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Coupled fixed point theorems for rational type contractions via C-class functions

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Abstract

The main purpose of the paper is to extend some results of the coupled fixed point theorems, based on some previous works [15, 16], by using C-class functions. First part of the paper is related to some fixed point theorems, the second part presents the uniqueness and existence for the solution of the coupled fixed point problem and in the third part we discuss data dependence, well-posedness, Ulam-Hyers stability and limit shadowing property of the coupled fixed point set.

Keywords: Fixed point, ordered metric space, rational type contraction, coupled fixed point, c-class functions, data dependence, well-posedness, Ulam-Hyers stability, limit shadowing property.
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1. Introduction

In this presented work, we consider both of the above research directions (b-metric spaces and coupled fixed point problem ([7, 6, 10, 13, 14, 12])), and many other results related to this kind of problem (see [3, 8, 12, 15, 16]). More precisely, by using C-class functions, we will prove some fixed point theorems for monotone rational contractions in ordered b-metric spaces. Also, some coupled fixed point theorems for operators $T : X \times X \to X$ satisfying some rational type assumptions on comparable elements. Finally, data dependence, well-posedness, Ulam- Hyers stability, limit shadowing properties for the coupled fixed point problem are presented.

We shall recall some well known notions and definition of the b-metric spaces.

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Definition 1.1. *Let X be a set and let $s \geq 1$ be a given real number. A functional $d : X \times X \to \mathbb{R}^+$ is said to be a b-metric if the following axioms are satisfied:*

(i) if $x, y \in X$, then $d(x, y) = 0$ if and only if $x = y$;

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(iii) $d(x, z) \leq s[d(x, y) + d(y, z)]$ for all $x, y, z \in X$.

A pair (X, d) with the above properties is called a b-metric space.

Let $(X, \leq)$ be a partially ordered set and d a metric on X. Notice that we can endow the product space $X \times X$ with the following partial order:

$$\text{for} \quad (x, y), (u, v) \in X \times X, \quad \text{we write} \quad (x, y) \leq_p (u, v) \Leftrightarrow x \leq u, y \geq v.$$

Definition 1.2. *Let $(X, \leq)$ be an partially ordered set and A, B be two nonempty subsets of X. Then we will wrote $A \leq_s B$ if and only for all $a \in A$ exists $b \in B$ satisfying $a \leq b$.*

Definition 1.3. *Let $(X, \leq)$ be a partially ordered set and let $T : X \times X \to X$. We say that T has the mixed monotone property if $T(\cdot, y)$ is monotone increasing for any $y \in X$ and $T(x, \cdot)$ is monotone decreasing for any $x \in X$.*

Lemma 1.4. Let (X, d) be a b-metric space. Then the sequence $\{x_n\} \in X$ is called:

(i) convergent if and only if there exists $x \in X$ such that $d(x_n, x) \to 0$ as $n \to \infty$. In this case we write $\lim\limits_{n\to\infty} x_n = x$;

(ii) Cauchy if and only if $d(x_n, x_m) \to 0$ as $n, m \to \infty$.

Let $(X, \leq)$ be a partially ordered set and $(x, y), (u, v) \in X \times X$. We have the partial order $(x, y) \leq_p (u, v)$ if and only if $x \leq u, y \geq v$.

If (X, d) is a metric space and $T : X \times X \to X$ is an operator, then by definition, a coupled fixed point for T is a pair $(x^*, y^*) \in X \times X$ satisfying

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*) \end{cases} \qquad (P_1)$$

We will denote by $CFix(T)$ the coupled fixed point set for T.

Definition 1.5. *([2]) A mapping $F : [0, \infty)^2 \to \mathbb{R}$ is called a C-class function if it is continuous and satisfies the following axioms:*

(i) $F(s, t) \leq s$ for all $s, t \in [0, \infty)$;

(ii) $F(s, t) = s$ implies that either $s = 0$ or $t = 0$.

Mention that some C-class function F verifies $F(0, 0) = 0$. We denote by $\mathcal{C}$ the set of C-class functions.

Remark 1.6. *([2]) Let Φ_u denote the class of the functions $\varphi : [0, \infty) \to [0, \infty)$ which satisfy the following conditions:*

(a) φ is continuous;

(b) $\varphi(t) > 0, t > 0$ and $\varphi(0) \geq 0$.

Definition 1.7. *([11]) A function $\psi : [0,\infty) \to [0,\infty)$ is called an altering distance function if the following properties are satisfied:*
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

We let Ψ denote the class of the altering distance functions.

Definition 1.8. *A function $\psi : \mathbb{R} \to \mathbb{R}$ is called an infinite altering distance function if the following properties are satisfied:*
(i) ψ is non-decreasing and continuous,
(ii) $\psi(t) = 0$ if and only if $t = 0$.

We let Ψ_{inf} denote the class of the infinite altering distance functions.

Definition 1.9. *A tripled (ψ, φ, F) where $\psi \in \Psi$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$ is say to be monotone if for any $x, y \in [0,\infty)$*

$$x \leqslant y \Longrightarrow F(\psi(x), \varphi(x)) \leqslant F(\psi(y), \varphi(y)).$$

Example 1.10. *Let $F(s,t) = s - t, \phi(x) = \sqrt{x}$*

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x > 1 \end{cases},$$

then (ψ, ϕ, F) is monotone.

Example 1.11. *Let $F(s,t) = s - t, \phi(x) = x^2$*

$$\psi(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ x^2, & \text{if } x > 1 \end{cases},$$

then (ψ, ϕ, F) is not monotone.

Lemma 1.12. *([1]) Let (X,d) be a b-metric space with $s \geq 1$, and suppose that $\{x_n\}$ and $\{y_n\}$ are b-convergent to x, y, respectively. Then we have*

$$\frac{1}{s^2} d(x,y) \leq \liminf_{n\to\infty} d(x_n, y_n) \leq \limsup_{n\to\infty} d(x_n, y_n) \leq s^2 d(x,y).$$

In particular, if $x = y$, then we have $\lim_{n\to\infty} d(x_n, y_n) = 0$. Moreover, for each $z \in X$, we have,

$$\frac{1}{s} d(x,z) \leq \liminf_{n\to\infty} d(x_n, z) \leq \limsup_{n\to\infty} d(x_n, z) \leq s d(x,z).$$

2. Fixed point theorems via C-class functions

In this section, we will present fixed point theorems in ordered b-metric spaces, under a rational type contraction condition and using C-class functions.

Theorem 2.1. *Let $(X, \leq)$ be a partially ordered set and $d : X \times X \to X$ be a complete b-metric with constant $s > 1$. Let $f : X \to X$ be an operator which has closed graph (in particular, it is continuous) with respect to d and increasing with respect to " $\leq$ ". Suppose that there exists $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$ such that (ψ, φ, F) is monotone and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ satisfying*

$$\begin{aligned} \psi(d(f(x), f(y))) \leq\ & F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta \cdot d(x, y)]), \\ & \varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(y, f(y))[1 + d(x, f(x))]}{1 + d(x, y)} + \beta \cdot d(x, y)])\Big), \end{aligned} \tag{2.1}$$

for $x, y \in X$ with $x \leq y$. If there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$, there exists $x^* \in X$ such that $x^* = f(x^*)$ and $f^n(x_0) \to x^*$, as $n \to \infty$.

Proof . We have two cases:

Case 1. If $f(x_0) = x_0$, then $Fix(f) \neq \emptyset$.

Case 2. Suppose that $x_0 < f(x_0)$. Using that f is an increasing operator and by mathematical induction, we have

$$x_0 < f(x_0) \leq f^2(x_0) \leq ... \leq f^n(x_0) \leq f^{n+1}(x_0) \leq ... . \tag{2.2}$$

Using (2.2), we define the sequence $(x_n) \in X$ by

$$x_{n+1} = f(x_n) = f(f(x_{n-1})) = f^2(x_{n-1}) = ... = f^n(x_1) = f^{n+1}(x_0)$$

for each $n \in \mathbb{N}$. If $x_{n+1} = x_n$ for some $n \in \mathbb{N}$, then f has a fixed point. In particular x_n is a fixed point of f, that is $Fix(f) \neq \emptyset$.

