Solving a nonlinear inverse system of Burgers equations

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Abstract

By applying finite difference formula to time discretization and the cubic B-splines for spatial variable, a numerical method for solving the inverse system of Burgers equations is presented. Also, the convergence analysis and stability for this problem are investigated and the order of convergence is obtained. By using two test problems, the accuracy of presented method is verified. Additionally, obtained numerical results of the cubic B-spline method are compared to trigonometric cubic B-spline method, exponential cubic B-spline method and radial basis function method. Implementation simplicity and less computational cost are the main advantages of proposed scheme compared to previous proposals.

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1. Introduction

The system of Burgers equations is a simple model for the understanding of physical flows and problems, such as hydrodynamic turbulence, shock wave theory, wave processes in thermo-elastic medium, vorticity transport and dispersion in porous media [1, 2, 3, 4]. This system was first derived by Esipov [1] to study the model of polydispersive sedimentation. Cole and Burgers [2, 4] found that this system of equations describe various kinds of phenomena such as a mathematical model of turbulence and the approximate theory of flow through a shock wave traveling in a viscous fluid. The systems of Burgers equations are usually difficult to solve analytically and so the numerical approaches are created to overcome the complexities of analytical methods [5, 6, 7, 8, 9, 10]. Inverse problems are encountered in many branches of engineering and science. For example, in the field of heat transfer, the inverse problem under certain conditions have

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been used to calculate the thermal properties of solids. Besides, several functions and parameters such as static and moving heating sources, material properties, initial conditions, boundary conditions, optimal shape etc, can be estimated from the inverse problem [11, 12, 13, 14, 15]. Mathematically, the inverse problems belong to the class of problems called the ill-posed problems, i.e. small errors in the measured data can lead to large deviations in the estimated quantities. As a consequence, their solutions do not satisfy the general requirement of existence, uniqueness and stability under small changes in the initial parameters. To simplify the inverse problem a variety of techniques have been proposed, where have been resulted from mathematical fields such as partial differential equations, numerical analysis, harmonic analysis, functional analysis, fourier analysis and etc. Tikhonov regularization [16], iterative regularization[17], base function [18] and the function specification methods [19] was used as solution to the inverse problem. The theory of spline functions is a very active field of approximation theory and boundary value problems (BVPs), when numerical aspects are considered. In a series of paper by Caglar et al. [22, 23, 24, 25, 26] BVPs of order two, third, fourth and fifth were solved using third, fourth and sixth-degree splines. We know that B-splines have some special features, which are useful in numerical work. One feature is that the continuity conditions are inherent, another special feature of B-splines is that they have small local support, that is, each B-spline function is only non-zero over a few mesh subintervals, so that the resulting matrix for the discretization equation is tightly banded. Due to their smoothness and capability to handle local phenomena, B-splines offer distinct advantages. In combination with collocation, it significantly simplifies the solution procedure of differential equations. The absence of integrations during the calculations (e.g. variational methods) leads to a great reduction of calculations cost. Obviously, some previous techniques using various transformations to reduce the equation to the simpler equation. Unlike some previous methods, the cubic B-splines collocation method does not require extra effort to deal with the nonlinear terms. Accordingly, the equations can be solved easily and daintily. In this work, using B-spline collocation method which is adopted to Tikhonov regularization method, the solution of the inverse system of Burgers equations will be investigated. In the following, convergence and stability analysis will be studied. To demonstrate accuracy, efficiency and applicability of the presented method, two test problems will be used. Also, obtained numerical results of the cubic B-spline method will be compared to the trigonometric cubic B-spline method, exponential cubic B-spline method and radial basis function method. Simplicity of implementation and less computational cost can be mentioned as main advantages of the proposed scheme compared to previous plan. The plan of this paper is as follows: Section 2 is devoted to formulate inverse problem. Description of the cubic B-splines collocation method and procedure for implementation of the present method are illustrated in Sections 3 and 4, respectively. In Section 5, procedure to obtain an initial vector which is required to start our method is explained. To regularize the resultant ill-posed linear system of equations, in Section 6, we apply the Tikhonov regularization (of 2nd order) method to obtain the stable numerical approximation of our solution. The uniform convergence and the conditional stability based on the Von-Neumann method are discussed in Sections 7 and 8, respectively. To explain the effectiveness and compare of the presented method with trigonometric cubic B-spline (TCBS) method, exponential cubic B-spline (ECBS) method and radial basis function (RBF) method, Section 9 gives some examples with analytical solution. Finally in Section 10, we will finish this paper with a brief conclusion.

2. Inverse system of Burgers equations

In this paper, we focus on the inverse system of Burgers equations and seek to determine the boundary conditions in this system as follows:

$$\begin{cases} u_t - u_{xx} + \eta u u_x + \alpha(uv)_x = 0, \ 0 < x < 1, \ 0 \le t \le T, \\ \\ v_t - v_{xx} + \eta u u_x + \beta(uv)_x = 0, \ 0 < x < 1, \ 0 \le t \le T, \end{cases} \tag{2.1}$$

with the initial conditions

$$u(x,0) = f_1(x), \qquad v(x,0) = f_2(x), \quad 0 < x < 1, \tag{2.2}$$

the boundary conditions

$$
\begin{aligned}
u(0,t) &= p_1(t), & v(0,t) &= p_2(t), & 0 \le t \le T, \\
u(1,t) &= q_1(t), & v(1,t) &= q_2(t), & 0 \le t \le T,
\end{aligned}
\tag{2.3}
$$

and the overspecified data

$$u(x^*,t) = g_1(t), \qquad v(x^*,t) = g_2(t), \qquad 0 < x^* < 1, \qquad 0 \le t \le T, \tag{2.4}$$

where η is real constant, α and β arbitrary constants depending on the system parameters such as Peclet number, stokes velocity of particles due to gravity and the Brownian diffusivity. Also, T represents the final time, $\omega = \{(x,t) : x \in [0,1] = \Omega, t \in [0,T]\}$ and $f_1(x), f_2(x), q_1(t), q_2(t)$ and $g_1(t), g_2(t)$ are given continuous functions. The boundary conditions $p_1(t), p_2(t)$ are unknown and are, in fact, to be determined from overspecified data. We seek the functions $u(x,t)$, $v(x,t)$ and $p_1(t), p_2(t)$. For two unknown boundary conditions $p_1(t), p_2(t)$ we must therefore provide additional information (2.4) to prepare a unique solution $\big(u(x,t), v(x,t), p_1(t), p_2(t)\big)$ to the inverse problem.

