Nash Equilibrium Strategy for Bi-matrix Games with L-R Fuzzy Payoffs

F. Madandar[a], S. Haghayeghi[b,*], S. M. Vaezpour[c]

[a]*Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran*
[b]*Department of Mathematics, Karaj Branch, Islamic Azad University, Karaj, Iran*
[c] *Department of Mathematics and Computer Sciences, Amirkabir University of Technology, Tehran, Iran*

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Abstract

In this paper, bi-matrix games are investigated based on L-R fuzzy variables. Also, based on the fuzzy max order several models in non-symmetrical L-R fuzzy environment is constructed and the existence condition of Nash equilibrium strategies of the fuzzy bi-matrix games is proposed. At last, based on the Nash equilibrium of crisp parametric bi-matrix games, we obtain the Pareto and weak Pareto Nash equilibrium strategies of the fuzzy bi-matrix games.

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1. Introduction

In 1951, John Nash [12] presented non-cooperative games which each player has a well-defined utility function on the set of the player's strategy. In this article we focus on a class of non-cooperative games namely bi-matrix games. However in the complex problems such as economics, social and political sciences due to complexity and uncertainty, each player can not give the exact payoffs. So the payoff function is not always represented by a crisp number. In this paper we use the fuzzy set theory for express the uncertainty which it is proposed by Zadeh[22], in 1987. So, the new concept of equilibrium strategy is defined and investigated the properties of the equilibrium strategy.

In 1978, Butnariu[3] proposed non-cooperative fuzzy games which the players have fuzzy preferences. Many of mathematicians including Zimmermann[23] in 1985 and Yazenin[19] in 1987 applied

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*Corresponding author

Email addresses: fatemeh.madandar@kiau.ac.ir (F. Madandar), s.haghayeghi@kiau.ac.ir (S. Haghayeghi), vaez@aut.ac.ir (S. M. Vaezpour)

the fuzzy theory to optimization problems. In 1989, two-person zero-sum games with fuzzy payoffs was modeled by Compos[4] using fuzzy programming and transformation into the linear programming problems. In 2000, Meada[11] constructed kind of concepts of minimax equilibrium strategies. In 2001, Nishizaki and Sakawa[14] investigated multiobjective games and they modeled a fractional programming problem and computed the fractional programming by a relaxed method. In 2003, Takashi[18] presented kinds of equilibrium strategies of fuzzy matrix games based on symmetric triangular fuzzy variable. In 2005, Bector and Chandra[2] provided fuzzy non-cooperative games in uncertainty and modeled by fuzzy programming problems. In 2011, Cunlin and Zhang Qiang[5] investigated two-person zero-sum games in the symmetric triangular fuzzy environment. Afterthen Cunlin[6] charactrized the equilibrium strategy of bi-matrix games with L-R fuzzy payoffs. In 2014, Bapi Dutta[7] investigated the matrix games in trapezoidal fuzzy environment.

In this paper we define the L-R fuzzy trapezoidal variables and generalize the method of Cunlin[5] and Bapi Dutta[7] for Nash equilibrium solution concepts. The paper is organized as follows: In section 2, the basic definitions and notations of L-R fuzzy variables are given. In section 3, we introduce the notation of bi-matrix games with L-R fuzzy payoffs and investigate existence conditions of equilibrium strategy for the fuzzy games. In section 4, crisp parametric bi-matrix games are characterized and several type of equilibrium strategy of fuzzy bi-matrix games are investigated.

2. Preliminaries

In this section, we suggest some basic definitions and concepts of L-R fuzzy variables and introduce some notations of fuzzy sets, such as α-level set for L-R fuzzy variable and pseudoinverse of the monoton function.

Definition 2.1. *A L-R fuzzy variable $\tilde{a}$ is a fuzzy set on the real line $\mathbb{R}$ whose membership function $\mu_{\tilde{a}}(x) : \mathbb{R} \longrightarrow [0,1]$ as following*

$$\mu_{\tilde{a}}(x) = \begin{cases} L(\frac{a-x}{h}) & , x \leq a, \ a-h > 0, \\ 1 & , a \leq x \leq c, \\ R(\frac{x-c}{k}) & , x \geq c, \ c+k > 0, \end{cases}$$

where $L, R : \mathbb{R} \longrightarrow [0,1]$ are not constant and left continuose function and they satisfy the following:

(i) $L(x) = L(-x), R(x) = R(-x)$,

(ii) $L(0) = R(0) = 1, L(1) = R(1) = 0$,

(iii) L, R are nonincreasing on $[0, \infty]$.