Let $x_{n+1} \neq x_n$ for $n \geq 0$. Since $x_n \leq x_{n+1}$ for any $n \in \mathbb{N}$, from (2.1) we have

$$\begin{aligned} &\psi(d(x_n, x_{n+1})) = \psi(d(f(x_{n-1}), f(x_n))) \\ &\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_n, f(x_n))[1 + d(x_{n-1}, f(x_{n-1}))]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)]), \\ &\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_n, f(x_n))[1 + d(x_{n-1}, f(x_{n-1}))]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)])\Big) \\ &= F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)]), \\ &\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_n, x_{n+1})[1 + d(x_{n-1}, x_n)]}{1 + d(x_{n-1}, x_n)} + \beta \cdot d(x_{n-1}, x_n)])\Big) \\ &= F\Big(\psi(\frac{1}{\alpha+\beta s}[\alpha \cdot d(x_n, x_{n+1}) + \beta \cdot d(x_{n-1}, x_n)]), \\ &\varphi(\frac{1}{\alpha+\beta s}[\alpha \cdot d(x_n, x_{n+1}) + \beta \cdot d(x_{n-1}, x_n)])) \\ &\leq \psi(\frac{1}{\alpha+\beta s}[\alpha \cdot d(x_n, x_{n+1}) + \beta \cdot d(x_{n-1}, x_n)]\Big). \end{aligned} \tag{2.3}$$

Therefore

$$d(x_n, x_{n+1}) \leq \frac{1}{\alpha + \beta s}[\alpha \cdot d(x_n, x_{n+1}) + \beta \cdot d(x_{n-1}, x_n)], \tag{2.4}$$

and (2.4) implies that

$$d(x_n, x_{n+1}) \leq \frac{1}{s} \cdot d(x_{n-1}, x_n) \leq d(x_{n-1}, x_n),$$

for any $n \in \mathbb{N}$.

Suppose that

$$d(x_n, x_{n+1}) \to r \geq 0,$$

then, with $n \to \infty$ in (2.3) we get

$$\psi(r) \leq F(\psi(r), \varphi(r)) \Longrightarrow \psi(r) = 0 \quad \text{or} \quad \varphi(r) = 0$$

that is,

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.5}$$

Now, we will prove that $x_n = f^n(x_0)$ is a b-Cauchy sequence. Suppose the contrary, i.e., that $\{x_m\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \text{ and } d(x_{m_i}, x_{n_i}) \geq \varepsilon. \tag{2.6}$$

This means that

$$d(x_{m_i}, x_{n_i - 1}) < \varepsilon. \tag{2.7}$$

From (2.6) and using the triangular inequality, we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+1}) + sd(x_{m_i+1}, x_{n_i}).$$

By taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{2.8}$$

Using the triangular inequality, we have

$$d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{n_i-1}) + sd(x_{n_i-1}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$ in the above inequality and using (2.7) we get

$$\limsup_{i\to\infty} d(x_{m_i}, x_{n_i}) \leq \varepsilon s. \tag{2.9}$$

Now, from (2.1) we have

$$\begin{aligned}
\psi(d(x_{m_{i+1}}, x_{n_i})) &= \psi(d(fx_{m_i}, fx_{n_{i-1}})) \\
&\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_{n_{i-1}}, fx_{n_{i-1}})[1 + d(x_{m_i}, fx_{m_i})]}{1 + d(x_{m_i}, x_{n_{i-1}})} \\
&\quad + \beta \cdot d(x_{m_i}, x_{n_{i-1}})]), \\
&\quad \varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_{n_{i-1}}, fx_{n_{i-1}})[1 + d(x_{m_i}, fx_{m_i})]}{1 + d(x_{m_i}, x_{n_{i-1}})} \\
&\quad + \beta \cdot d(x_{m_i}, x_{n_{i-1}})])\Big).
\end{aligned}$$

Again, if $i \to \infty$ by (2.5), (2.7) and (2.8), we obtain

$$\begin{aligned}
\psi(\frac{\beta}{\alpha+\beta s}\varepsilon) &\leq \psi(\frac{\varepsilon}{s}) \\
&\leq \psi(\limsup_{i\to\infty} d(x_{m_{i+1}}, x_{n_i})) = \limsup_{i\to\infty} \psi(d(x_{m_{i+1}}, x_{n_i})) \\
&\leq \limsup_{i\to\infty} F\Big(\psi(\frac{\beta}{\alpha+\beta s}[\frac{\alpha \cdot d(x_{n_{i-1}}, fx_{n_{i-1}})[1+d(x_{m_i}, fx_{m_i})]}{1+d(x_{m_i}, x_{n_{i-1}})} \\
&\quad +\beta \cdot d(x_{m_i}, x_{n_{i-1}})]), \\
&\quad \varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_{n_{i-1}}, fx_{n_{i-1}})[1+d(x_{m_i}, fx_{m_i})]}{1+d(x_{m_i}, x_{n_{i-1}})} \\
&\quad +\beta \cdot d(x_{m_i}, x_{n_{i-1}})])\Big) \\
&= F\Big(\limsup_{i\to\infty} \psi(\frac{\beta}{\alpha+\beta s}[\frac{\alpha \cdot d(x_{n_{i-1}}, fx_{n_{i-1}})[1+d(x_{m_i}, fx_{m_i})]}{1+d(x_{m_i}, x_{n_{i-1}})} \\
&\quad +\beta \cdot d(x_{m_i}, x_{n_{i-1}})]), \\
&\quad \liminf_{i\to\infty} \varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot d(x_{n_{i-1}}, fx_{n_{i-1}})[1+d(x_{m_i}, fx_{m_i})]}{1+d(x_{m_i}, x_{n_{i-1}})} \\
&\quad +\beta \cdot d(x_{m_i}, x_{n_{i-1}})])\Big),
\end{aligned}$$

which implies that

$$\psi(\frac{\beta}{\alpha+\beta s}\varepsilon) \leq F\Big(\psi(\frac{\beta}{\alpha+\beta s}\varepsilon), \varphi(\frac{\beta}{\alpha+\beta s}\varepsilon))\Big) \leq \psi(\frac{\beta}{\alpha+\beta s}\varepsilon).$$

So, $\psi(\frac{\beta}{\alpha+\beta s}\varepsilon) = 0$ or $\varphi(\frac{\beta}{\alpha+\beta s}\varepsilon) = 0$, that is, ε which is a contradiction. Thus, $\{f^n(x_0)\}$ is a b-Cauchy sequence. Completeness of X yields that $\{f^n(x_0)\}$ converges to a point $x^* \in X$, that is, $f^n(x_0) \to x^*$ as $n \to \infty$.

Because f has closed graph, then $x^* \in Fix(f)$, which implies $Fix(f) \neq \emptyset$. Or f is continuous, we have

$$f(x^*) = f(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} x_{n+1} = x^*.$$

This ends the proof. □

Next, we extended the previous result to a global version:

Theorem 2.2. *Suppose that all the hypotheses of Theorem 2.1 are satisfied. Additionally, suppose that for all $x, y \in X$ there exists $z \in X$ such that $z \leq x$ and $z \leq y$. Then $Fix(f) = \{x^*\}$.*

Proof . Let $x^*, y^* \in X$ be two fixed points of f. We have two cases:

i) We suppose that x^* and y^* are comparable. That is $x^* \le y^*$ (or $y^* \le x^*$ is the same)

$$\begin{aligned}
\psi(\frac{\beta}{\alpha+\beta s}\cdot d(x^*,y^*)) &\le \psi(d(x^*,y^*)) \\
&= \psi(d(f(x^*),f(y^*))) \\
&\le F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(y^*,f(y^*))[1+d(x^*,f(x^*))]}{1+d(x^*,y^*)} \\
&\quad +\beta\cdot d(x^*,y^*)]), \\
&\quad \varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(y^*,f(y^*))[1+d(x^*,f(x^*))]}{1+d(x^*,y^*)} \\
&\quad +\beta\cdot d(x^*,y^*)])\Big) \\
&= F\Big(\psi(\frac{\beta}{\alpha+\beta s}\cdot d(x^*,y^*)),\varphi(\frac{\beta}{\alpha+\beta s}\cdot d(x^*,y^*))\Big) \\
&\le \psi(\frac{\beta}{\alpha+\beta s}\cdot d(x^*,y^*)),
\end{aligned}$$

so, $\psi(\frac{\beta}{\alpha+\beta s}\cdot d(x^*,y^*))=0$ or $\varphi(\frac{\beta}{\alpha+\beta s}\cdot d(x^*,y^*))=0$, that is, $d(x^*,y^*)=0$. This implies that $x^*=y^*$, so $Fix(f)=\{x^*\}$.

ii) Now, we suppose that x^* and y^* are not comparable. From the hypotheses of theorem, we have that there exists $z\in X$ with $z\le x^*$ and $z\le y^*$.