3. Description of the cubic B-spline functions

In cubic B-splines collocation method the approximate solution can be written as a linear combination of basis functions which organized constitute a basis for the approximation space under consideration. To construct numerical solution, we introduce a uniformly distributed set of nodes $0 = x_0 < x_1 < \ldots < x_N = 1$ over the spatial domain $[0,1]$ and the spacial step length is denoted by h, $h = x_{i+1} - x_i$, $i = 0,1,\ldots,N-1$. To construct the cubic B-spline, we need to extend the set of nodal points to

$$x_{-3} < x_{-2} < x_{-1} < x_0 \quad and \quad x_N < x_{N+1} < x_{N+2} < x_{N+3}.$$

The cubic B-spline B_i, $i = -1, 0, \ldots, N+1$, are defined in the following way

$$
B_i(x) = \frac{1}{h^3}
\begin{cases}
(x - x_{i-2})^3, & x \in [x_{i-2}, x_{i-1}], \\
h^3 + 3h^2(x - x_{i-1}) + 3h(x - x_{i-1})^2 - 3(x - x_{i-1})^3, & x \in [x_{i-1}, x_i], \\
h^3 + 3h^2(x_{i+1} - x) + 3h(x_{i+1} - x)^2 - 3(x_{i+1} - x)^3, & x \in [x_i, x_{i+1}], \\
(x_{i+2} - x)^3, & x \in [x_{i+1}, x_{i+2}], \\
0, & \text{otherwise,}
\end{cases}
\tag{3.1}
$$

where $B_i(x)\,(i = -1, \cdots, N+1)$ form a basis for functions defined on the interval $[0,1]$. Each cubic B-splines covers four elements so that an element is covered by four cubic B-splines. All other B-splines are zero in this region. By using splines defined in (3.1), the value of $B_i(x)$ and its derivatives at the nodes x_i's are given by

$$
B_m(x_i) =
\begin{cases}
4, & \text{if } m = i, \\
1, & \text{if } |m - i| = 1, \\
0, & \text{if } |m - i| \ge 2,
\end{cases}
\qquad
B'_m(x_i) =
\begin{cases}
0, & \text{if } m = i, \\
-\frac{3}{h}, & \text{if } m = i - 1, \\
\frac{3}{h}, & \text{if } m = i + 1, \\
0, & \text{if } |m - 1| \ge 2,
\end{cases}
\tag{3.2}
$$

$$
B''_m(x_i) =
\begin{cases}
-\frac{12}{h^2}, & \text{if } m = i, \\
\frac{6}{h^2}, & \text{if } |m - 1| = 1, \\
0, & \text{if } |m| \ge 2.
\end{cases}
$$

4. Implementation of method

In this section, we first present our method based on the cubic B-spline functions for solving the Eqs. (2.1)-(2.4). To apply the proposed method, expressing $u(x,t)$ and $v(x,t)$ by using cubic B-spline functions. Let

$$U_n(x,t) = \sum_{i=-1}^{N+1} c_i^n(t)B_i(x), \qquad V_n(x,t) = \sum_{i=-1}^{N+1} d_i^n(t)B_i(x), \tag{4.1}$$

be the approximate solutions of boundary value problem and the overspecified condition (2.1)-(2.4), where c_i and d_i are unknown time dependent quantities to be determined. To apply the proposed method, discretizing the time derivative in the usual finite difference way to Eq. (2.1), we get

$$\begin{cases} \left[\frac{u^{n+1}-u^n}{\Delta t}\right] - \left(u_{xx}\right)^n + \eta\left(uu_x\right)^n + \alpha\left(uv\right)^n_x = 0, \\[2mm] \left[\frac{v^{n+1}-v^n}{\Delta t}\right] - \left(v_{xx}\right)^n + \eta\left(vv_x\right)^n + \beta\left(uv\right)^n_x = 0. \end{cases} \tag{4.2}$$

Now, by substituting (4.1) in (4.2) at the point $x = x_m$, the cubic B-spline functions and their derivatives up to second order which are determined in (3.2), we have

$$\left.\begin{array}{c}3\\ \hline h\alpha\left(d_{m-1}^n+4d_m^n+d_{m+1}^n\right)\left(c_{m+1}^n-c_{m-1}^n\right)\end{array}\right\} = \left(c_{m-1}^n+4c_m^n+c_{m+1}^n\right),$$

$$\left.\begin{array}{c}3\\ \hline h\beta\left(d_{m-1}^n+4d_m^n+d_{m+1}^n\right)\left(c_{m+1}^n-c_{m-1}^n\right)\end{array}\right\} = \left(d_{m-1}^n+4d_m^n+d_{m+1}^n\right),$$

for simplifying, we set

$$\begin{array}{c}3\\ \hline h\alpha\left(d_{m-1}^n+4d_m^n+d_{m+1}^n\right)\left(c_{m+1}^n-c_{m-1}^n\right)\end{array}, \Gamma_2 = \frac{6}{h^2}\left(d_{m-1}^n-2d_m^n+d_{m+1}^n\right) - \frac{3}{h}\eta\left(d_{m-1}^n+4d_m^n+d_{m+1}^n\right)\left(d_{m+1}^n-d_{m-1}^n\right) - \frac{3}{h}\beta\left(c_{m-1}^n+4c_m^n+c_{m+1}^n\right)\left(d\right.$$

$$\begin{array}{c}3\\ \hline h\beta\left(d_{m-1}^n+4d_m^n+d_{m+1}^n\right)\left(c_{m+1}^n-c_{m-1}^n\right)\end{array},$$

then, we have

$$\left(c_{m-1}^{n+1}+4c_m^{n+1}+c_{m+1}^{n+1}\right) = \Delta t\Gamma_1 + \left(c_{m-1}^n+4c_m^n+c_{m+1}^n\right) = \mathbf{X}_m^n,$$

$$\left(d_{m-1}^{n+1}+4d_m^{n+1}+d_{m+1}^{n+1}\right) = \Delta t\Gamma_2 + \left(d_{m-1}^n+4d_m^n+d_{m+1}^n\right) = \mathbf{R}_m^n. \tag{4.3}$$

The system (4.3) consists of $2(N+1)$ equations in $2(N+3)$ unknown coefficients, but to obtain a unique solution to the resulting system four additional constraints are required. To this end, by imposing the boundary conditions (2.3) and the overspecified condition (2.4), we have

$$U_{n+1}(x_s) = c_{s-1}^{n+1} + 4c_s^{n+1} + c_{s+1}^{n+1}, \qquad\qquad U_{n+1}(x_N) = c_{N-1}^{n+1} + 4c_N^{n+1} + c_{N+1}^{n+1},$$

$$V_{n+1}(x_s) = d_{s-1}^{n+1} + 4d_s^{n+1} + d_{s+1}^{n+1}, \qquad\qquad V_{n+1}(x_N) = d_{N-1}^{n+1} + 4d_N^{n+1} + d_{N+1}^{n+1},$$

where $x_s = x^*$, $1 \le s \le N-1$. Then a system of $2(N+3)$ linear equations in the $2(N+3)$ unknown coefficients is obtained. This system can be written in the matrix vector form as follows

$$AX = B, \tag{4.4}$$

where

$$X = \left[c_{-1}^{n+1}, c_0^{n+1}, c_1^{n+1}, \ldots, c_{N+1}^{n+1}, d_{-1}^{n+1}, d_0^{n+1}, d_1^{n+1}, \ldots, d_{N+1}^{n+1} \right]^T,$$

$$B = \left[\mathbf{X}_{-1}^n, \mathbf{X}_0^n, \mathbf{X}_1^n, \ldots, \mathbf{X}_{N+1}^n, \mathbf{R}_{-1}^n, \mathbf{R}_0^n, \mathbf{R}_1^n \ldots, \mathbf{R}_{N+1}^n \right]^T,$$

where $\mathbf{X}_{-1}^n = g_1(t_n)$, $\mathbf{X}_{N+1}^n = q_1(t_n)$, $\mathbf{R}_{-1}^n = g_2(t_n)$ and $\mathbf{R}_{N+1}^n = q_2(t_n)$. A is an $2(N+3) \times 2(N+3)$ dimensional matrix given by

$$A = \begin{pmatrix} M & | & O \\ - & - & - \\ O & | & M \end{pmatrix}. \tag{4.5}$$