The L-R fuzzy variable is denoted by $\tilde{a} = (a, c, h, k)_{L-R}$ where the interval $[a, c]$ is called the center of $\tilde{a}$ and h, k are said left and right extension of $\tilde{a}$, respectively.

We can choose different functions for $L(x)$ and $R(x)$. For instance, consider the following examples.

Example 2.2. *Let* $L(x) = max\{0, 1 - x^2\}$, $R(x) = \frac{1}{1+x^2}$ *and* $a = 5, c = 9, h = 2$ *and* $k = 3$. *Then* $(5, 9, 2, 3)_{L-R}$ *denotes an L-R fuzzy number with membership function (see Fig.1)*

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{-x^2+10x-21}{4} & , 3 \le x < 5, \\ 1 & , 5 \le x \le 9, \\ \frac{9}{x^2-18x+90} & , 9 \le x \le 12. \end{cases}$$

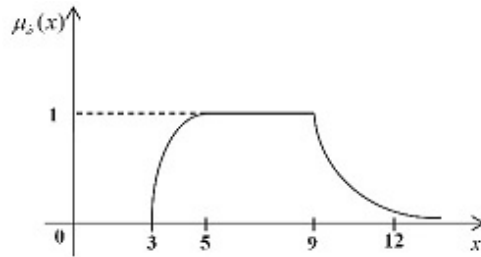Figure 1: The membership function of $(5, 9, 2, 3)_{L-R}$

Example 2.3. *Let* $L(x) = \frac{1}{1+x}$, $R(x) = e^{-x}$ *and* $a = 2, c = 4, h = 1$ *and* $k = 3$. *Then* $(2, 4, 1, 3)_{L-R}$ *denotes an L-R fuzzy number with membership function (see Fig.2)*

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{1}{3-x} & , x \le 2, \\ 1 & , 2 < x \le 4, \\ e^{\frac{4-x}{3}} & , x > 4. \end{cases}$$
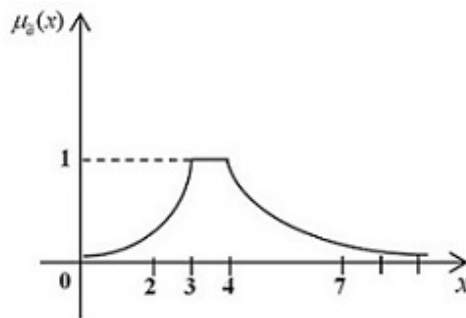

Figure 2: The membership function of $(2, 4, 1, 3)_{L-R}$

. In the rest of the paper, for simplicity, the L-R fuzzy variables set is denoted by $\mathfrak{F}$. The α-level of fuzzy variables have an important role in parametric ordering of fuzzy numbers. Let $\tilde{a} \in \mathfrak{F}$ and $\alpha \in [0, 1]$, $\tilde{a}_\alpha \triangleq \{x | \mu_{\tilde{a}}(x) \ge \alpha, x \in \mathbb{R}\}$ is called α-level of $\tilde{a}$ and denoted by $\tilde{a}_\alpha = [a_\alpha^L, a_\alpha^U]$ where $a_\alpha^U \triangleq \sup \tilde{a}_\alpha$ and $a_\alpha^L \triangleq \inf \tilde{a}_\alpha$. If $\alpha = 0$, $\tilde{a}_0 \triangleq \{x | \mu_{\tilde{a}}(x) > 0, x \in \mathbb{R}\}$ is called support of $\tilde{a}$.

Definition 2.4. *[6] Let $f : [a, b] \longrightarrow [c, d]$ be a monotone function, where $[a, b]$ and $[c, d]$ are closed subintervals of extended real line $[-\infty, +\infty]$. The pseudoinverse $f^{(-1)} : [c, d] \longrightarrow [a, b]$ of f is defined by*

$$f^{(-1)}(y) = \begin{cases} \sup\{x \in [a, b] | f(x) < y\} & , f(a) < f(b), \\ \sup\{x \in [a, b] | f(x) \geq y\} & , f(a) > f(b), \\ a & , f(a) = f(b). \end{cases}$$

Remark 2.5. *Let $L(t)$ and $R(t)$ be functions which are defined in Definition(2.1). The domains of $L^{(-1)}(t)$ and $R^{(-1)}(t)$ are $[0, 1]$.*