Since $z\le x^*$, then $f^n(z)\le f^n(x^*)=x^*$ for any $n\in\mathbb{N}$. Therefore,

$$\begin{aligned}
&\psi(d(f^n(z),x^*)) = \psi(d(f^n(z),f^n(x^*))) \\
&\le F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(f^{n-1}(x^*),f^n(x^*))[1+d(f^{n-1}(z),f^n(z)]}{1+d(f^{n-1}(z),f^{n-1}(x^*))} \\
&+\beta\cdot d(f^{n-1}(z),f^{n-1}(x^*))]), \\
&\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(f^{n-1}(x^*),f^n(x^*))[1+d(f^{n-1}(z),f^n(z)]}{1+d(f^{n-1}(z),f^{n-1}(x^*))} \\
&+\beta\cdot d(f^{n-1}(z),f^{n-1}(x^*))])\Big) \\
&= F\Big(\psi(\frac{\beta}{\alpha+\beta s}\cdot d(f^{n-1}(z),f^{n-1}(x^*))),\varphi(\frac{\beta}{\alpha+\beta s}\cdot d(f^{n-1}(z),f^{n-1}(x^*)))\Big) \\
&\le \psi(\frac{\beta}{\alpha+\beta s}\cdot d(f^{n-1}(z),f^{n-1}(x^*))) \le \psi(\frac{1}{s}\cdot d(f^{n-1}(z),f^{n-1}(x^*))),
\end{aligned}$$

which implies that

$$d(f^n(z),x^*)\le \frac{1}{s}\cdot d(f^{n-1}(z),x^*)\le (\frac{1}{s})^2\cdot d(f^{n-2}(z),x^*)\le ...\le (\frac{1}{s})^n\cdot d(z,x^*).$$

Since $(\frac{1}{s})<1$, $(\frac{1}{s})^n\to 0$,

$$\lim_{n\to\infty} d(f^n(z),x^*)=0.$$

This implies $\lim_{n\to\infty} f^n(z)=x^*$. In the same manner, we get that $\lim_{n\to\infty} f^n(z)=y^*$. Then $x^*=y^*$. □

Next, we extended the previous result to a global version:

Theorem 2.3. *Let (X,d) be a complete b-metric space with constant $s>1$, $f:X\to X$ be an operator of X with the following condition:*

Suppose that there exists $\psi\in\Psi_{\inf}$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$ such that (ψ,φ,F) is monotone and $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ satisfying

$$\begin{aligned}\psi(d(f(x),f(y))) \;\leq\; & F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(y,f(y))[1+d(x,f(x))]}{1+d(x,y)}+\beta\cdot d(x,y)]),\\ & \varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(y,f(y))[1+d(x,f(x))]}{1+d(x,y)}+\beta\cdot d(x,y)])\Big),\end{aligned} \tag{2.10}$$

for $x,y\in X$. Then f has a unique fixed point.

Proof . For an arbitrary point $x_0\in X$, we define the sequence (x_n) by $x_{n+1}=f(x_n)$ using the same method as in previous proof. We know that it is a Cauchy sequence.

Since (X,d) is a complete b-metric space, there exists $x^*\in X$ such that $\lim\limits_{n\to\infty}x_n=x^*$. Using (2.10) with $x=x_n$ and $y=x^*$, we get

$$\begin{aligned}&\psi(\frac{d(x^*,f(x^*))-s\cdot d(x^*,f(x_n))}{s})\leq\psi(d(f(x_n),f(x^*)))\\ &\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(x^*,f(x^*))[1+d(x_n,f(x_n))]}{1+d(x_n,x^*)}+\beta\cdot d(x_n,x^*)]),\\ &\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(x^*,f(x^*))[1+d(x_n,f(x_n))]}{1+d(x_n,x^*)}+\beta\cdot d(x_n,x^*)])\Big)\\ &=F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(x^*,f(x^*))[1+d(x_n,x_{n+1})]}{1+d(x_n,x^*)}+\beta\cdot d(x_n,x^*)]),\\ &\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(x^*,f(x^*))[1+d(x_n,x_{n+1})]}{1+d(x_n,x^*)}+\beta\cdot d(x_n,x^*)])\Big).\end{aligned} \tag{2.11}$$

Taking the limit as $n\to\infty$ in (2.11), and since F is monotone, we obtain

$$\begin{aligned}\psi(\frac{d(x^*,f(x^*))}{s}) \;\leq\; & F\Big(\psi(\frac{\alpha}{\alpha+\beta s}[d(x^*,f(x^*))]),\varphi(\frac{\alpha}{\alpha+\beta s}[d(x^*,f(x^*))])\Big)\\ \leq\; & F\Big(\psi(\frac{d(x^*,f(x^*))}{s}),\varphi(\frac{d(x^*,f(x^*))}{s})\Big)\\ \leq\; & \psi(\frac{d(x^*,f(x^*))}{s}).\end{aligned} \tag{2.12}$$

From (2.12), $\psi(\frac{d(x^*,f(x^*))}{s})=0$ or $\varphi(\frac{d(x^*,f(x^*))}{s})=0$, that is, $d(x^*,f(x^*))=0$. So $f(x^*)=x^*$, i.e. $Fix(f)\neq\emptyset$.

Now, we prove that x^* is the unique fixed point of f. Let y^* be another fixed point of f, i.e. $f(y^*) = y^*$. Thus

$$\begin{aligned}
&\psi(\frac{\beta}{\alpha+\beta s}\cdot d(y^*,x^*)) \leq \psi(d(y^*,x^*)) = \psi(d(f(y^*),f(x^*)))\\
\leq & F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(x^*,f(x^*))[1+d(y^*,f(y^*))]}{1+d(x^*,y^*)}+\beta\cdot d(y^*,x^*)]),\\
&\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot d(x^*,f(x^*))[1+d(y^*,f(y^*))]}{1+d(x^*,y^*)}+\beta\cdot d(y^*,x^*)])\Big)\\
&= F\Big(\psi(\frac{\beta}{\alpha+\beta s}\cdot d(y^*,x^*)),\varphi(\frac{\beta}{\alpha+\beta s}\cdot d(y^*,x^*))\Big)\\
&\leq \psi(\frac{\beta}{\alpha+\beta s}\cdot d(y^*,x^*)).
\end{aligned}$$

Hence,

$$\psi(\frac{\beta}{\alpha+\beta s}\cdot d(y^*,x^*)) = 0 \quad \text{or} \quad \varphi(\frac{\beta}{\alpha+\beta s}\cdot d(y^*,x^*)) = 0.$$

So, $d(y^*,x^*) = 0$, that is $y^* = x^*$.Therefore x^* is the unique fixed point of f. □

3. *C*-class functions for coupled fixed point theorems

In this part of the paper our results obtained in Section 2 are applied to coupled fixed point problem, in order to obtain new theorems.

Theorem 3.1. *Let $(X,\leq)$ be a partially ordered set and $d : X\times X \to \mathbb{R}^+$ be a complete b-metric on X with constant $s>1$. Let $T : X\times X \to X$ be an operator with closed graph (or in particular, it is continuous) which has the mixed monotone property on $X\times X$. Assume that the following conditions are satisfied:*

i) Suppose that there exists $\psi\in\Psi_{\inf}$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$ such that (ψ,φ,F) is monotone and $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ such that

$$\begin{aligned}
&\psi(d(T(x,y),T(u,v))+d(T(y,x),T(v,u))) \qquad (3.1)\\
&\leq F\Bigg(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}\\
&+\beta\cdot[d(x,u)+d(y,v)]]),\\
&\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}\\
&+\beta\cdot[d(x,u)+d(y,v)]]\Bigg),
\end{aligned}$$

for all $(x,y),(u,v)\in X\times X$ with $x\leq u, y\geq v$;

ii) There exist $x_0,y_0\in X$ such that $x_0\leq T(x_0,y_0), y_0\geq T(y_0,x_0)$, i.e. $(x_0,y_0)\leq_p (T(x_0,y_0),T(y_0,x_0))$.

Then, there exists $(x^*,y^*)\in X\times X$ a solution of the coupled fixed point problem (P_1), such that the sequences $(x_n),(y_n)$ in X defined by

$$\begin{cases} x_{n+1} = T(x_n,y_n),\\ y_{n+1} = T(y_n,x_n), \quad \forall n\in\mathbb{N}, \end{cases}$$

have the property that $x_n\to x^*, y_n\to y^*$ as $n\to\infty$.

Proof . Using ii) we have that $z_0 = (x_0, y_0) \leq_p (T(x_0, y_0), T(y_0, x_0)) = (x_1, y_1) = z_1$. So, $z_0 \leq_p z_1$.

If $x_2 = T(x_1, y_1)$ and $y_2 = T(y_1, x_1)$, it follow that $x_2 = T(x_1, y_1) = T^2(x_0, y_0)$ and $y_2 = T(y_1, x_1) = T^2(y_0, x_0)$. From the mixed monotone property of T,

$$x_2 = T(x_1, y_1) \geq T(x_0, y_0) = x_1,$$
$$y_2 = T(y_1, x_1) \leq T(y_0, x_0) = y_1.$$

Hence $z_1 = (x_1, y_1) \leq_p (x_2, y_2) = z_2$. Using this method, we construct the sequences $(x_n), (y_n)$ in X by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n). \end{cases}$$

From mathematical induction, we have $z_n = (x_n, y_n) \leq_p (x_{n+1}, y_{n+1}) = z_{n+1}$, which implies that (z_n) is a monotone increasing sequence in $(Z, \leq_p)$, where $Z = X \times X$.

Now, we consider the metric $\widetilde{d} : Z \times Z \to \mathbb{R}^+$, defined by $\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v)$. Then $\widetilde{d}$ is a b-metric on Z with the same constant $s > 1$. If (X, d) is complete, $(Z, \widetilde{d})$ is complete, too.