The matrices O and M have the same size $(N+3) \times (N+3)$, where O is a zero matrix and matrix M is defined as follows

$$M = \begin{pmatrix} 0 & \ldots & 0 & 1 & 4 & 1 & 0 & \ldots & 0 \\ 1 & 4 & 1 & & & & & & \\ & 1 & 4 & 1 & & & & & \\ & & \ldots & \ldots & \ldots & & & & \\ & & & \ldots & \ldots & \ldots & & & \\ & & & & \ldots & \ldots & \ldots & & \\ & & & & & \ldots & \ldots & \ldots & \\ \vdots & & & & & & 1 & 4 & 1 \\ 0 & \ldots & & & & & 1 & 4 & 1 \end{pmatrix},$$

that $M[1, s+1] = 1$, $M[1, s+2] = 4$, $M[1, s+3] = 1$. With solving (4.4) by Tikhonov regularization method, the coefficients c_j and d_j are obtained and with these coefficients, we can obtain the approximate solutions, i.e.

$$
\begin{aligned}
p_1(t^{(n)}) &= c_{-1}^{(n)} + 4c_0^{(n)} + c_1^{(n)}, & n &= 0, 1, \ldots, \\
p_2(t^{(n)}) &= d_{-1}^{(n)} + 4d_0^{(n)} + d_1^{(n)}, & n &= 0, 1, \ldots, \\
U(x_j, t^{(n)}) &= c_{j-1}^{(n)} + 4c_j^{(n)} + c_{j+1}^{(n)}, & n &= 0, 1, \ldots, & j &= 0, 1, \ldots, N, \\
V(x_j, t^{(n)}) &= d_{j-1}^{(n)} + 4d_j^{(n)} + d_{j+1}^{(n)}, & n &= 0, 1, \ldots, & j &= 0, 1, \ldots, N.
\end{aligned}
$$

5. The initial vector C^0

The initial vector C^0 can be found from the initial condition (2.2), boundary and overspecified conditions (2.3) and (2.4) as the following expressions

$$
\begin{aligned}
u(x_s,0) &= c_{s-1}^{(0)} + 4c_s^{(0)} + c_{s+1}^{(0)} = g_1(0), \\
v(x_s,0) &= d_{s-1}^{(0)} + 4d_s^{(0)} + d_{s+1}^{(0)} = g_2(0), \\
u(x_j,0) &= c_{j-1}^{(0)} + 4c_j^{(0)} + c_{j+1}^{(0)} = f_1(x_j), \qquad 0 \le j \le N, \\
v(x_j,0) &= d_{j-1}^{(0)} + 4d_j^{(0)} + d_{j+1}^{(0)} = f_2(x_j), \qquad 0 \le j \le N, \\
u(x_N,0) &= c_{N-1}^{(0)} + 4c_N^{(0)} + c_{N+1}^{(0)} = q_1(0), \\
v(x_N,0) &= d_{N-1}^{(0)} + 4d_N^{(0)} + d_{N+1}^{(0)} = q_2(0).
\end{aligned}
$$

This yields a $2(N+3) \times 2(N+3)$ system of equations, of the form

$$
A^* X^0 = B^*, \tag{5.1}
$$

where $A^* = A$ and

$$
X^0 = \left[c_{-1}^0, c_0^0, c_1^0, \ldots, c_{N+1}^0, d_{-1}^0, d_0^0, d_1^0, \ldots, d_{N+1}^0 \right]^T,
$$
$$
B^* = \left[g_1(0), f_1(x_0), f_1(x_1), \ldots, f_1(x_N), q_1(0), g_2(0), f_2(x_0), f_2(x_1), \ldots, f_2(x_N), q_2(0) \right]^T.
$$

The solution of (5.1) can be obtained by the Tikhonov regularization method.

6. Tikhonov regularization method

The matrix A is singular and ill-posed, thus the estimate of X^0 by (5.1) will be unstable so that the Tikhonov regularization method must be used to control this singularity. In our computation, we adapt the Tikhonov regularization method to solve the matrix equations (4.4) and (5.1). The Tikhonov regularized solutions to the systems of linear algebraic equations (4.4) and (5.1) are given by

$$
F_\sigma(X) = \|AX - B\|_2^2 + \sigma \|R^{(z)} X\|_2^2,
$$

$$
F_\sigma(X^0) = \|AX^0 - B^*\|_2^2 + \sigma \|R^{(z)} X^0\|_2^2.
$$

On the case of the first- and second- order Tikhonov regularization method the matrix $R^{(z)}$, for $z = 1, 2$, is given by, see e.g. [27],

$$
R^{(1)} = \begin{pmatrix}
-1 & 1 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & \ldots & 0 & -1 & 1
\end{pmatrix} \in \mathbb{R}^{(M-1) \times (M)},
$$

$$R^{(2)} = \begin{pmatrix} 1 & -2 & 1 & \ldots & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & \ldots & 1 & -2 & 1 \end{pmatrix} \in \mathbb{R}^{(M-2)\times(M)},$$

where $M = 2(N+3)$. Therefore the Tikhonov regularized solutions to the systems of linear algebraic equations (4.4) and (5.1) are given by

$$X_\sigma = [A^T A + \sigma (R^{(z)})^T R^{(z)}]^{-1} A^T B, \tag{6.1}$$

$$X_\sigma^0 = [A^T A + \sigma (R^{(z)})^T R^{(z)}]^{-1} A^T B^*. \tag{6.2}$$

In our computation, we use the generalized cross-validation (GCV) scheme to determine a suitable value of σ ([28, 29, 30]).

7. Convergence analysis

Theorem 7.1. *The collocation approximations $U_n(x)$ and $V_n(x)$ for the solutions $u_n(x)$ and $v_n(x)$ of the inverse problem $(2.1) - (2.4)$ satisfy the following error estimate*

$$\left\| (u_n - U_n, v_n - V_n) \right\|_\infty \leq \mu h^2, \tag{7.1}$$

for sufficiently small h (i.e. for sufficiently large N) where μ is a positive constant.