Let $\mathfrak{F}(X)$ and $\mathfrak{F}(Y)$ be two fuzzy variable set defined on X and Y where X, Y are crisp sets. The function $f : X \longrightarrow Y$ induces another function $\tilde{f} : \mathfrak{F}(X) \longrightarrow \mathfrak{F}(Y)$ defined on each set $\tilde{u}$ on X by

$$\tilde{f}(\tilde{u})(y) = \sup_{x \in X, f(x) = y} u(x).$$

Also, let X_i and Y be crisp sets for $i = 1, 2, ..., n$, $\mathfrak{F}(\prod_{i=1}^{n} X_i)$ and $\mathfrak{F}(Y)$ are two fuzzy variable sets defined on $\prod_{i=1}^{n} X_i$ and Y. Then the function $f : X \longrightarrow Y$ induces another function $\tilde{f} : \mathfrak{F}(\prod_{i=1}^{n} X_i) \longrightarrow \mathfrak{F}(Y)$ defined on each fuzzy set on $\prod_{i=1}^{n} X_i$ by

$$\tilde{f}(\tilde{u}_1, \tilde{u}_2, ..., \tilde{u}_n)(y) = \sup_{f(x_1, x_2, ..., x_n) = y} \min_{i} u_i(x_i).$$

Using the pseudoinverse of L and R we have the following lemma.

Lemma 2.6. *Let $\tilde{a} = (a, c, h, k)_{L-R}$ be a L-R fuzzy variable. Then for all $\alpha \in (0, 1]$, the α-level of $\tilde{a}$ is, $\tilde{a}_\alpha = [a - hL^{(-1)}(\alpha), c + kR^{(-1)}(\alpha)]$.*

Let $\tilde{a} = (a, c, h, k)_{L-R}$ and $\tilde{b} = (b, d, z, w)_{L-R}$ are two L-R fuzzy variables and $\lambda \in \mathbb{R}^+$ then

- Addition: $\tilde{a} + \tilde{b} = (a + b, c + d, h + z, k + w)_{L-R}$,

- Scalar Multiplication : $\lambda \tilde{a} = (\lambda a, \lambda c, \lambda h, \lambda k)_{L-R}$.

We recall that x, y are defined component wise i.e if $x = (\xi_1, \xi_2, ..., \xi_n)$ and $y = (\eta_1, \eta_2, ..., \eta_n)$ be vectors in $\mathbb{R}^n$ then

(i) $x \geqq y$ if and only if $\xi_i \geq \eta_i$ for all $i = 1, 2, ..., n$,

(ii) $x \geq y$ if and only if $x \geqq y$ and $x \neq y$.

3. Bi-matrix Games with *L-R* Fuzzy Payoffs

In this section, we shall consider bi-matrix games with L-R fuzzy payoffs. Let $P = \{1, 2, ..., p\}$ and $Q = \{1, 2, ..., q\}$ be the sets of pure strategies of player I and player J, respectively. The mixed strategies of players I and player J are probability distributions on the set of pure strategies. The set of mixed strategies for player I is represented by

$$S_I = \{(\xi_1, \xi_2, ..., \xi_p) \in \mathbb{R}^p | \xi_i \geq 0, i = 1, 2, ..., p, \sum_{i=1}^{p} \xi_i = 1\},$$

where $\mathbb{R}^p$ is a set of p-dimensional real numbers space. Similarly, the set of mixed strategies for player J is represented by

$$S_J = \{(\eta_1, \eta_2, ..., \eta_q) \in \mathbb{R}^q | \eta_j \geq 0, i = 1, 2, ..., q, \sum_{j=1}^{q} \eta_j = 1\},$$

where $\mathbb{R}^q$ is a set of q-dimensional real numbers space.

In this section, the payoffs of the pair $(x, y) \in S_I \times S_J$ are modeled by L-R fuzzy terapeziodal variables. Let player I chooses a mixed strategy $x \in S_I$ and player J chooses mixed strategy $y \in S_J$. The L-R fuzzy variable $\tilde{a}_{ij} = (a_{ij}, c_{ij}, h_{ij}, k_{ij})$ indicates the payoffs that player I receives and the L-R fuzzy variable $\tilde{b}_{ij} = (b_{ij}, d_{ij}, z_{ij}, w_{ij})$ indicates the payoffs that player J receives. The fuzzy bi-matrix game is denoted by $\tilde{G} \equiv (I, J, S_I, S_J, \tilde{A}, \tilde{B})$ where $\tilde{a}_{ij}$ representes the income of player I and $\tilde{b}_{ij}$ representes the income of player J. Also

$$E(x, y) = x^T \tilde{A} y = \sum_{i=1}^{p} \sum_{j=1}^{q} \xi_i \tilde{a}_{ij} \eta_j, \quad E(x, y) = x^T \tilde{B} y = \sum_{i=1}^{p} \sum_{j=1}^{q} \xi_i \tilde{b}_{ij} \eta_j,$$

are called the expected value of players and payoff matrix of player I and J is given by

$$\tilde{A} = \begin{pmatrix} \tilde{a}_{11} & \cdots & \tilde{a}_{1q} \\ \vdots & \ddots & \vdots \\ \tilde{a}_{p1} & \cdots & \tilde{a}_{pq} \end{pmatrix}, \tilde{B} = \begin{pmatrix} \tilde{b}_{11} & \cdots & \tilde{b}_{1q} \\ \vdots & \ddots & \vdots \\ \tilde{b}_{p1} & \cdots & \tilde{b}_{pq} \end{pmatrix},$$

respectively.