Suppose that $G : Z \to Z$ is an operator defined by $G(x, y) = (T(x, y), T(y, x))$ for all $(x, y) \in Z$.

Consider the sequence $z_{n+1} = G(z_n)$, for $n \geq 0$ where $z_0 = (x_0, y_0)$. Using the mixed monotone property of T, then the operator G is monotone increasing with respect to " $\leq_p$ " i.e. $(x, y), (u, v) \in Z$, with $(x, y) \leq_p (u, v) \Rightarrow G(x, y) \leq_p G(u, v)$.

Because T has a closed graph (or respectively is continuous on $X \times X$), then G has a closed graph (or respectively is continuous on Z).

G is a contraction in $(Z, \widetilde{d})$ on all comparable elements of Z. Let $z = (x, y) \leq_p (u, v) = w \in Z$, so

$$\begin{aligned}
&\psi(\widetilde{d}(G(z), G(w))) = \psi(\widetilde{d}((T(x, y), T(y, x)), (T(u, v), T(v, u)))\\
&= \psi(d(T(x, y), T(u, v)) + d(T(y, x), T(v, u)))\\
&\leq F\Big(\psi(\frac{1}{\alpha + \beta s}[\frac{\alpha \cdot [d(u, T(u, v)) + d(v, T(v, u))][1 + d(x, T(x, y)) + d(y, T(y, x))]}{1 + d(x, u) + d(y, v)}\\
&+ \beta \cdot [d(x, u) + d(y, v)]]),\\
&\varphi(\frac{1}{\alpha + \beta s}[\frac{\alpha \cdot [d(u, T(u, v)) + d(v, T(v, u))][1 + d(x, T(x, y)) + d(y, T(y, x))]}{1 + d(x, u) + d(y, v)}\\
&+ \beta \cdot [d(x, u) + d(y, v)]])\Big)\\
&= F\Big(\psi(\frac{\beta}{\alpha + \beta s}[\frac{\alpha \cdot \widetilde{d}(w, G(w))[1 + \widetilde{d}(z, G(z))]}{1 + \widetilde{d}(z, w)} + \beta \cdot \widetilde{d}(z, w)]),\\
&\varphi(\frac{\beta}{\alpha + \beta s}[\frac{\alpha \cdot \widetilde{d}(w, G(w))[1 + \widetilde{d}(z, G(z))]}{1 + \widetilde{d}(z, w)} + \beta \cdot \widetilde{d}(z, w)])\Big).
\end{aligned}$$

The operator $G : Z \to Z$ has the following property:

1) $G : Z \to Z$ has a closed graph (or continuous) on Z;
2) $G : Z \to Z$ is increasing on Z;
3) There exists $z_0 = (x_0, y_0) \in Z$ such that $z_0 \leq_p G(z_0)$;

4) there exists $\psi \in \Psi_{\inf}$, $\varphi \in \Phi_u$ and $F \in \mathcal{C}$ such that (ψ, φ, F) is monotone and $\alpha, \beta \geq 0$ with $\alpha + \beta > 0$ such that

$$\begin{aligned}\psi(\widetilde{d}(G(z), G(w))) \;\leq\; & F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}(w, G(w))[1+\widetilde{d}(z, G(z))]}{1+\widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)]), \\ & \varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}(w, G(w))[1+\widetilde{d}(z, G(z))]}{1+\widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)])\Big).\end{aligned}$$

We can apply the conclusion of the Theorem 2.1 and we get that G has at least one fixed point. Hence, there exists $z^* \in Z$ with $G(z^*) = z^*$. Let $z^* = (x^*, y^*) \in Z$, so we have $G(x^*, y^*) = (x^*, y^*)$.

This implies

$$(T(x^*, y^*), T(y^*, x^*)) = (x^*, y^*) \Rightarrow \begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*), \end{cases}$$

and the sequences $(x_n), (y_n)$ in X defined by

$$\begin{cases} x_{n+1} = T(x_n, y_n) \\ y_{n+1} = T(y_n, x_n) \quad \text{for } n \in \mathbb{N}, \end{cases}$$

have the property that $x_n \to x^*$ and $y_n \to y^*$ as $n \to \infty$. □

The purpose of the next theorem is to obtain the uniqueness of the coupled fixed point.

Theorem 3.2. *Suppose that all the hypotheses of Theorem 3.1 are satisfied. Moreover, suppose that for all $(x, y), (u, v) \in X \times X$ there exists $(z, w) \in X \times X$ such that $(z, w) \leq_p (x, y)$ and $(z, w) \leq_p (u, v)$. Then $CFix(T) = \{(x^*, y^*)\}$.*

Proof . As a result by property of the operator T in Theorem 3.1, we get $(x^*, y^*) \in Z := X \times X$ such that

$$\begin{cases} x^* = T(x^*, y^*) \\ y^* = T(y^*, x^*). \end{cases}$$

Suppose that $(\overline{x}, \overline{y}) \in CFix(T)$ and $\widetilde{d} : Z \times Z \to \mathbb{R}^+$ defined by $\widetilde{d}((x, y), (u, v)) = d(x, u) + d(y, v)$, where $Z = X \times X$.

We have two cases:

i) Let (x^*, y^*) and $(\overline{x}, \overline{y})$ be comparable, that is $(x^*, y^*) \leq_p (\overline{x}, \overline{y})$. This implies that

$$
\begin{aligned}
&\psi(\tfrac{\beta}{\alpha+\beta s}\widetilde{d}((x^*,y^*),(\overline{x},\overline{y})))\\
&\leq \psi(\widetilde{d}((x^*,y^*),(\overline{x},\overline{y}))) = \psi(\widetilde{d}((T(x^*,y^*),T(y^*,x^*)),(T(\overline{x},\overline{y}),T(\overline{y},\overline{x}))))\\
&= \psi(d(T(x^*,y^*),T(\overline{x},\overline{y})) + d(T(y^*,x^*),T(\overline{y},\overline{x})))\\
&\leq F\Big(\psi(\tfrac{1}{\alpha+\beta s}[\tfrac{\alpha\cdot[d(\overline{x},T(\overline{x},\overline{y}))+d(\overline{y},T(\overline{y},\overline{x}))][1+d(x^*,T(x^*,y^*))+d(y^*,T(y^*,x^*))]}{1+d(x^*,\overline{x})+d(y^*,\overline{y})}\\
&+\beta\cdot[d(x^*,\overline{x})+d(y^*,\overline{y})]]),\\
&\varphi(\tfrac{1}{\alpha+\beta s}[\tfrac{\alpha\cdot[d(\overline{x},T(\overline{x},\overline{y}))+d(\overline{y},T(\overline{y},\overline{x}))][1+d(x^*,T(x^*,y^*))+d(y^*,T(y^*,x^*))]}{1+d(x^*,\overline{x})+d(y^*,\overline{y})}\\
&+\beta\cdot[d(x^*,\overline{x})+d(y^*,\overline{y})]])\Big)\\
&= F\Big(\psi(\tfrac{\beta}{\alpha+\beta s}[d(x^*,\overline{x})+d(y^*,\overline{y})]),\varphi(\tfrac{\beta}{\alpha+\beta s}[d(x^*,\overline{x})+d(y^*,\overline{y})])\Big)\\
&= F\Big(\psi(\tfrac{\beta}{\alpha+\beta s}\widetilde{d}((x^*,y^*),(\overline{x},\overline{y}))),\varphi(\tfrac{\beta}{\alpha+\beta s}\widetilde{d}((x^*,y^*),(\overline{x},\overline{y})))\Big)\\
&\leq \psi(\tfrac{\beta}{\alpha+\beta s}\widetilde{d}((x^*,y^*),(\overline{x},\overline{y}))).
\end{aligned}
$$

Hence,

$$
\psi(\frac{\beta}{\alpha+\beta s}\widetilde{d}((x^*,y^*),(\overline{x},\overline{y}))) = 0 \quad\text{or}\quad \varphi(\frac{\beta}{\alpha+\beta s}\widetilde{d}((x^*,y^*),(\overline{x},\overline{y}))) = 0,
$$

which yields $\widetilde{d}((x^*,y^*),(\overline{x},\overline{y})) = 0$. Therefore, $(x^*,y^*) = (\overline{x},\overline{y})$.