Proof . Let $u_n(x,t)$ and $v_n(x,t)$ be the exact solutions of the problem (2.1) with the boundary conditions, initial conditions, overspecific conditions and also $U_n(x,t) = \sum_{i=-1}^{N+1} c_i^n(t) B_i(x)$ and $V_n(x,t) = \sum_{i=-1}^{N+1} d_i^n(t) B_i(x)$ be the B-spline collocation approximations to $u_n(x,t)$ and $v_n(x,t)$. Due to round off errors in computations we assume that $\hat{U}_n(x,t)$ and $\hat{V}_n(x,t)$ be the computed splines for $U_n(x,t)$ and $V_n(x,t)$ so that $\hat{U}_n(x,t) = \sum_{j=-1}^{N+1} \hat{c}_j^n(t) B_j(x)$ and $\hat{V}_n(x,t) = \sum_{j=-1}^{N+1} \hat{d}_j^n(t) B_j(x)$. To estimate the errors $\|u_n(x,t) - U_n(x,t)\|_\infty$ and $\|v_n(x,t) - V_n(x,t)\|_\infty$ we must estimate the errors $\|u_n(x,t) - \hat{U}_n(x,t)\|_\infty$, $\|\hat{U}_n(x,t) - U_n(x,t)\|_\infty$, $\|v_n(x,t) - \hat{V}_n(x,t)\|_\infty$ and $\|\hat{V}_n(x,t) - V_n(x,t)\|_\infty$ separately. Following (4.4) for $\hat{U}_n$ and $\hat{V}_n$ we have

$$A\hat{X} = \hat{B}, \tag{7.2}$$

where

$$\hat{X} = \left[\hat{c}_{-1}^{n+1}, \hat{c}_0^{n+1}, \hat{c}_1^{n+1}, \ldots, \hat{c}_{N+1}^{n+1}, \hat{d}_{-1}^{n+1}, \hat{d}_0^{n+1}, \hat{d}_1^{n+1}, \ldots, \hat{d}_{N+1}^{n+1} \right]^T,$$

$$\hat{B} = \left[\mathbf{X}_{-1}^n, \hat{\mathbf{X}}_0^n, \hat{\mathbf{X}}_1^n, \ldots, \hat{\mathbf{X}}_N^n, \mathbf{X}_{N+1}^n, \mathbf{R}_{-1}^n, \hat{\mathbf{R}}_0^n, \hat{\mathbf{R}}_1^n, \ldots, \hat{\mathbf{R}}_N^n, \mathbf{R}_{N+1}^n \right]^T,$$

where $\mathbf{X}_{-1}^n = g_1(t_n)$, $\mathbf{X}_{N+1}^n = q_1(t_n)$, $\mathbf{R}_{-1}^n = g_2(t_n)$ and $\mathbf{R}_{N+1}^n = q_2(t_n)$. By subtracting (4.4) and (7.2), we have

$$A\left(X - \hat{X} \right) = \left(B - \hat{B} \right), \tag{7.3}$$

where

$$B - \hat{B} = \left[0, \mathbf{X}_0^n - \hat{\mathbf{X}}_0^n, \mathbf{X}_1^n - \hat{\mathbf{X}}_1^n, \ldots, \mathbf{X}_N^n - \hat{\mathbf{X}}_N^n, 0, 0, \mathbf{R}_0^n - \hat{\mathbf{R}}_0^n, \mathbf{R}_1^n - \hat{\mathbf{R}}_1^n, \ldots, \mathbf{R}_N^n - \hat{\mathbf{R}}_N^n, 0 \right], \qquad (7.4)$$

and for every $0 \leq m \leq N$,

$$\mathbf{X}_m^n = \Delta t \left[U_n''(x_n) - \eta U_n(x_m) U_n'(x_m) - \alpha \left(U_n(x_m) V_n(x_m) \right)_x \right] + U_n(x_m),$$

$$\hat{\mathbf{X}}_m^n = \Delta t \left[\hat{U}_n''(x_n) - \eta \hat{U}_n(x_m) \hat{U}_n'(x_m) - \alpha \left(\hat{U}_n(x_m) \hat{V}_n(x_m) \right)_x \right] + \hat{U}_n(x_m),$$

$$\mathbf{R}_m^n = \Delta t \left[V_n''(x_n) - \eta V_n(x_m) V_n'(x_m) - \beta \left(U_n(x_m) V_n(x_m) \right)_x \right] + V_n(x_m),$$

$$\hat{\mathbf{R}}_m^n = \Delta t \left[\hat{V}_n''(x_n) - \eta \hat{V}_n(x_m) \hat{V}_n'(x_m) - \beta \left(\hat{U}_n(x_m) \hat{V}_n(x_m) \right)_x \right] + \hat{V}_n(x_m).$$

So

$$\left| \mathbf{X}_m^n - \hat{\mathbf{X}}_m^n \right| = \left| \Delta t \left[\left(U_n''(x_n) - \hat{U}_n''(x_n) \right) - \eta U_n(x_m) U_n'(x_m) + \eta \hat{U}_n(x_m) \hat{U}_n'(x_m) \right. \right.$$

$$\left. - \alpha \left(U_n(x_m) V_n(x_m) \right)_x + \alpha \left(\hat{U}_n(x_m) \hat{V}_n(x_m) \right)_x \right]$$

$$\left. + \left(U_n(x_m) - \hat{U}_n(x_m) \right) \right|.$$

By using the Cauchy-Schwarz inequality, we have

$$\left| \mathbf{X}_m^n - \hat{\mathbf{X}}_m^n \right| \leq \Delta t \left| \left(U_n''(x_n) - \hat{U}_n''(x_n) \right) \right| + \left| \left(U_n(x_m) - \hat{U}_n(x_m) \right) \right| +$$

$$\Delta t \underbrace{\left| - \eta U_n(x_m) U_n'(x_m) + \eta \hat{U}_n(x_m) \hat{U}_n'(x_m) - \alpha \left(U_n(x_m) V_n(x_m) \right)_x + \alpha \left(\hat{U}_n(x_m) \hat{V}_n(x_m) \right)_x \right|}_{(I)},$$

$$(I) = \left\| \left[\frac{\eta}{2} \left(U_n^2(x_m) - \hat{U}_n^2(x_m) \right) + \alpha \left(U_n(x_m) V_n(x_m) \right) - \alpha \left(\hat{U}_n(x_m) \hat{V}_n(x_m) \right) \right]_x \right\|$$

$$= \left\| \left[\frac{\eta}{2} \left(U_n(x_m) \right) \left(U_n(x_m) + \frac{2\alpha}{\eta} V_n(x_m) \right) - \frac{\eta}{2} \left(\hat{U}_n(x_m) \right) \left(\hat{U}_n(x_m) + \frac{2\alpha}{\eta} \hat{V}_n(x_m) \right) \right]_x \right\|$$

$$= \left\| \left[\frac{\eta}{2} \left(U_n(x_m) \right) \left(U_n(x_m) - \hat{U}_n(x_m) \right) + \alpha \left(U_n(x_m) \right) \left(V_n(x_m) - \hat{V}_n(x_m) \right) + \right. \right.$$
$$\left. \left. \frac{\eta}{2} \left(U_n(x_m) - \hat{U}_n(x_m) \right) \left(\hat{U}_n(x_m) + \frac{2\alpha}{\eta} \hat{V}_n(x_m) \right) \right]_x \right\|$$

$$= \left\| \left[\eta \left(U_n(x_m) \right) \left(U_n(x_m) - \hat{U}_n(x_m) \right) + \alpha \left(U_n(x_m) \right) \left(V_n(x_m) - \hat{V}_n(x_m) \right) - \right. \right.$$
$$\frac{\eta}{2} \left(U_n(x_m) - \hat{U}_n(x_m) \right)^2 + \alpha \left(V_n(x_m) \right) \left(U_n(x_m) - \hat{U}_n(x_m) \right) -$$
$$\left. \left. \alpha \left(V_n(x_m) - \hat{V}_n(x_m) \right) \left(U_n(x_m) - \hat{U}_n(x_m) \right) \right]_x \right\|,$$