Definition 3.1. *[5] Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then*

(i) $\tilde{a} \gtrapprox \tilde{b}$ *if and only if* $(a_\alpha^L, a_\alpha^U) \geqq (b_\alpha^L, b_\alpha^U)$, *for all* $\alpha \in [0, 1]$,

(ii) $\tilde{a} \gtrsim \tilde{b}$ *if and only if* $(a_\alpha^L, a_\alpha^U) \geq (b_\alpha^L, b_\alpha^U)$, *for all* $\alpha \in [0, 1]$,

(iii) $\tilde{a} \succ \tilde{b}$ *if and only if* $(a_\alpha^L, a_\alpha^U) > (b_\alpha^L, b_\alpha^U)$, *for all* $\alpha \in [0, 1]$.

The following theorem characterize the orders for L-R fuzzy terapeziodal variables.

Lemma 3.2. *Let $\tilde{a} = (a, c, h, k)_{L-R}$, $\tilde{b} = (b, d, z, w)_{L-R}$ be two L-R fuzzy variables. Then*

(i) $\tilde{a} \precapprox \tilde{b}$ *if and only if* $max\{z - h, 0\} \leq b - a$ *and* $max\{k - w, 0\} \leq d - c$,

(ii) $\tilde{a} \prec \tilde{b}$ *if and only if* $max\{z - h, 0\} < b - a$ *and* $max\{k - w, 0\} < d - c$.

Proof. By using Definition (3.1) $\tilde{a} \precapprox \tilde{b}$ if and only if for all $\alpha \in [0, 1]$, $(a_\alpha^L, a_\alpha^U) \leqq (b_\alpha^L, b_\alpha^U)$ or equivalently $a_\alpha^L \leqslant b_\alpha^L$ and $a_\alpha^U \leqslant b_\alpha^U$. But by Lemma (2.6) $a_\alpha^L \leqslant b_\alpha^L$ if and only if

$$a - hL^{(-1)}(\alpha) \leqslant b - zL^{(-1)}(\alpha) \quad for \quad all \quad \alpha \in [0, 1],$$

which are equivalent to

$$(z - h)L^{(-1)}(\alpha) \leqslant b - a \quad for \quad all \quad \alpha \in [0, 1],$$

and equivalently $max\{z - h, 0\} \leq b - a$. Also, by using Lemma(2.6) it can be conclude $a_\alpha^U \leqslant b_\alpha^U$ if and only if $max\{k - w, 0\} \leq d - c$ and the proof of part (i) is complete. Part (ii) can be proved, similarly. $\square$

Definition 3.3. *[7] A pair $(x^*, y^*) \in S_I \times S_J$ is called a Nash equilibrium strategy for a game $\tilde{G}$ if*

(i) $x^T \tilde{A} y^* \precapprox x^{*T} \tilde{A} y^*, \quad \forall x \in S_I$,

(ii) $x^{*T} \tilde{B} y \precapprox x^{*T} \tilde{B} y^*, \quad \forall y \in S_J$.

. Let $\tilde{A} = (\tilde{a}_{ij})_{p \times q}$ be one of fuzzy payoff matrix of bi-matrix game $\tilde{G}$, then for $x \in S_I$, $y \in S_J$ and $\alpha \in [0, 1]$, $x^T \tilde{A} y$ is a L-R fuzzy variable and $(x^T \tilde{A} y)_\alpha = [x^T A_\alpha^L y, x^T A_\alpha^U y]$, for more details see [6].

Theorem 3.4. *Let $\tilde{G} = (\{I, J\}, S_I, S_J, \tilde{A}, \tilde{B})$ be a bi-matrix game with fuzzy payoffs, the pair $(x^*, y^*) \in S_I \times S_J$ is the expected Nash equilibrium strategy of $\tilde{G}$ if and only if for all $x \in S_I, y \in S_J$ the following inequalities hold*

(i) $x^T A y^* \leq x^{*T} A y^*, x^T C y^* \leq x^{*T} C y^*,$

(ii) $x^{*T} B y \leq x^{*T} B y^*, x^{*T} D y \leq x^{*T} D y^*,$

(iii) $x^T (A - H) y^* \leq x^{*T} (A - H) y^*, x^{*T} (B - Z) y \leq x^{*T} (B - Z) y^*,$