ii) Let (x^*,y^*) and $(\overline{x},\overline{y})$ be not comparable. So, there exists $(z,w)\in Z$, such that $(z,w)\leq_p (x^*,y^*)$, implies $G^n(z,w)\leq_p G^n(x^*,y^*)$ because G is an increasing operator. Also, since G is an increasing operator and $(z,w)\leq_p(\overline{x},\overline{y})$, we have $G^n(z,w)\leq_p G^n(\overline{x},\overline{y})$. From (3.1), we have

$$
\begin{aligned}
&\psi(\widetilde{d}(G^n(z,w),(x^*,y^*))) = \psi(\widetilde{d}(G^n(z,w),G^n(x^*,y^*)))\\
&= \psi(\widetilde{d}(G(G^{n-1}(z,w)),G(G^{n-1}(x^*,y^*))))\\
&\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(G^{n-1}(x^*,y^*),G^n(x^*,y^*))[1+\widetilde{d}(G^{n-1}(z,w),G^n(z,w))]}{1+\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*))}\\
&+\beta\cdot\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*))]),\\
&\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(G^{n-1}(x^*,y^*),G^n(x^*,y^*))[1+\widetilde{d}(G^{n-1}(z,w),G^n(z,w))]}{1+\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*))}\\
&+\beta\cdot\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*))])\Big)\\
&= F\Big(\psi(\frac{1}{\alpha+\beta s}\beta\cdot\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*))),\\
&\varphi(\frac{1}{\alpha+\beta s}\beta\cdot\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*)))\Big)\\
&\leq \psi(\frac{1}{\alpha+\beta s}\beta\cdot\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*)))\\
&\leq \psi(\frac{1}{s}\cdot\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*))).
\end{aligned}
$$

Using mathematical induction and since ψ is non-decreasing, we obtain

$$\begin{aligned}\widetilde{d}(G^n(z,w),G^n(x^*,y^*)) &\leq (\frac{1}{s})\cdot\widetilde{d}(G^{n-1}(z,w),G^{n-1}(x^*,y^*)\\ &\leq ... \leq (\frac{1}{s})^n\cdot\widetilde{d}((z,w),(x^*,y^*))\to 0,\ \text{as}\quad n\to\infty.\end{aligned}$$

Therefore

$$\lim_{n\to\infty} G^n(z,w)=(x^*,y^*). \tag{3.2}$$

Since $(z,w)\leq_p(\overline{x},\overline{y})$, $G^n(z,w)\leq_p G^n(\overline{x},\overline{y})=(\overline{x},\overline{y})$. Similarly, we get

$$\widetilde{d}(G^n(z,w),(\overline{x},\overline{y}))\leq(\frac{1}{s})^n\cdot\widetilde{d}((z,w),(\overline{x},\overline{y}))\to 0\quad\text{as}\quad n\to\infty$$

which implies that

$$\lim_{n\to\infty} G^n(z,w)=(\overline{x},\overline{y}). \tag{3.3}$$

Hence, from (3.2) and (3.3), we obtain that $(x^*,y^*)=(\overline{x},\overline{y})$. □

Theorem 3.3. *Let (X,d) be a complete b-metric with constant $s>1$, $T:X\times X\to X$ be an operator with the following condition:*

There exists $\psi\in\Psi_{\inf}$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$ such that (ψ,φ,F) is monotone and $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ such that

$$\begin{aligned}&\psi(d(T(x,y),T(u,v))+d(T(y,x),T(v,u)))\\ &\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}\\ &+\beta\cdot[d(x,u)+d(y,v)]]),\\ &\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}\\ &+\beta\cdot[d(x,u)+d(y,v)]])\Big),\end{aligned} \tag{3.4}$$

for all $(x,y),(u,v)\in X\times X$ with $x\leq u, y\geq v$.

Then, there exists a unique solution $(x^*,y^*)\in X\times X$ of the coupled fixed point problem (P_1), and for any initial point $(x_0,y_0)\in X\times X$ the sequence $z_{n+1}=(x_{n+1},y_{n+1})=(T(x_n,y_n),T(y_n,x_n))\in X\times X$ converge to (x^*,y^*).

Proof . Let $Z=X\times X$ and the functional $\widetilde{d}:Z\times Z\to\mathbb{R}^+$, such that $\widetilde{d}((x,y),(u,v))=d(x,u)+d(y,v)$.

We know that $\widetilde{d}$ is a b-metric on Z with the same constant $s>1$. If (X,d) is a complete b-metric space, then $(Z,\widetilde{d})$ is a complete b-metric space too.

Consider the operator $G:Z\to Z$ defined by $G(x,y)=(T(x,y),T(y,x))$ for $(x,y)\in Z$.

Let $z=(x,y)\in Z$ and $w=(u,v)\in Z$.

We have

$$\psi(\widetilde{d}(G(z),G(w))) = \psi(\widetilde{d}((T(x,y),T(y,x)),(T(u,v),T(v,u)))) =$$
$$= \psi(d(T(x,y),T(u,v)) + d(T(y,x),T(v,u)))$$
$$\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}$$
$$+\beta\cdot[d(x,u)+d(y,v)]]),$$
$$\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}$$
$$+\beta\cdot[d(x,u)+d(y,v)]])\Big)$$
$$= F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}((u,v),(T(u,v),T(v,u)))[1+\widetilde{d}((x,y),(T(x,y),T(y,x)))]}{1+\widetilde{d}((x,y),(u,v))}$$
$$+\beta\cdot\widetilde{d}((x,y),(u,v))]),$$
$$\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}((u,v),(T(u,v),T(v,u)))[1+\widetilde{d}((x,y),(T(x,y),T(y,x)))]}{1+\widetilde{d}((x,y),(u,v))}$$
$$+\beta\cdot\widetilde{d}((x,y),(u,v))]\Big)$$
$$= F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(w,G(w))[1+\widetilde{d}(z,G(z))]}{1+\widetilde{d}(z,w)}+\beta\cdot\widetilde{d}(z,w)]),$$
$$\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(w,G(w))[1+\widetilde{d}(z,G(z))]}{1+\widetilde{d}(z,w)}+\beta\cdot\widetilde{d}(z,w)]\Big).$$

Therefore,

$$\psi(\widetilde{d}(G(z),G(w))) \leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(w,G(w))[1+\widetilde{d}(z,G(z))]}{1+\widetilde{d}(z,w)}+\beta\cdot\widetilde{d}(z,w)]),$$
$$\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(w,G(w))[1+\widetilde{d}(z,G(z))]}{1+\widetilde{d}(z,w)}+\beta\cdot\widetilde{d}(z,w)]\Big).$$

From Theorem 2.3, we have that $Fix(F) = \{(x^*,y^*)\}$, so the coupled fixed point theorem (P_1) has a unique solution $(x^*,y^*) \in Z$. □

Theorem 3.4. *If we suppose that we have the hypotheses of Theorem 3.2, then for the unique coupled fixed point (x^*,y^*) of T we have that $x^* = y^*$ i.e. $T(x^*,x^*) = x^*$.*

Proof . From Theorem 3.2, there exists a unique coupled fixed point of T, $(x^*,y^*) \in X \times X$.

We have two cases:

Case 1. If x^* and y^* are comparable, $x^* \leq y^*$.

Then we have

$$\psi((T(x,y),T(u,v))+d(T(y,x),T(v,u)))$$
$$\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}$$
$$+\beta\cdot[d(x,u)+d(y,v)]]),$$
$$\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T(u,v))+d(v,T(v,u))][1+d(x,T(x,y))+d(y,T(y,x))]}{1+d(x,u)+d(y,v)}$$
$$+\beta\cdot[d(x,u)+d(y,v)]])\Big).$$

Let $x=v=x^*$ and $y=u=y^*$.

So, we obtain

$$\psi(2\cdot d(T(x^*,y^*),T(y^*,x^*)))\leq$$
$$F\Big(\psi(\frac{1}{\alpha+\beta s}\cdot[\frac{\alpha\cdot[d(y^*,T(y^*,x^*)+d(x^*,T(x^*,y^*)][1+d(x^*,T(x^*,y^*)+d(y^*,T(y^*,x^*)]}{1+2d(x^*,y^*)}]$$
$$+\beta\cdot 2\cdot d(x^*,y^*),$$
$$\varphi(\frac{1}{\alpha+\beta s}\cdot[\frac{\alpha\cdot[d(y^*,T(y^*,x^*)+d(x^*,T(x^*,y^*)][1+d(x^*,T(x^*,y^*)+d(y^*,T(y^*,x^*)]}{1+2d(x^*,y^*)}]$$
$$+\beta\cdot 2\cdot d(x^*,y^*))\Big)$$
$$=F\Big(\psi(\frac{2\beta}{\alpha+\beta s}\cdot d(x^*,y^*)),\varphi(\frac{2\beta}{\alpha+\beta s}\cdot d(x^*,y^*))\Big)$$
$$\leq\psi(\frac{2\beta}{\alpha+\beta s}\cdot d(x^*,y^*)).$$

This yields to $d(x^*,y^*)\leq\dfrac{\beta}{\alpha+\beta s}\cdot d(x^*,y^*)$.

So, $(\dfrac{\alpha+\beta s-\beta}{\alpha+\beta s})\cdot d(x^*,y^*)\leq 0$, follows that $x^*=y^*$.

Case 2. If x^* and y^* are not comparable.

Hence, there exists $z\in X$ such that $z\leq x^*$ and $z\leq y^*$, satisfying the following:

$(z,y^*)\leq_p(y^*,z)$ follows that $z\leq y^*, y^*\geq z$ that is true
$(z,y^*)\leq_p(x^*,y^*)$ follows that $z\leq x^*, y^*\geq y^*$ that is true
$(y^*,x^*)\leq_p(y^*,z)$ follows that $y^*\leq y^*,\ x^*\geq z$ that is true.