then, after simplifying and differential, we have

$$\left| \mathbf{X}_m^n - \hat{\mathbf{X}}_m^n \right| \le \Delta t \left| \left(U_n''(x_n) - \hat{U}_n''(x_n) \right) \right| + \left| \left(U_n(x_m) - \hat{U}_n(x_m) \right) \right| +$$

$$\Delta t \left[\eta \left| U_n'(x_m) \right| \left| U_n(x_m) - \hat{U}_n(x_m) \right| + \eta \left| U_n(x_m) \right| \left| U_n'(x_m) - \hat{U}_n'(x_m) \right| + \right.$$

$$\alpha \left| U_n'(x_m) \right| \left| V_n(x_m) - \hat{V}_n(x_m) \right| + \alpha \left| U_n(x_m) \right| \left| V_n'(x_m) - \hat{V}_n'(x_m) \right| +$$

$$\eta \left| U_n(x_m) - \hat{U}_n(x_m) \right| \left| U_n'(x_m) - \hat{U}_n'(x_m) \right| +$$

$$\alpha \left| V_n'(x_m) \right| \left| U_n(x_m) - \hat{U}_n(x_m) \right| +$$

$$\alpha \left| V_n(x_m) \right| \left| U_n'(x_m) - \hat{U}_n'(x_m) \right| +$$

$$\alpha \left| V_n'(x_m) - \hat{V}_n'(x_m) \right| \left| U_n(x_m) - \hat{U}_n(x_m) \right| +$$

$$\left. \alpha \left| V_n(x_m) - \hat{V}_n(x_m) \right| \left| U_n'(x_m) - \hat{U}_n'(x_m) \right| \right].$$

Now, first we need the following theorem:

Theorem 7.2. *Suppose $u \in C^4[a,b]$ and $|u^{(4)}(x)| \le L$ for $x \in [a,b]$. Let Δ be a partition $\Delta = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of the interval $[a,b]$ with step size h. If $\hat{U}$ is the spline function which interpolates the values of the function f at the knots $x_0, \cdots, x_n \in \Delta$, then there exist constants $\lambda_j \le 2$, which do not depend on the partition Δ, such that for $x \in [a,b]$,*

$$\| u^{(j)}(x) - \hat{U}^{(j)}(x) \| \le \lambda_j L h^{4-j}, \qquad j = 0, 1, 2, 3, \tag{7.5}$$

where $\| . \|$ represents the ∞-norm.

Proof . For the proof see Stoer and Bulirsch [31]. $\square$

Now, using Theorem 7.2

$$
\left| \mathbf{X}_m^n - \hat{\mathbf{X}}_m^n \right| \leq \Delta t \left[h^2 \lambda_2 \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty + h^4 \eta \lambda_0 \left\| U_n' \right\|_\infty \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty + h^3 \lambda_1 \eta \left\| U_n \right\|_\infty \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty + \right.
$$

$$
h^4 \lambda_0 \alpha \left\| U_n' \right\|_\infty \left\| \frac{d^4 V_n}{dx^4} \right\|_\infty + h^3 \lambda_1 \alpha \left\| U_n' \right\|_\infty \left\| \frac{d^4 V_n}{dx^4} \right\|_\infty + h^7 \eta \lambda_0 \lambda_1 \left\| \frac{d^4 V_n}{dx^4} \right\|_\infty^2 +
$$

$$
h^4 \alpha \lambda_0 \left\| V_n' \right\|_\infty \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty +
$$

$$
h^3 \alpha \lambda_1 \left\| V_n \right\|_\infty \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty + h^7 \alpha \lambda_0 \lambda_1 \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty \left\| \frac{d^4 V_n}{dx^4} \right\|_\infty +
$$

$$
\left. h^7 \alpha \lambda_0 \lambda_1 \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty \left\| \frac{d^4 V_n}{dx^4} \right\|_\infty \right] + h^4 \lambda_0 \left\| \frac{d^4 U_n}{dx^4} \right\|_\infty, \tag{7.6}
$$

after simplifying, we get

$$
\left| \mathbf{X}_m^n - \hat{\mathbf{X}}_m^n \right| \leq h^2 \left[\Delta t \left(\lambda_2 L + h^2 \eta \lambda_0 ML + h \eta \lambda_1 ML + h^2 \alpha \lambda_0 ML + h^1 \alpha \lambda_1 ML \right. \right. \tag{7.7}
$$

$$
\left. \left. + h^5 \eta \lambda_0 \lambda_1 L^2 + h^2 \alpha \lambda_0 ML + h \alpha \lambda_1 ML + 2 h^5 \alpha \lambda_0 \lambda_1 L^2 \right) + h^2 \lambda_0 L \right].
$$

We can rewrite (7.7) as follows

$$
\left| \mathbf{X}_m^n - \hat{\mathbf{X}}_m^n \right| \leq h^2 M_1, \tag{7.8}
$$

where

$$
M_1 = \Delta t \left(\lambda_2 L + h^2 \eta \lambda_0 ML + h \eta \lambda_1 ML + h^2 \alpha \lambda_0 ML + h^1 \alpha \lambda_1 ML \right.
$$

$$
\left. + h^5 \eta \lambda_0 \lambda_1 L^2 + h^2 \alpha \lambda_0 ML + h \alpha \lambda_1 ML + 2 h^5 \alpha \lambda_0 \lambda_1 L^2 \right) + h^2 \lambda_0 L.
$$

Similar results can be obtained for $\mathbf{R}_m^n - \hat{\mathbf{R}}_m^n$, i.e.

$$
\left| \mathbf{R}_m^n - \hat{\mathbf{R}}_m^n \right| \leq h^2 M_2, \tag{7.9}
$$

where

$$
M_2 = \Delta t \left(\lambda_2 L + h^2 \eta \lambda_0 ML + h \eta \lambda_1 ML + h^2 \beta \lambda_0 ML + h^1 \beta \lambda_1 ML \right.
$$

$$
\left. + h^5 \eta \lambda_0 \lambda_1 L^2 + h^2 \beta \lambda_0 ML + h \beta \lambda_1 ML + 2 h^5 \beta \lambda_0 \lambda_1 L^2 \right) + h^2 \lambda_0 L.
$$

Setting $\mathbf{M} = \max\{M_1, M_2\}$, we have

$$
\left| \mathbf{X}_m^n - \hat{\mathbf{X}}_m^n \right| \leq \mathbf{M} h^2, \tag{7.10}
$$

$$
\left| \mathbf{R}_m^n - \hat{\mathbf{R}}_m^n \right| \leq \mathbf{M} h^2. \tag{7.11}
$$