(iv) $x^T (C + K) y^* \leq x^{*T} (C + K) y^*, x^{*T} (D + W) y \leq x^{*T} (D + W) y^*.$

Proof . Let $\tilde{G}$ be a bi-matrix game with the L-R fuzzy payoff matrix $\tilde{A} = (A, C, H, K)$ and $\tilde{B} = (B, D, Z, W)$ for player I and J, respectively. Let $(x^*, y^*) \in S_I \times S_J$ be the Nash equilibrium strategy of the game $\tilde{G}$. Therefore by Definition (3.3) we have

(i) $x^T \tilde{A} y^* \precapprox x^{*T} \tilde{A} y^*, \quad \forall x \in S_I,$

(ii) $x^{*T} \tilde{B} y \precapprox x^{*T} \tilde{B} y^*, \quad \forall y \in S_J.$

Since

$$\begin{aligned} x^T \tilde{A} y^* &= (x^T A y^*, x^T C y^*, x^T H y^*, x^T K y^*), \\ x^{*T} \tilde{A} y^* &= (x^{*T} A y^*, x^{*T} C y^*, x^{*T} H y^*, x^{*T} K y^*). \end{aligned}$$

So, by Lemma (3.2), $x^T \tilde{A} y^* \precapprox x^{*T} \tilde{A} y^*$ if and only if

$$\begin{aligned} max\{x^{*T} H y^* - x^T H y^*, 0\} &\leq x^{*T} A y^* - x^T A y^*, \\ max\{x^T K y^* - x^{*T} K y^*, 0\} &\leq x^{*T} C y^* - x^T C y^*. \end{aligned}$$

Consequently $x^T \tilde{A} y^* \precapprox x^{*T} \tilde{A} y^*$ if and only if

$$x^T (A - H) y^* \leq x^{*T} (A - H) y^*, x^T A y^* \leq x^{*T} A y^*, \tag{3.1}$$

$$x^T (C + K) y^* \leq x^{*T} (C + K) y^*, x^T C y^* \leq x^{*T} C y^*. \tag{3.2}$$

Also, since

$$\begin{aligned} x^{*T} \tilde{B} y &= (x^{*T} B y, x^{*T} D y, x^{*T} Z y, x^{*T} W y), \\ x^{*T} \tilde{B} y^* &= (x^{*T} B y^*, x^{*T} D y^*, x^{*T} Z y^*, x^{*T} W y^*), \end{aligned}$$

similary by Lemma (3.2), $x^{*T} \tilde{B} y \precapprox x^{*T} \tilde{B} y^*$ if and only if

$$x^{*T} (B - Z) y \leq x^{*T} (B - Z) y^*, x^{*T} B y \leq x^{*T} B y^*, \tag{3.3}$$

$$x^{*T} (D + W) y \leq x^{*T} (D + W) y^*, x^{*T} D y \leq x^{*T} D y^*. \tag{3.4}$$

Hence, we have the required inequalities (i)-(iv) of the theorem, by rearranging the inequalities. $\square$ In the rest of this paper, we set

$$A = (a_{ij})_{p \times q}, C = (c_{ij})_{p \times q}, H = (h_{ij})_{p \times q}, K = (k_{ij})_{p \times q}, A_0^L = A - H, A_0^U = C + K,$$

and

$$B = (b_{ij})_{p \times q}, D = (d_{ij})_{p \times q}, Z = (z_{ij})_{p \times q}, W = (w_{ij})_{p \times q}, B_0^L = B - Z, B_0^U = D + W.$$

Using these notations Theorem(3.4) can be rewrite as follows.

Corollary 3.5. *Let $\tilde{G}$ be a bi-matrix game with L-R fuzzy payoffs, the pair (x^*, y^*) is the Nash equilibrium strategy of $\tilde{G}$ if and only if the followings hold*

$$x^T(A, C, A_0^L, A_0^U)y^* \leq x^{*T}(A, C, A_0^L, A_0^U)y^*,$$
$$x^{*T}(B, D, B_0^L, B_0^U)y \leq x^{*T}(B, D, B_0^L, B_0^U)y^*.$$

Obviously, the equilibrium strategy of fuzzy bi-matrix games $\tilde{G}$ is equilibrum strategy of four crisp bi-matrix games. It is difficult that these conditions satisfy simultaneously. However, it holdes in the following conditions.