Let $G:Z\to Z$ be defined by $G(x,y)=(T(x,y),T(y,x))\ \forall(x,y)\in Z$. Then,

$$\psi(d(x^*,y^*)) = \psi(\frac{1}{2}\cdot\widetilde{d}((y^*,x^*),(x^*,y^*))) = \psi(\frac{1}{2}\cdot\widetilde{d}(G(G^{n-1}(y^*,x^*)),G(G^{n-1}(x^*,y^*)))) \leq$$
$$\psi(\widetilde{d}(G(G^{n-1}(y^*,x^*)),G(G^{n-1}(x^*,y^*)))) \leq$$
$$F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(G^{n-1}(x^*,y^*),G(G^{n-1}(x^*,y^*)))[1+\widetilde{d}(G^{n-1}(y^*,x^*),G(G^{n-1}(y^*,x^*)))]}{1+\widetilde{d}(G^{n-1}(y^*,x^*),G^{n-1}(x^*,y^*))}$$
$$+\beta\cdot\widetilde{d}(G^{n-1}(y^*,x^*)),G^{n-1}(x^*,y^*))]),$$
$$\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot\widetilde{d}(G^{n-1}(x^*,y^*),G(G^{n-1}(x^*,y^*)))[1+\widetilde{d}(G^{n-1}(y^*,x^*),G(G^{n-1}(y^*,x^*)))]}{1+\widetilde{d}(G^{n-1}(y^*,x^*)),G^{n-1}(x^*,y^*))}]$$
$$+\beta\cdot\widetilde{d}(G^{n-1}(y^*,x^*),G^{n-1}(x^*,y^*))]])\Big)$$
$$\leq F\Big(\psi(\frac{\beta}{\alpha+\beta s}\widetilde{d}(G^{n-1}(y^*,x^*)),G^{n-1}(x^*,y^*)),\varphi(\frac{\beta}{\alpha+\beta s}\widetilde{d}(G^{n-1}(y^*,x^*)),G^{n-1}(x^*,y^*))\Big)$$
$$\leq \psi(\frac{\beta}{\alpha+\beta s}\widetilde{d}(G^{n-1}(y^*,x^*)),G^{n-1}(x^*,y^*)))$$
$$\leq \psi(\frac{1}{s}\widetilde{d}(G^{n-1}(y^*,x^*)),G^{n-1}(x^*,y^*))).$$

Using mathematical induction and since ψ is non-decreasing, we obtain

$$d(x^*,y^*) \leq \widetilde{d}(G^n(y^*,x^*),G^n(x^*,y^*)) \leq (\frac{1}{s})\cdot\widetilde{d}(G^{n-1}(y^*,x^*),G^{n-1}(x^*,y^*))$$
$$\leq \ldots \leq (\frac{1}{s})^n\cdot\widetilde{d}((y^*,x^*),(x^*,y^*)) \to 0, \text{ as } \quad n\to\infty.$$

Hence, we have that $x^* = T(x^*,x^*)$. □

4. Properties of the *C*-class functions for coupled fixed point problem

In this part of the paper we shall present data dependence, well-posedness, Ulam- Hyers stability, limit shadowing of the C-class functions for the coupled fixed point problem.

In this first theorem we shall prove the data dependence propertie:

Theorem 4.1. *Let (X,d) be a complete b-metric space with constant $s>1$ and $T_i : X\times X \to X \quad (i\in\{1,2\})$ be two operators which satisfy the following conditions:*

i) There exists $\psi\in\Psi_{\inf}$, $\varphi\in\Phi_u$ and $F\in\mathcal{C}$ such that (ψ,φ,F) is monotone and $\alpha,\beta\geq 0$ with $\alpha+\beta>0$ and $\max\{\alpha,\frac{\beta}{\alpha(1-\alpha)}\}<\frac{1}{s}$, such that

$$\psi(d(T_1(x,y),T_1(u,v))+d(T_1(y,x),T_1(v,u)))$$
$$\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T_1(u,v))+d(v,T_1(v,u))][1+d(x,T_1(x,y))+d(y,T_1(y,x))]}{1+d(x,u)+d(y,v)}$$
$$+\beta\cdot[d(x,u)+d(y,v)]]),$$
$$\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(u,T_1(u,v))+d(v,T_1(v,u))][1+d(x,T_1(x,y))+d(y,T_1(y,x))]}{1+d(x,u)+d(y,v)}$$
$$+\beta\cdot[d(x,u)+d(y,v)]])\Big),$$

for all $(x,y),(u,v) \in X \times X$ *with* $x \leq u, y \geq v$

ii) $CFix(T_2) \neq \emptyset$;

iii) There exists $\eta > 0$ *such that* $d(T_1(x,y), T_2(x,y)) \leq \eta$ *for all* $(x,y) \in X \times X$.

In the above conditions, if $(x^*, y^*) \in X \times X$ *is the unique coupled fixed point for* T_1*, then* $d(x^*, \overline{x}) + d(y^*, \overline{y}) \leq \frac{2\eta\alpha}{\beta}(\frac{\alpha+\beta s}{\alpha-\beta s})$*, where* $(\overline{x}, \overline{y}) \in CFix(T_2)$.

Proof . From Theorem 3.3, there exists $(x^*, y^*) \in X \times X$ such that $\begin{cases} x^* = T_1(x^*, y^*) \\ y^* = T_1(y^*, x^*). \end{cases}$

Let $(\overline{x}, \overline{y}) \in CFix(T_2)$, i.e. $\begin{cases} \overline{x} = T_2(\overline{x}, \overline{y}) \\ \overline{y} = T_2(\overline{y}, \overline{x}). \end{cases}$

Consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}^+$, defined by $\widetilde{d}((x,y),(u,v)) = d(x,u) + d(y,v)$ for $(x,y),(u,v) \in Z$, where $Z = X \times X$.

Consider two operators $G_i : Z \to Z$ defined by $G_i(x,y) = (T_i(x,y), T_i(y,x))$, for $(x,y) \in Z$, $i = \{1,2\}$. We denote by $z = (x^*, y^*) \in Z$, which means $G_1(z) = z$ and $w = (\overline{x}, \overline{y}) \in Z$, which means $G_2(w) = w$. Then,

$$\begin{aligned}
&\psi(\widetilde{d}(G_1(z), G_1(w))) \\
&\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}(w, G_1(w))[1+\widetilde{d}(z, G_1(z))]}{1+\widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)]), \\
&\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}(w, G_1(w))[1+\widetilde{d}(z, G_1(z))]}{1+\widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)])\Big) \\
&= F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}(w, G_1(w))}{1+\widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)]), \\
&\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}(w, G_1(w))}{1+\widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)])\Big) \\
&\leq \psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}(w, G_1(w))}{1+\widetilde{d}(z,w)} + \beta \cdot \widetilde{d}(z,w)]) \\
&\leq \psi(\frac{1}{\alpha+\beta s}[\alpha \cdot \widetilde{d}(w, G_1(w)) + \beta \cdot \widetilde{d}(z,w)]) \\
&\leq \psi(\frac{\alpha}{\beta s} \cdot \widetilde{d}(w, G_1(w)) + \frac{\beta}{\alpha} \cdot \widetilde{d}(z,w)).
\end{aligned}$$

So,

$$\begin{aligned}
\widetilde{d}(G_1(z), G_1(w)) &\leq \frac{\alpha}{\beta s} \cdot \widetilde{d}(w, G_1(w)) + \frac{\beta}{\alpha} \cdot \widetilde{d}(z,w) \\
&\leq \frac{2 \cdot \alpha}{\beta s} \cdot \eta + \frac{\beta}{\alpha} \cdot \widetilde{d}(z,w).
\end{aligned}$$

But

$$\begin{aligned}
&\widetilde{d}(z,w) = \widetilde{d}(G_1(z), G_2(w)) \leq s \cdot [\widetilde{d}(G_1(z), G_1(w)) + \widetilde{d}(G_1(w), G_2(w))] \\
&\leq s \cdot [\frac{2\alpha}{\beta s} \cdot \eta + \frac{\beta}{\alpha} \cdot \widetilde{d}(z,w)] + 2s \cdot \eta.
\end{aligned}$$

Using this condition, we will obtain

$$(1-\frac{s\beta}{\alpha})\cdot\widetilde{d}(z,w)\leq 2\eta(\frac{\alpha}{\beta}+s).$$

From $\max\{\alpha,\frac{\beta}{\alpha(1-\alpha)}\}<\frac{1}{s}$, we get that $1-\frac{s\beta}{\alpha}>0$. Therefore,

$$\widetilde{d}(z,w)\leq\frac{2\eta(\frac{\alpha}{\beta}+s)}{1-\frac{s\beta}{\alpha}}=\frac{2\eta\alpha}{\beta}(\frac{\alpha+\beta s}{\alpha-\beta s}),$$

and by definition of the metric $\widetilde{d}$, we have

$$d(x^*,\overline{x})+d(y^*,\overline{y})\leq\frac{2\eta\alpha}{\beta}(\frac{\alpha+\beta s}{\alpha-\beta s}).$$

This finishes the proof. □

Definition 4.2. *([16]) Let (X,d) be a b-metric space with constant $s\geq 1$ and $T:X\times X\to X$ be an operator. By definition , the coupled fixed point problem (P_1) is said to be well-posed if:*

(i) $CFix(T)=\{(x^*,y^*)\}$;

(ii) For any sequence $(x_n,y_n)_{n\in\mathbb{N}}\in X\times X$ for which $d(x_n,T(x_n,y_n))\to 0$ and $d(y_n,T(y_n,x_n))\to 0$ as $n\to\infty$, we have that $(x_n)_{n\in\mathbb{N}}\to x^*$ and $(y_n)_{n\in\mathbb{N}}\to y^*$ as $n\to\infty$.