From (7.4), (7.10) and (7.11), it is deduced that

$$\left\| B - \hat{B} \right\|_\infty \leq \mathbf{M} h^2. \tag{7.12}$$

Since, the matrix A in (7.3) is an ill-posed matrix, from the Tikhonov regularized solution (6.1), we get

$$(X - \hat{X}) = [A^T A + \alpha (R^{(z)})^T R^{(z)}]^{-1} A^T (B - \hat{B}).$$

Using the relation (7.12) and taking the infinity norm, we find

$$
\begin{aligned}
\|X - \hat{X}\|_\infty &\leq \|(A^T A + \alpha (R^{(z)})^T R^{(z)})^{-1} A^T \|_\infty \|B - \hat{B}\|_\infty \\
&\leq \|(A^T A + \alpha (R^{(z)})^T R^{(z)})^{-1} A^T \|_\infty \mathbf{M} h^2 \\
&\leq \mathbf{M}_1 h^2,
\end{aligned} \tag{7.13}
$$

where $\mathbf{M}_1 = \|(A^T A + \alpha (R^{(z)})^T R^{(z)})^{-1} A^T \|_\infty \mathbf{M}$. Now, we compute $\left\| \left(u_n - U_n, v_n - V_n \right) \right\|_\infty$ as the following

$$
\begin{aligned}
\left\| \left(u_n - U_n, v_n - V_n \right) \right\|_\infty &= \left\| u_n - U_n \right\|_\infty + \left\| v_n - V_n \right\|_\infty \\
&\leq \left\| u_n - \hat{U}_n \right\|_\infty + \left\| \hat{U}_n - U_n \right\|_\infty + \left\| v_n - \hat{V}_n \right\|_\infty + \left\| \hat{V}_n - V_n \right\|_\infty
\end{aligned}
$$

such that

$$U_n(x) - \hat{U}_n(x) = \sum_{i=-1}^{N+1} (c_i^n - \hat{c}_i^n) B_i(x),$$

$$\left| U_n(x_m) - \hat{U}_n(x_m) \right| \leq \max_{-1 \leq i \leq N+1} \left\{ \left| c_i^n - \hat{c}_i^n \right| \right\} \sum_{i=-1}^{N+1} \left| B_i(x_m) \right|, \qquad 0 \leq m \leq N,$$

and

$$V_n(x) - \hat{V}_n(x) = \sum_{i=-1}^{N+1} (d_i^n - \hat{d}_i^n) B_i(x),$$

$$\left| V_n(x_m) - \hat{V}_n(x_m) \right| \leq \max_{-1 \leq i \leq N+1} \left\{ \left| d_i^n - \hat{d}_i^n \right| \right\} \sum_{i=-1}^{N+1} \left| B_i(x_m) \right|, \qquad 0 \leq m \leq N.$$

By using the values of $B_i(x_m)$'s given in Section 3, one can easily see that $\sum_{i=-1}^{N+1} |B_i(x_m)| \leq 10$, $0 \leq m \leq N$ [32], therefore

$$\left\| U_n(x_m) - \hat{U}_n(x_m) \right\|_\infty \leq 10 \mathbf{M}_1 h^2, \qquad \left\| V_n(x_m) - \hat{V}_n(x_m) \right\|_\infty \leq 10 \mathbf{M}_1 h^2. \tag{7.14}$$

So, according to (7.5) and (7.14), we obtain

$$
\begin{aligned}
\left\| \left(u_n - U_n, v_n - V_n \right) \right\|_\infty &\leq \lambda_0 L h^4 + 10 \mathbf{M}_1 h^2 + \lambda_0 L h^4 + 10 \mathbf{M}_1 h^2 \\
&= h^2 (2 \lambda_0 L h^2 + 20 \mathbf{M}_1).
\end{aligned}
$$

Setting $\gamma = 2 \lambda_0 L h^2 + 20 \mathbf{M}_1$, we have

$$\left\| \left(u_n - U_n, v_n - V_n \right) \right\|_\infty \leq \gamma h^2.$$

$\square$

Theorem 7.3. *Let $u(x,t)$ and $v(x,t)$ be the solutions of the initial boundary value problem (2.1)-(2.4). Also, suppose that $U_n(x)$ and $V_n(x)$ are the collocation approximation to the solutions $u_n(x)$ and $v_n(x)$ after the temporal discretization stage. Then the error estimate of the totally discrete scheme is given by*

$$\left\|\left(u_n - U_n, v_n - V_n\right)\right\|_\infty \leq \tau(\Delta t + h^2),$$

where τ is some finite constant.

Proof . The time discretization process (4.3) that we use to discretize the system (2.1)-(2.4) in time variable is of the one order convergence (see, [33]). So, according to the Theorem 1, we have

$$\left\|\left(u_n - U_n, v_n - V_n\right)\right\|_\infty \leq \tau(\Delta t + h^2),$$

where τ is some finite constant. Thus the order of convergence of our process is $O(\Delta t + h^2)$.
□

8. The stability analysis

For stability analysis, we use the Von-Neumann technique. For this purpose, we get

$$
\left(c_{m-1}^{n+1} + 4c_m^{n+1} + c_{m+1}^{n+1}\right) = \Delta t \left\{ \frac{6}{h^2}\left(c_{m-1}^n - 2c_m^n + c_{m+1}^n\right) - \right.
$$
$$
\frac{3}{h}\eta\left(c_{m-1}^n + 4c_m^n + c_{m+1}^n\right)\left(c_{m+1}^n - c_{m-1}^n\right) -
$$
$$
\frac{3}{h}\alpha\left(c_{m-1}^n + 4c_m^n + c_{m+1}^n\right)\left(d_{m+1}^n - d_{m-1}^n\right) -
$$
$$
\left. \frac{3}{h}\alpha\left(d_{m-1}^n + 4d_m^n + d_{m+1}^n\right)\left(c_{m+1}^n - c_{m-1}^n\right) \right\} +
$$
$$
\left(c_{m-1}^n + 4c_m^n + c_{m+1}^n\right), \tag{8.1}
$$

$$
\left(d_{m-1}^{n+1} + 4d_m^{n+1} + d_{m+1}^{n+1}\right) = \Delta t \left\{ \frac{6}{h^2}\left(d_{m-1}^n - 2d_m^n + d_{m+1}^n\right) - \right.
$$
$$
\frac{3}{h}\eta\left(d_{m-1}^n + 4d_m^n + d_{m+1}^n\right)\left(d_{m+1}^n - d_{m-1}^n\right) -
$$
$$
\frac{3}{h}\beta\left(c_{m-1}^n + 4c_m^n + c_{m+1}^n\right)\left(d_{m+1}^n - d_{m-1}^n\right) -
$$
$$
\left. \frac{3}{h}\beta\left(d_{m-1}^n + 4d_m^n + d_{m+1}^n\right)\left(c_{m+1}^n - c_{m-1}^n\right) \right\} +
$$
$$
\left(d_{m-1}^n + 4d_m^n + d_{m+1}^n\right). \tag{8.2}
$$