Definition 3.6. *A bi-matrix fuzzy game $\tilde{G} = (\{I, J\}, S_I, S_J, \tilde{A}, \tilde{B})$ is called to be a proportional bi-matrix fuzzy game if there exists $\gamma_n \in (0, 1]; n = 1, ..., 4$ such that $h_{ij} = \gamma_1 a_{ij}$, $k_{ij} = \gamma_2 c_{ij}$, $z_{ij} = \gamma_3 b_{ij}$ and $w_{ij} = \gamma_4 d_{ij}$ for all $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$.*

Theorem 3.7. *A pair of mixed strategies $(x^*, y^*) \in S_I \times S_J$ is a Nash equilibrium strategy of the proportional fuzzy matrix game $\tilde{G} = (\{I, J\}, S_I, S_J, \tilde{A}, \tilde{B})$ if and only if $(x^*, y^*) \in S_I \times S_J$ is the Nash equilibrium of crisp bi-matrix games $G_a = (\{I, J\}, S_I, S_J, A, C)$, $G_b = (\{I, J\}, S_I, S_J, B, D)$.*

Proof . Let $\tilde{G} = (\{I, J\}, S_I, S_J, \tilde{A}, \tilde{B})$ be a proportional fuzzy bi-matrix game. Therefore by Definition (3.6) $\tilde{A} = (A, C, \gamma_1 A, \gamma_2 C)$ is the payoff matrix of the player I and $\tilde{A} = (B, D, \gamma_3 B, \gamma_4 D)$ is the payoff matrix of the player J . By Theorem (3.4), $(x^*, y^*) \in S_I \times S_J$ is a Nash equilibrium of $\tilde{G}$ if and only if

(i) $x^T A y^* \leq x^{*T} A y^*, x^T C y^* \leq x^{*T} C y^*,$

(ii) $x^{*T} B y \leq x^{*T} B y^*, x^{*T} D y \leq x^{*T} D y^*,$

because the other inequalities came to these one. Equivalently, $(x^*, y^*) \in S_I \times S_J$ is a Nash equilibrium of crisp bi-matrix games $G_a = (\{I, J\}, S_I, S_J, A, C)$,
$G_b = (\{I, J\}, S_I, S_J, B, D)$. The proof is complete. $\square$

Definition 3.8. *The bi-matrix fuzzy game $\tilde{G}$ is called constant fuzzy game if and only if there exist $h, k, z, w > 0$ such that $h_{ij} = h, k_{ij} = k, z_{ij} = z, w_{ij} = w$ for all $i = 1, 2, ..., p$ and $j = 1, 2, ..., q$.*

Lemma 3.9. *Let $\tilde{G} = (\{I, J\}, S_I, S_J, \tilde{A}, \tilde{B})$ be a constant fuzzy game. A pair of mixed strategies $(x^*, y^*) \in S_I \times S_J$ is the Nash equilibrium strategy for $\tilde{G}$ if and only if (x^*, y^*) is a Nash equilibrium of bi-matrix games Γ_a, Γ_b .*

Proof . By Definition(3.8) H, K, Z and W are constant matrices which all the entries are h, k, z and w , respectively. Hence $x^T H y = h, x^T K y = k, X^T Z y = z$ and $x^T W y = w$ for all $x \in S_I, y \in S_J$. By Theorem(3.4) the result can be obtained, directly. $\square$

Theorem 3.10. *Let $\tilde{G} = (\{I, J\}, S_I, S_J, \tilde{A}, \tilde{B})$ be a fuzzy bi-matrix game, $T_I(\tilde{G})$ and $T_J(\tilde{G})$ are the sets of the strategy of player I and player J, respectively. Then $T_I(\tilde{G})$ and $T_J(\tilde{G})$ are closed convex sets.*

Proof . Let $x^* \in T_I(\tilde{G})$ and $y^* \in T_J(\tilde{G})$ and (x^*, y^*) be a equilibrium strategy of the fuzzy bi-matrix games. From Definition (3.3) we have

(i) $x^T \tilde{A} y^* \precsim x^{*T} \tilde{A} y^*, \quad \forall x \in S_I,$

(ii) $x^{*T}\tilde{B}y \prec pprox x^{*T}\tilde{B}y^*, \quad \forall y \in S_J.$

By Theorem (3.7), the pair (x^*, y^*) is the equilibrium strategy of two crisp bi-matrix games $G_a = (\{I, J\}, S_I, S_J, A, C)$ and $G_b = (\{I, J\}, S_I, S_J, B, D)$. Consequently, $x^* \in T_I(G_a)$ and $x^* \in T_I(G_b)$ then $x^* \in T_I(G_a) \cap T_I(G_b)$. Moreover, $T_I(G_a), T_I(G_b)$ are closed convex sets. Therefore $T_I(\tilde{G})$ is a closed convex set. Similarly, it can be result that $T_J(\tilde{G})$ is a closed and convex set. □

