcal{B}, \mu, T) = \mu(A)\, h(A, \mathcal{B} \cap A, \mu_A, T_A),$$

where A is any finite-measure sweep-out set, satisfying $\bigcup_{n=0}^{\infty} T^{-n}A = X$. This set always exists when T is of type $\mathbf{II}_\infty$ [10].

The Krengel entropy coincides with the Kolmogorov-Sinai entropy when restricted to finite systems. This follows from Abramov formula: if $S : \Omega \to \Omega$ is an ergodic probability measure preserving (p.m.p.) transformation on $(\Omega, \mathcal{F}, p)$, and $A \in \mathcal{B}$, then

$$h(A, \mathcal{F} \cap A, p(\cdot \mid A), S_A) = \frac{1}{p(A)} h(\Omega, \mathcal{F}, p, S).$$

3.2. Poisson entropy

Poisson suspensions are studied extensively in Physics and ergodic theory and probability [6, 7, 8, 18, 19] (see also, [16] and [21]). Let us formally define the Poisson suspension $(X^*, \mathcal{B}^*, \mu^*, T_*)$ of a standard, σ-finite invertible measure preserving transformation $(X, \mathcal{B}, \mu, T)$.

Let X^* denote the space of measures on X, and let $\mathcal{B}^*$ denote the σ-algebra generated by the collection of sets

$$\Big\{\{\gamma \in X^* : \ \gamma(B) \in [a, b]\} : B \in \mathcal{B},\ 0 \le a \le b \le \infty \ \Big\}.$$

The probability measure μ^* on $(X^*, \mathcal{B}^*)$ is defined by

$$\mu^*\left(\gamma(A) = k\right) = e^{-\mu(A)} \frac{\mu(A)^k}{k!}.$$

A measure preserving map $T : (X, \mathcal{B}, \mu) \to (Y, \mathcal{C}, \nu)$ naturally gives rise to a measure preserving map $T_* : (X^*, \mathcal{B}^*, \mu^*) \to (Y^*, \mathcal{C}^*, \nu^*)$ by $T_*\gamma = \gamma \circ T^{-1}$. If T is an endomorphism, the dynamical system $(X^*, \mathcal{B}^*, \mu^*, T_*)$ is the Poisson suspension of $(X, \mathcal{B}, \mu, T)$.

Following [16], the Poisson entropy of an infinite measure preserving transformation is defined as the Kolmogorov entropy of the Poisson suspension. If S is a factor of T, its Poisson entropy is less than the Poisson entropy of T, and Poisson entropy of T^n is $|n|$ times Poisson entropy of T (c.f., [10]).

It is proved in [16] that the Poisson entropy of a probability measure preserving (p.m.p.) transformation is equal to its Kolmogorov entropy. Indeed, for any quasi-finite transformation, the Poisson entropy is equal to Parry and Krengel entropies (this holds in particular for finite measure preserving systems) [10].

If $(X, \mathcal{B}, \mu, T)$ is conservative, there exists a unique partition of X into T-invariant sets X_1 and X_∞, which are the measurable union of finite (resp. infinite) ergodic components of μ. If $\mu(X_1) = 0$, T is said to be of type $\mathbf{II}_\infty$, and if $\mu(X_\infty) = 0$, it is said to be of type $\mathbf{II}_1$ [10]. Only $\mathbf{II}_\infty$ systems are of interest in this context, since the $\mathbf{II}_1$ case reduces to the finite measure case [10].

A factor of T is a σ-finite sub-σ-algebra $\mathcal{F}$ satisfying $T^{-1}\mathcal{F} = \mathcal{F}$. The trivial σ-algebra is not a factor of a $\mathbf{II}_\infty$-system and if T is of type $\mathbf{II}_\infty$, then μ is continuous; any factor of T is of type $\mathbf{II}_\infty$; any σ-finite sub-σ-algebra $\mathcal{A}$ satisfying $T^{-1}\mathcal{A} \subset \mathcal{A}$ has no atom; and $(X^*, \mathcal{B}^*, \mu^*, T_*)$ is ergodic [10].