Theorem 4.3. *Assume that all the hypotheses of Theorem 3.3 take place. Then the coupled fixed problem (P_1) is well-posed.*

Proof . We denote by $Z=X\times X$. By Theorem 3.3, we have $CFix(T)=\{(x^*,y^*)\}$.

Let $(x_n,y_n)_{n\in\mathbb{N}}$ be a sequence on Z. We know that $d(x_n,T(x_n,y_n))\to 0$ and $d(y_n,T(y_n,x_n))\to 0$ as $n\to\infty$.

Consider the b-metric $\widetilde{d}:Z\times Z\to\mathbb{R}^+$, such that $\widetilde{d}((x,y),(u,v))=d(x,u)+d(y,v)$ for all $(x,y),(u,v)\in Z$.

Let $G:Z\to Z$ be an operator defined by $G(x,y)=(T(x,y),T(y,x))$ for all $(x,y)\in Z$. We know that $G(x^*,y^*)=(x^*,y^*)$, so we have

$$\begin{aligned}&\widetilde{d}((x_n,y_n),(x^*,y^*))=d(x_n,x^*)+d(y_n,y^*)\\&\leq s\cdot d(x_n,T(x_n,y_n))+s\cdot d(T(x_n,y_n),T(x^*,y^*))+s\cdot d(y_n,T(y_n,x_n))\\&+s\cdot d(T(y_n,x_n),T(y^*,x^*))\\&=s\cdot[d(x_n,T(x_n,y_n))+d(y_n,T(y_n,x_n))]+s\cdot[d(T(x_n,y_n),T(x^*,y^*))\\&+d(T(y_n,x_n),T(y^*,x^*))].\end{aligned}$$

Thus by taking $n\to\infty$, we get

$$\lim_{n\to\infty}\widetilde{d}((x_n,y_n),(x^*,y^*))\leq\lim_{n\to\infty}s\cdot[d(T(x_n,y_n),T(x^*,y^*))+d(T(y_n,x_n),T(y^*,x^*))].$$

Hence,

$$\begin{aligned}
&\lim_{n\to\infty}\psi(\frac{1}{s}\widetilde{d}((x_n,y_n),(x^*,y^*)))\\
&\quad\le \lim_{n\to\infty}\psi(d(T(x_n,y_n),T(x^*,y^*))+d(T(y_n,x_n),T(y^*,x^*)))\\
&\quad\le \lim_{n\to\infty}F\Big(\\
&\quad\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(x^*,T(x^*,y^*))+d(y^*,T(y^*,x^*))][1+d(x_n,T(x_n,y_n))+d(y_n,T(y_n,x_n))]}{1+d(x_n,x^*)+d(y_n,y^*)}\\
&\quad+\beta\cdot[d(x_n,x^*)+d(y_n,y^*)]),\\
&\quad\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(x^*,T(x^*,y^*))+d(y^*,T(y^*,x^*))][1+d(x_n,T(x_n,y_n))+d(y_n,T(y_n,x_n))]}{1+d(x_n,x^*)+d(y_n,y^*)}\\
&\quad+\beta\cdot[d(x_n,x^*)+d(y_n,y^*)])\Big)\\
&\quad=\lim_{n\to\infty}F\Big(\psi(\frac{\beta}{\alpha+\beta s}[d(x_n,x^*)+d(y_n,y^*)]),\varphi(\frac{\beta}{\alpha+\beta s}[d(x_n,x^*)+d(y_n,y^*)])\Big)\\
&\quad=\lim_{n\to\infty}F\Big(\psi(\frac{\beta}{\alpha+\beta s}[\widetilde{d}((x_n,y_n),(x^*,y^*))]),\varphi(\frac{\beta}{\alpha+\beta s}[\widetilde{d}((x_n,y_n),(x^*,y^*))])\Big)\\
&\quad\le\lim_{n\to\infty}F\Big(\psi(\frac{1}{s}[\widetilde{d}((x_n,y_n),(x^*,y^*))]),\varphi(\frac{1}{s}[\widetilde{d}((x_n,y_n),(x^*,y^*))])\Big)\\
&\quad\le\lim_{n\to\infty}\psi(\frac{1}{s}[\widetilde{d}((x_n,y_n),(x^*,y^*))]).
\end{aligned}$$

Which implies that

$$\lim_{n\to\infty}\psi(\frac{1}{s}[\widetilde{d}((x_n,y_n),(x^*,y^*))])=0 \quad\text{or}\quad \lim_{n\to\infty}\varphi(\frac{1}{s}[\widetilde{d}((x_n,y_n),(x^*,y^*))])=0.$$

Therefore, $(x_n,y_n)\to(x^*,y^*)$ as $n\to\infty$. □

Definition 4.4. *([16]) Let (X,d) be a b-metric space with constant $s\ge 1$ and $T:X\times X\to X$ be an operator. By definition, the coupled fixed point problem (P_1) is said to be Ulam-Hyers stable if there exists an operator $\phi:\mathbb{R}^+\to\mathbb{R}^+$ increasing, continuous in 0 with $\phi(0)=0$, such that for each $\varepsilon>0$ and for each ε-solution $(\overline{x},\overline{y})\in X\times X$ of the inequality $d(x,T(x,y))+d(y,T(y,x))\le\varepsilon$, there exists a solution $(x^*,y^*)\in X\times X$ of the coupled fixed point problem (P_1) such that $d(\overline{x},x^*)+d(\overline{y},y^*)\le\phi(\varepsilon)$.*

Theorem 4.5. *Assume that all the hypotheses of Theorem 3.3 take place and $\alpha>0$. Then the coupled fixed point problem (P_1) is Ulam-Hyers stable.*

Proof . Let $Z=X\times X$. By Theorem 3.3, we have $CFix(T)=\{(x^*,y^*)\}$. Let any $\varepsilon>0$ and let $(\overline{x},\overline{y})\in Z$ such that $d(\overline{x},T(\overline{x},\overline{y}))+d(\overline{y},T(\overline{y},\overline{x}))\le\varepsilon$.

Consider the b-metric $\widetilde{d}:Z\times Z\to\mathbb{R}^+$, such that $\widetilde{d}((x,y),(u,v))=d(x,u)+d(y,v)$ for all $(x,y),(u,v)\in Z$ and $G:Z\to Z$ an operator defined by $G(x,y)=(T(x,y),T(y,x))$ for all $(x,y)\in Z$.

We have

$$\begin{aligned}
&\widetilde{d}((\overline{x},\overline{y}),(x^*,y^*)) = d(\overline{x},x^*) + d(\overline{y},y^*) = d(\overline{x},T(x^*,y^*)) + d(\overline{y},T(y^*,x^*)) \\
&\leq s\cdot[d(\overline{x},T(\overline{x},\overline{y})) + d(T(\overline{x},\overline{y}),T(x^*,y^*))] + s\cdot[d(\overline{y},T(\overline{y},\overline{x})) + d(T(\overline{y},\overline{x}),T(y^*,x^*))] \\
&\leq s\cdot[d(\overline{x},T(\overline{x},\overline{y})) + d(\overline{y},T(\overline{y},\overline{x}))] + s[d(T(\overline{x},\overline{y}),T(x^*,y^*)) + d(T(\overline{y},\overline{x}),T(y^*,x^*))] \\
&\leq s\cdot\varepsilon + s[d(T(\overline{x},\overline{y}),T(x^*,y^*)) + d(T(\overline{y},\overline{x}),T(y^*,x^*))].
\end{aligned}$$

Hence,

$$\frac{\widetilde{d}((\overline{x},\overline{y}),(x^*,y^*)) - \varepsilon\cdot s}{s} \leq d(T(\overline{x},\overline{y}),T(x^*,y^*)) + d(T(\overline{y},\overline{x}),T(y^*,x^*))$$