Definition 3.11. *[7] A pair of mixed strategies $(x^*, y^*) \in S_I \times S_J$ is called a Pareto Nash equilibrium strategy of the game $\tilde{G}$ if*

(i) *there does not exist any $x \in S_I$ such that $x^{*T}\tilde{A}y^* \precsim x^T\tilde{A}y^*$,*

(ii) *there does not exist any $y \in S_J$ such that $x^{*T}\tilde{B}y^* \precsim x^{*T}\tilde{B}y$.*

Theorem 3.12. *Let $\tilde{G} = (\{I, J\}, S_I, S_J, \tilde{A}, \tilde{B})$ be a fuzzy bi-matrix game. A pair $(x^*, y^*) \in S_I \times S_J$ is the Pareto Nash equilibrium strategy for the game $\tilde{G}$ if and only if*

(i) *there exist no $x \in S_I$ such that $x^{*T}Ay^* \leq x^T Ay^*$, $x^{*T}Cy^* \leq x^T Cy^*$ and*

$$(x^{*T}A_0^L y^*, x^{*T}A_0^U y^*) \leq (x^T A_0^L y^*, x^T A_0^U y^*), \tag{3.5}$$

(ii) *there exist no $y \in S_J$ such that $x^{*T}By^* \leq x^T By$, $x^{*T}Dy^* \leq x^T Dy$ and*

$$(x^{*T}B_0^L y^*, x^{*T}B_0^U y^*) \leq (x^{*T}B_0^L y, x^{*T}B_0^U y). \tag{3.6}$$

Proof . By contradiction, let $(x^*, y^*) \in S_I \times S_J$ be the Pareto Nash equilibrium strategy of $\tilde{G}$. Suppose that there exist $\hat{x} \in S_I$ such that $x^{*T}Ay^* \leq \hat{x}^T Ay^*$, $x^{*T}Cy^* \leq \hat{x}^T Cy^*$ and $(x^{*T}A_0^L y^*, x^{*T}A_0^U y^*) \leq (\hat{x}^T A_0^U y^*, \hat{x}^T A_0^U y^*)$. It implies that

$$x^{*T}(A - H)y^* \leq \hat{x}^T(A - H)y^*, x^{*T}(C + K)y^* \leq \hat{x}^T(C + K)y^*.$$

But, by the above inequalities do not occur simultaneously. Let $\alpha \in [0, 1]$ and

$$\mu_{\tilde{a}_{ij}}(x) = \begin{cases} L(\frac{a_{ij}-x}{h_{ij}}) & , x \leq a_{ij}, \ a_{ij} - h_{ij} > 0, \\ 1 & , a \leq x \leq c, \\ R(\frac{x-c_{ij}}{k_{ij}}) & , x \geq c_{ij}, \ c_{ij} + k_{ij} > 0, \end{cases}$$

then $L^{-1}(\alpha), R^{-1}(\alpha) \in [0, 1]$. Therefore, from above inequalities we get

$$\left(x^{*T}((1 - L^{(-1)}(\alpha))A + L^{(-1)}(\alpha)(A - H))y^*, x^{*T}((1 - R^{(-1)}(\alpha))C + R^{(-1)}(\alpha)(C + K))y^*\right)$$

$$\leq \left(\hat{x}^T((1 - L^{(-1)}(\alpha))A + L^{(-1)}(\alpha)(A - H))y^*, \hat{x}^T((R^{(-1)}(\alpha))C + R^{(-1)}(\alpha)(C + K))y^*\right),$$

by rearrenging we obtain

$$\left(x^{*T}(A - L^{(-1)}(\alpha)H)y^*, x^{*T}(C + KR^{(-1)}(\alpha))y^*\right)$$

$$\leq \left(\hat{x}^T(A - L^{(-1)}(\alpha)H)y^*, \hat{x}^T(C + KR^{(-1)}(\alpha))y^*\right),$$

Using Definition (3.11) it implies that $x^{*T}\tilde{A}y^{*} \preceq x_{1}^{T}\tilde{A}y^{*}$. This is a contradiction.
Conversely, we assume that the pair of mixed strategy $(x^{*}, y^{*}) \in S_{I} \times S_{J}$ be satisfy (3.5) and (3.6).
Suppose that there exists a strategy $\hat{x} \in S_{I}$ such that $x^{*T}\tilde{A}y^{*} \preceq x_{1}^{T}\tilde{A}y^{*}$. So, we have for all $\alpha \in [0, 1]$

$$(x^{*T}A_{\alpha}^{L}y^{*}, x^{*T}A_{\alpha}^{U}y^{*}) \leq (\hat{x}^{T}A_{\alpha}^{L}y^{*}, \hat{x}^{T}A_{\alpha}^{U}y^{*}),$$

which $(A - L^{(-1)}(\alpha)H) = A_{\alpha}^{L}, \quad (C + KR^{(-1)}(\alpha)) = A_{\alpha}^{U}$. Set $\alpha = 0$, then

$$x^{*T}(A_{0}^{L}, A_{0}^{U})y^{*} \leq \hat{x}^{T}(A_{0}^{L}, A_{0}^{U})y^{*}, x^{*T}Ay^{*} \leq \hat{x}^{T}Ay^{*}.$$