Setting $c_m^n = \xi_1^n e^{im\psi h}$ and $d_m^n = \xi_2^n e^{im\psi h}$, then by substituting in the Eqs. (8.1), (8.2) and simplifying them, it is obtained that

where ξ_1 and ξ_2 are the amplification factors for the scheme, ψ is the mode number, h is the element size and $i = \sqrt{-1}$ and $\theta = \psi h$. We consider the solution of (**??**) in the vector notation

$$
T(\xi_1, \xi_2) = \begin{pmatrix} T_1(\xi_1, \xi_2) \\ \\ T_2(\xi_1, \xi_2) \end{pmatrix}
$$

$$
= \begin{pmatrix} \left(4 + 2\cos(\theta)\right)\left(\xi_1 - 1\right) - \Delta t \left\{ \dfrac{6}{h^2}\left(2\cos(\theta) - 2\right) - \\ \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\left(\dfrac{3\eta}{h}\xi_1^n + \dfrac{6\alpha}{h}\xi_2^n\right)\right\} \\ \\ \left(4 + 2\cos(\theta)\right)\left(\xi_2 - 1\right) - \Delta t \left\{ \dfrac{6}{h^2}\left(2\cos(\theta) - 2\right) - \\ \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\left(\dfrac{3\eta}{h}\xi_2^n + \dfrac{6\beta}{h}\xi_1^n\right)\right\} \end{pmatrix}.
$$

It is convenient to introduce the Jacobian matrix for the functions T_1 and T_2

$$
T'(\xi_1, \xi_2) = \begin{pmatrix} \dfrac{\partial T_1}{\partial \xi_1} & \dfrac{\partial T_1}{\partial \xi_2} \\ \\ \dfrac{\partial T_2}{\partial \xi_1} & \dfrac{\partial T_2}{\partial \xi_2} \end{pmatrix}
$$

$$
= \begin{pmatrix} \dfrac{\left(4 + 2\cos(\theta)\right)}{\Delta t} + & \\ \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{3n\eta}{h}\xi_1^{n-1} & \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{6n\alpha}{h}\xi_2^{n-1} \\ \\ \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{6n\beta}{h}\xi_1^{n-1} & \dfrac{\left(4 + 2\cos(\theta)\right)}{\Delta t} + \\ & \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{3n\eta}{h}\xi_2^{n-1} \end{pmatrix}.
$$

We know that if $\|T'\| \leq 1$ then T has a unique solution (ξ_1, ξ_2), (see [34]) and if $|\xi_1|, |\xi_2| \leq 1$ then the scheme (4.3) is stable. Now, to investigate the above-mentions points, we have

$$
\left\|T'\right\| = \max \left\{ \left| \dfrac{\left(4 + 2\cos(\theta)\right)}{\Delta t} + \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{3n\eta}{h}\xi_1^{n-1} \right| + \right.
$$

$$
\left| \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{6n\beta}{h}\xi_1^{n-1} \right|,
$$

$$
\left| \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{6n\alpha}{h}\xi_2^{n-1} \right| +
$$

$$
\left. \left| \dfrac{\left(4 + 2\cos(\theta)\right)}{\Delta t} + \left(2i\sin(\theta)\right)\left(4 + 2\cos(\theta)\right)\dfrac{3n\eta}{h}\xi_2^{n-1} \right| \right\}.
$$

At first, suppose that

$$\left\| T' \right\| = \left| \frac{(4+2cos(\theta))}{\Delta t} + \left(2isin(\theta)\right)\left(4+2cos(\theta)\right)\frac{3n\eta}{h}\xi_1^{n-1} \right| +$$

$$\left| \left(2isin(\theta)\right)\left(4+2cos(\theta)\right)\frac{6n\beta}{h}\xi_1^{n-1} \right|,$$

then, from triangle inequality, we have

$$\left\| T' \right\| \leq \left| \frac{6}{\Delta t} \right| + 12\left| \frac{3n\eta}{h} \right|\left| \xi_1^{n-1} \right| + 12\left| \frac{6n\beta}{h} \right|\left| \xi_1^{n-1} \right|$$

$$\leq \left(\frac{6}{\Delta t} \right) + \left(\frac{72n}{h} \right)|\xi_1|^{n-1}\left(|\eta| + |\beta| \right).$$

If $\left\| T' \right\| \leq 1$, then

$$|\xi_1| \leq \sqrt[n-1]{\frac{\Delta t - 6}{\Delta t C_1}},$$

where $C_1 = \left(\frac{72n}{h} \right)\left(|\eta| + |\beta| \right)$, and if $\sqrt[n-1]{\frac{\Delta t - 6}{\Delta t C_1}} < 1$ then $|\xi_1| < 1$ and the scheme is stable. We conclude similarly that $|\xi_2| < 1$ when $\sqrt[n-1]{\frac{\Delta t - 6}{\Delta t C_2}} < 1$, where $C_2 = \left(\frac{72n}{h} \right)\left(|\eta| + |\alpha| \right)$.

9. Numerical results and discussion

In this Section, we are going to study numerically the inverse problem (2.1)–(2.4) with the unknown boundary conditions. The main aim here is to show the applicability of the present method for solving inverse problems. As expected the inverse problems are ill-posed and therefore it is necessary to investigate the stability of the present method by giving a test problem.

Remark 9.1. *In an inverse problem, there are two sources of error in the estimation. The first source is the unavoidable bias deviation or deterministic error, and the second source of error is the variance due to the amplification of measurement errors or stochastic error. The global effect of deterministic and stochastic errors is considered in the root mean square or total error [35]. Therefore, we compute total error S by using following formula*

$$RMS = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(p(t_i)^{exact} - p(t_i)^{numerical} \right)^2},$$

where N is the total number of estimated values.

The comparison between the exact solutions of $p_1(t)$, $p_2(t)$ and numerical solutions of the cubic B-spline (CBS) method, trigonometric cubic B-spline (TCBS) method, exponential cubic B-spline (ECBS) method and radial basis function (RBF) method, with noisy data are presented. Also, in all calculations, we put $T = 1$, $N = \frac{1}{10}$ and $\Delta t = \frac{1}{1000}$.

Example 9.2. *In this example we solve inverse parabolic system (2.1)-(2.4) satisfying,*

$$\begin{cases} u_t - u_{xx} + 2uu_x - 2(uv)_x = 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq T, \\ \\ v_t - v_{xx} + 2vv_x - \frac{1}{2}(uv)_x = 0, & 0 \leq x \leq 1, \quad 0 \leq t \leq T, \end{cases}$$

time	Exact	CBS	TCBS	ECBS	RBF
t	$p_1(t)$	method	method	method	method
0.1	0.094162	0.091238	0.091632	0.091004	0.093230
0.2	0.098807	0.094954	0.095098	0.094312	0.095370
0.3	0.103940	0.099564	0.099661	0.098584	0.097129
0.4	0.109613	0.104749	0.104844	0.103529	0.098319
0.5	0.115883	0.110497	0.110605	0.109079	0.098618
0.6	0.122812	0.116853	0.116981	0.115247	0.098683
0.7	0.130469	0.123877	0.124031	0.122076	0.098750
0.8	0.138933	0.131641	0.131827	0.129627	0.098820
0.9	0.148286	0.140221	0.140448	0.137973	0.098894
1	0.158623	0.149703	0.149980	0.147198	0.098973
RMS		5.8282×10^{-3}	5.6672×10^{-3}	7.3314×10^{-3}	2.2802×10^{-2}
Execution time (second)		5.438	5.5	5.678	12.078
Condition number		Inf	Inf	Inf	4.1354×10^{29}
Regularization parameter		3.6633	0.61175	0.79852	7.3538×10^{-2}

Table 1: The comparison among exact and numerical solutions $u(x,0) = p_1(t)$ of Example 9.2 with noisy data.

with initial conditions

$$u(x,0) = (0.05) + (0.04)\, e^{-x}, \qquad v(x,0) = (0.025) + (0.02)\, e^{-x}, \qquad 0 \le x \le 1,$$

and boundary conditions as follows

$$\begin{cases} u(0.2,t) = (0.05) + (0.04)\, e^{-0.2+t}, & u(1,t) = (0.05) + (0.04)\, e^{-1+t}, \qquad 0 \le t \le T, \\ \\ v(0.2,t) = (0.025) + (0.02)\, e^{-0.2+t}, & v(1,t) = (0.025) + (0.02)\, e^{-1+t}, \qquad 0 \le t \le T. \end{cases}$$

The exact solution of this problem is

$$u(x,t) = (0.05) + (0.04)\, e^{-x+t}, \quad v(x,t) = (0.025) + (0.02)\, e^{-x+t}, \quad 0 \le x \le 1, \quad 0 \le t \le T.$$

Figure 1: The plots of approximation and exact solutions of $p_1(t)$ and $p_2(t)$ for Example 9.2 with noisy data.