So,

$$\begin{aligned}
&\psi(\frac{\widetilde{d}((\overline{x},\overline{y}),(x^*,y^*)) - \varepsilon\cdot s}{s}) \\
&\leq \psi(d(T(\overline{x},\overline{y}),T(x^*,y^*)) + d(T(\overline{y},\overline{x}),T(y^*,x^*))) \\
&\leq F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(x^*,T(x^*,y^*)) + d(y^*,T(y^*,x^*))][1 + d(\overline{x},T(\overline{x},\overline{y})) + d(\overline{y},T(\overline{y},\overline{x}))]}{1 + d(\overline{x},x^*) + d(\overline{y},y^*)} \\
&+ \beta\cdot[d(\overline{x},x^*) + d(\overline{y},y^*)]), \\
&\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha\cdot[d(x^*,T(x^*,y^*)) + d(y^*,T(y^*,x^*))][1 + d(\overline{x},T(\overline{x},\overline{y})) + d(\overline{y},T(\overline{y},\overline{x}))]}{1 + d(\overline{x},x^*) + d(\overline{y},y^*)} \\
&+ \beta\cdot[d(\overline{x},x^*) + d(\overline{y},y^*)])\Big) \\
&= F\Big(\psi(\frac{\beta}{\alpha+\beta s}[d(\overline{x},x^*) + d(\overline{y},y^*)]),\varphi(\frac{\beta}{\alpha+\beta s}[d(\overline{x},x^*) + d(\overline{y},y^*)])\Big) \\
&\leq \psi(\frac{\beta}{\alpha+\beta s}[d(\overline{x},x^*) + d(\overline{y},y^*)]) = \psi(\frac{\beta}{\alpha+\beta s}[\widetilde{d}((\overline{x},\overline{y}),(x^*,y^*))]).
\end{aligned}$$

We obtain

$$\begin{aligned}
\frac{\widetilde{d}((\overline{x},\overline{y}),(x^*,y^*)) - \varepsilon\cdot s}{s} &\leq \frac{\beta}{\alpha+\beta s}[\widetilde{d}((\overline{x},\overline{y}),(x^*,y^*))] \\
\widetilde{d}((\overline{x},\overline{y}),(x^*,y^*)) &\leq \frac{s(\alpha+\beta s)}{\alpha}\cdot\varepsilon.
\end{aligned}$$

Therefore, the coupled fixed point problem (P_1) is Ulam-Hyers stable, with $\phi:\mathbb{R}^+\to\mathbb{R}^+$, $\phi(t)=ct$ where $c=\frac{s(\alpha+\beta s)}{\alpha}>0$. □

Definition 4.6. *([16]) Let (X,d) be a b-metric space with constant $s\geq 1$ and $T:X\times X\to X$ be an operator. By definition, the coupled fixed point problem (P_1) has the limit shadowing property, if for any sequence $(x_n,y_n)_{n\in\mathbb{N}}\in X\times X$ for which $d(x_{n+1},T(x_n,y_n))\to 0$ and respectively $d(y_{n+1},T(y_n,x_n))\to 0$ as $n\to\infty$, there exists a sequence $(T^n(x,y),T^n(y,x))_{n\in\mathbb{N}}$ such that $d(x_n,T^n(x,y))\to 0$ and $d(y_n,T^n(y,x))\to 0$ as $n\to\infty$.*

Theorem 4.7. *Assume that the hypotheses from Theorem 3.3 take place. Then the coupled fixed point problem (P_1) for T has the limit shadowing property.*

Proof . By Theorem 3.3, we have $CFix(T) = \{(x^*, y^*)\}$ and for any initial point $(x, y) \in X \times X$ the sequence $z_{n+1} = (T^n(x, y), T^n(y, x)) \in X \times X$ converge to (x^*, y^*) as $n \to \infty$.

Let $(x_n, y_n)_{n\in\mathbb{N}}$ be a sequence on $Z = X\times X$ such that $d(x_{n+1}, T(x_n, y_n)) \to 0$ and $d(y_{n+1}, T(y_n, x_n)) \to 0$ as $n \to \infty$.

We consider the b-metric $\widetilde{d} : Z \times Z \to \mathbb{R}^+$, defined by $\widetilde{d}((x, y),(u, v)) = d(x, u) + d(y, v)$ for all $(x, y),(u, v) \in Z$.

Let $G : Z \to Z$ be an operator defined by $G(u, v) = (T(u, v), T(v, u))$ for all $(u, v) \in Z$. We know that $G(x^*, y^*) = (x^*, y^*)$. Then for every $(x, y) \in Z$ we have:

$$\begin{aligned}&\widetilde{d}((x_{n+1}, y_{n+1}),(T^{n+1}(x, y), T^{n+1}(y, x)))\\ &\le s \cdot [\widetilde{d}((x_{n+1}, y_{n+1}),(x^*, y^*)) + \widetilde{d}((x^*, y^*),(T^{n+1}(x, y), T^{n+1}(y, x)))]),\end{aligned}$$

and by letting $n \to \infty$, we have

$$\begin{aligned}&\lim_{n\to\infty} \widetilde{d}((x_{n+1}, y_{n+1}),(T^{n+1}(x, y), T^{n+1}(y, x)))\\ &\le \lim_{n\to\infty} s \cdot [\widetilde{d}((x_{n+1}, y_{n+1}),(x^*, y^*)).\end{aligned} \tag{4.1}$$

But

$$\begin{aligned}&\lim_{n\to\infty} \widetilde{d}((x_{n+1}, y_{n+1}),(x^*, y^*)) \le \lim_{n\to\infty} s \cdot [\widetilde{d}((x_{n+1}, y_{n+1}),(T(x_n, y_n), T(y_n, x_n)))+\\ &\widetilde{d}(G(x_n, y_n), G(x^*, y^*))]\\ &= \lim_{n\to\infty} s \cdot [\widetilde{d}(G(x_n, y_n), G(x^*, y^*))].\end{aligned}$$

So,

$$\begin{aligned}&\lim_{n\to\infty} \psi(\frac{1}{s}\widetilde{d}((x_{n+1}, y_{n+1}),(x^*, y^*)))\\ &\le \lim_{n\to\infty} \psi(\widetilde{d}(G(x_n, y_n), G(x^*, y^*)))\\ &\le \lim_{n\to\infty} F\Big(\psi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}((x^*, y^*), G(x^*, y^*))[1 + \widetilde{d}((x_n, y_n), G(x_n, y_n))]}{1 + \widetilde{d}((x_n, y_n),(x^*, y^*))}\\ &+ \beta \cdot \widetilde{d}((x_n, y_n),(x^*, y^*))]),\\ &\varphi(\frac{1}{\alpha+\beta s}[\frac{\alpha \cdot \widetilde{d}((x^*, y^*), G(x^*, y^*))[1 + \widetilde{d}((x_n, y_n), G(x_n, y_n))]}{1 + \widetilde{d}((x_n, y_n),(x^*, y^*))}\\ &+ \beta \cdot \widetilde{d}((x_n, y_n),(x^*, y^*))])\Big)\\ &= \lim_{n\to\infty} F\Big(\psi(\frac{\beta}{\alpha+\beta s}\widetilde{d}((x_n, y_n),(x^*, y^*))), \varphi(\frac{\beta}{\alpha+\beta s}\widetilde{d}((x_n, y_n),(x^*, y^*)))\Big)\\ &\le \lim_{n\to\infty} \psi(\frac{\beta}{\alpha+\beta s}\widetilde{d}((x_n, y_n),(x^*, y^*))).\end{aligned}$$

This implies that

$$\lim_{n\to\infty} \widetilde{d}((x_{n+1}, y_{n+1}),(x^*, y^*)) \le \lim_{n\to\infty} s\Big(\frac{\beta}{\alpha+\beta s}\Big)\widetilde{d}((x_n, y_n),(x^*, y^*)).$$

Using mathematical induction, we obtain

$$\begin{aligned}\lim_{n\to\infty} \widetilde{d}((x_{n+1}, y_{n+1}), (x^*, y^*)) &\leq \lim_{n\to\infty} s\Big(\frac{\beta}{\alpha+\beta s}\Big)\widetilde{d}((x_n, y_n), (x^*, y^*)) \\ &\leq \lim_{n\to\infty} s^2\Big(\frac{\beta}{\alpha+\beta s}\Big)^2\widetilde{d}((x_{n-1}, y_{n-1}), (x^*, y^*)) \\ &\vdots \\ &\leq \lim_{n\to\infty} s^{n+1}\Big(\frac{\beta}{\alpha+\beta s}\Big)^{n+1}\widetilde{d}((x_0, y_0), (x^*, y^*)) \\ &= 0. \end{aligned} \tag{4.2}$$

From (4.1) and (4.2), we have $\widetilde{d}((x_{n+1}, y_{n+1}), (T^{n+1}(x, y), T^{n+1}(y, x))) \to 0$ as $n \to \infty$, so there exists a sequence $(T^n(x, y), T^n(y, x)) \in Z$ with

$$\widetilde{d}((x_n, y_n), (T^n(x, y), T^n(y, x))) = d(x_n, T^n(x, y)) + d(y_n, T^n(y, x)) \to 0 \quad \text{as } n \to \infty.$$

□

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