For each $A \in \mathcal{B}$ and $N \in X^*$, following [10], let us denote by $N(A) : X^* \to \mathbb{N}$ the random variable on the probability space $(X^*, \mathcal{B}^*, \mu^*)$ which is the (random) measure of the set A. If A has finite measure, $N(A)$ is Poisson distributed with parameter $\mu(A)$, and if $\mu(A) = \infty$, $N(A) = \infty$, μ^*-almost surely.

For a finite or countable partition α, let $N(\alpha) = (N(A))_{A \in \alpha}$ be the random vector of Poisson random variables corresponding to α. By definition of Poisson suspension, the coordinates of $N(\alpha)$ are independent [10].

If $\mathcal{C} \subset \mathcal{B}$ is a σ-algebra, $\mathcal{C}^* := \sigma\Big(\{N(A) : \ A \in \mathcal{C}\}\Big)$ is the sub-σ-algebra of $\mathcal{B}^*$ generated by the Poisson random variables of $\mathcal{C}$. Also, for a measurable partition α of X, $\alpha^* := (\sigma(\alpha))^*$.

The lack of atoms for a measure implies no "multiplicities" in the corresponding Poisson space [10]. Formally, if there are no atoms of positive measure in $(X, \mathcal{B}, \mu)$, μ^*-almost surely, there are no multiplicities, i.e.,

$$\mu^*\Big(\{\exists\, x \in X :\ N(\{x\}) \geq 2\}\Big) = 0.$$

In general, the equality $(\mathcal{C}_1 \vee \mathcal{C}_2)^* = \mathcal{C}_1^* \vee \mathcal{C}_2^*$ does not hold. This is however true if the intersection of the σ-algebras is non-atomic [17]: Let α, β and $\mathcal{C}$ be sub-σ-algebras of $\mathcal{B}$. Assume that $\mathcal{C}$ is σ-finite and non-atomic. Then

$$(\mathcal{C} \vee \alpha \vee \beta)^* = (\mathcal{C} \vee \alpha)^* \vee (\mathcal{C} \vee \beta)^* \pmod{\mu^*}.$$

A similar result is proved in [16], using the corresponding projections in L^2-spaces.

3.3. Parry entropy

In this section we recall Parry definition of entropy for a measure preserving transformation.

Parry [14] defines the entropy of a measure preserving transformation by

$$h_{\mathrm{Pa}}(X, \mathcal{B}, \mu, T) := \sup_{T^{-1}\mathcal{C} \subset \mathcal{C}} H_\mu(\mathcal{C} \mid T^{-1}\mathcal{C}),$$

where the supremum is taken over all σ-finite sub-σ-algebras $\mathcal{C}$ of $\mathcal{B}$ such that $T^{-1}\mathcal{C} \subset \mathcal{C}$. For probability measure preserving (p.m.p.) transformations, this definition coincides with the standard definition of Kolmogorov-Sinai entropy.

It was proved by Parry [14, Theorem 10.11] that for a measure preserving conservative transformation $(X, \mathcal{B}, \mu, T)$,

$$h_{\mathrm{Pa}}(X, \mathcal{B}, \mu, T) \leq h_{\mathrm{Kr}}(X, \mathcal{B}, \mu, T).$$

Replacing Krengel entropy by Poisson entropy, one could prove a similar result [10]: Let $(X, \mathcal{B}, \mu, T)$ be a $\mathbf{II}_\infty$ transformation, then

$$h_{\mathrm{Pa}}(X, \mathcal{B}, \mu, T) \leq h(X^*, \mathcal{B}^*, \mu^*, T_*).$$

4. concluding remarks

We recall the list of open problems on entropy of infinite systems, as recorded in [10]. The main open question left at this point, as stated in the beginning, is the following: Are Krengel, Parry and Poisson entropies equal for *every* conservative measure preserving transformation?

The other important questions are as follows: Is there an inequality between Poisson entropy and Krengel entropy which holds in general? Are the properties of having zero Poisson entropy and having zero Krengel entropy equivalent?

Related to this is the following question of Danilenko and Rudolph [5]: Does any conservative transformation have a factor with arbitrarily small Poisson or Krengel entropy? A positive answer to this question would imply a positive answer to the main question above. However one does not even know if there always exists a factor with *finite* Poisson or Krengel entropy.

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