Figure 2: The plots of approximation and exact solutions of $u(x,t)$ and $v(x,t)$ for Example 9.2 with noisy data.

time t	Exact $p_2(t)$	CBS method	TCBS method	ECBS method	RBF method
0.1	0.047081	0.045744	0.045828	0.045784	0.046615
0.2	0.049403	0.047529	0.047563	0.047372	0.047685
0.3	0.051970	0.049821	0.049844	0.049470	0.048564
0.4	0.054806	0.052410	0.052432	0.051920	0.049159
0.5	0.057941	0.055283	0.055308	0.054683	0.049309
0.6	0.061406	0.058460	0.058490	0.057760	0.049341
0.7	0.065234	0.061972	0.062009	0.061171	0.049375
0.8	0.069466	0.065854	0.065898	0.064944	0.049410
0.9	0.074142	0.070144	0.070197	0.069116	0.049447
1	0.079311	0.074885	0.074950	0.073727	0.049486
RMS		2.8759×10^{-3}	2.8396×10^{-3}	3.5250×10^{-3}	1.4013×10^{-2}

Table 2: The comparison among exact and numerical solutions $v(x,0) = p_2(t)$ of Example 9.2 with noisy data.

Example 9.3. *In this example let us consider the following inverse problem*

$$\begin{cases} u_t - u_{xx} + 2uu_x - (\frac{5}{2})(uv)_x = 0, & 0 \le x \le 1, \quad 0 \le t \le T, \\ v_t - v_{xx} + 2vv_x + (\frac{3}{10})(uv)_x = 0, & 0 \le x \le 1, \quad 0 \le t \le T, \end{cases}$$

with initial conditions

$$u(x,0) = (0.05) + \frac{2}{1+e^x}, \qquad v(x,0) = (0.01) + \frac{0.4}{1+e^x}, \qquad 0 \le x \le 1,$$

and boundary conditions as follows

$$\begin{cases} u(0.2,t) = (0.05) + \frac{2}{1+e^{0.2-1.05t}}, & u(1,t) = (0.05) + \frac{2}{1+e^{1-1.05t}}, & 0 \le t \le T, \\ v(0.2,t) = (0.01) + \frac{0.4}{1+e^{0.2-1.05t}}, & v(1,t) = (0.01) + \frac{0.4}{1+e^{1-1.05t}}, & 0 \le t \le T. \end{cases}$$

The exact solution of this problem is

$$u(x,t) = (0.05) + \frac{2}{1+e^{x-1.05t}}, \quad v(x,t) = (0.01) + \frac{0.4}{1+e^{x-1.05t}}, \qquad 0 \le x \le 1, \quad 0 \le t \le T.$$

Figure 3: The plots of approximation and exact solutions of $p_1(t)$ and $p_2(t)$ for Example 9.3 with noisy data.

time	Exact	CBS	TCBS	ECBS	RBF
t	$p_1(t)$	method	method	method	method
0.1	1.101928	1.101864	1.079279	1.070033	1.094579
0.2	1.154096	1.155921	1.135741	1.126184	1.131712
0.3	1.205698	1.210353	1.190105	1.181980	1.175817
0.4	1.256463	1.263973	1.243532	1.236249	1.220568
0.5	1.306141	1.316642	1.295947	1.289273	1.264546
0.6	1.354502	1.368154	1.347084	1.340972	1.299295
0.7	1.401342	1.418199	1.396686	1.391133	1.326525
0.8	1.446488	1.466456	1.444526	1.439534	1.352196
0.9	1.489793	1.512658	1.490409	1.485976	1.374065
1	1.531146	1.556606	1.534177	1.530290	1.393470
RMS		1.3432×10^{-2}	1.3235×10^{-2}	1.9313×10^{-2}	1.3396×10^{-2}
Execution time (second)		5.312	6.062	5.859	10.678
Condition number		Inf	Inf	Inf	4.1354×10^{29}
Regularization parameter		3.6156	0.5287	1.0001	2.6518×10^{-4}

Table 3: The comparison among exact and numerical solutions $u(x,0) = p_1(t)$ of Example 9.3 with noisy data.

time	Exact	CBS	TCBS	ECBS	RBF
t	$p_2(t)$	method	method	method	method
0.1	0.220386	0.220594	0.218419	0.217274	0.218915
0.2	0.230819	0.231192	0.229213	0.227109	0.226342
0.3	0.241139	0.241958	0.240105	0.237639	0.235163
0.4	0.251293	0.252702	0.250864	0.248301	0.244113
0.5	0.261228	0.263304	0.261412	0.258863	0.252909
0.6	0.270901	0.273667	0.271689	0.269207	0.259859
0.7	0.280269	0.283714	0.281646	0.279256	0.265305
0.8	0.289298	0.293387	0.291233	0.288953	0.270439
0.9	0.297959	0.302639	0.300414	0.298254	0.274813
1	0.306229	0.311435	0.309157	0.307124	0.278694
RMS		2.7323×10^{-3}	1.5662×10^{-3}	2.4290×10^{-2}	1.3396×10^{-2}

Table 4: The comparison among exact and numerical solutions $v(x,0) = p_2(t)$ of Example 9.3 with noisy data.

Figure 4: The plots of approximation and exact solutions of $u(x,t)$ and $v(x,t)$ for Example 9.3 with noisy data.

10. Conclusion

A numerical method, to estimate unknown boundary conditions is proposed and the following results are obtained.

- The present study successfully applies the numerical method to inverse problems.

- Simplicity of implementation and less computational cost can be mentioned as main advantages of the proposed scheme compared to previous proposals.

- Unlike some previous techniques using various transformations to reduce the equation in to more simple equation, the current method does not require extra attempt to deal with the nonlinear terms. Therefore, the equations are solved easily and daintily using the present method.

- Numerical results show that our approximations of unknown functions using some the B-spline functions in collocation method are more accurate than the numerical results of the radial basis function method, also the execution time in the cubic B-spline method is faster than the other methods.

- Numerical examples also verified the efficiency and accuracy of the method that can be obtained within a couple of minutes CPU time at Core(i5)–2.67 GHz PC.

- The present method has been found stable with respect to small perturbation in the input data.

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