This is contradict (i). Similarly, we can show that there does not exist any $y \in S_{J}$ such that $x^{*T}\tilde{B}y^{*} \preceq x^{*T}\tilde{B}y$. $\square$

Definition 3.13. *A pair of mixed strategies $(x^{*}, y^{*}) \in S_{I} \times S_{J}$ is a weak Pareto Nash equilibrium strategy of the game $\tilde{G}$ if*

(i) there does not exist any $x \in S_{I}$ such that $x^{*T}\tilde{A}y^{*} \prec x^{T}\tilde{A}y^{*}$,

(ii) there does not exist any $y \in S_{J}$ such that $x^{*T}\tilde{B}y^{*} \prec x^{*T}\tilde{B}y$.

Following theorem is obtaine directly from Definition (3.13) and Theorem (3.12).

Theorem 3.14. *Let $\tilde{G} = (\{I, J\}, S_{I}, S_{J}, \tilde{A}, \tilde{B})$ be a fuzzy bi-matrix game. A pair $(x^{*}, y^{*}) \in S_{I} \times S_{J}$ is the weak Pareto Nash equilibrium strategy for the game $\tilde{G}$ if and only if*

(i) there exist no $x \in S_{I}$ such that $x^{*T}Ay^{*} < x^{T}Ay^{*}, x^{*T}Cy^{*} < x^{T}Cy^{*}$ and

$$(x^{*T}A_{0}^{L}y^{*}, x^{*T}A_{0}^{U}y^{*}) < (x^{T}A_{0}^{L}y^{*}, x^{T}A_{0}^{U}y^{*}), \tag{3.7}$$

(ii) there exist no $y \in S_{J}$ such that $x^{*T}By^{*} < x^{T}By, x^{*T}Dy^{*} < x^{T}Dy$ and

$$(x^{*T}B_{0}^{L}y^{*}, x^{*T}B_{0}^{U}y^{*}) < (x^{*T}B_{0}^{L}y, x^{*T}B_{0}^{U}y). \tag{3.8}$$

4. Parametric Bi-Matrix Games

In this section we characterize the crisp parametric matrix games and investigate other types of Nash equilibrium strategies for fuzzy bi-matrix games. Let $S^{p} = \{\xi_{1}, \xi_{2}, ..., \xi_{p}\}$ and $S^{q} = \{\eta_{1}, \eta_{2}, ..., \eta_{q}\}$ be sets of pure strategies of player I and player J, respectively. While player I chooses the pure strategy ξ_{i} and player J chooses the pure strategy η_{i}, suppose $(1 - \rho)(a_{ij} - h_{ij}) + \rho(c_{ij} + k_{ij})$ be the payoff of player I and $-[(1 - \nu)(b_{ij} - z_{ij}) + \nu(dij + w_{ij})]$ be the payoff of player J, where $\rho, \nu \in [0, 1]$. The payoff matrices of player I and J are

$$A(\rho) = (1 - \rho)(A - H) + \rho(C + K), \quad B(\nu) = (1 - \nu)(B - Z) + \nu(D + W). \tag{4.1}$$

We consider the crisp parametric bi-matrix game $G(\rho, \nu) = (S_{I}, S_{J}, A(\rho), B(\nu))$.

Definition 4.1. *[12] Let $G(\rho, \nu)$ be a crisp parametric bi-matrix game. For $\rho, \nu \in [0, 1]$, a pair of mixed strategies $(x^{*}, y^{*}) \in S_{I} \times S_{J}$ is a Nash equilibrium strategy of G if it holds that*

(i) $x^{T}A(\rho)y^{*} \leq x^{*T}A(\rho)y^{*}$ for all $x \in S_{I}$,

(ii) $x^{*T}B(\nu)y \leq x^{*T}B(\nu)y^{*}$ for all $y \in S_{J}$.

Lemma 4.2. *There exists at least one Nash equilibrium strategy for all parametric games $G(\rho, \nu)$ for each $\rho, \nu \in [0, 1]$.*

Theorem 4.3. *Let $G(\rho, \nu)$ be a crisp prametric bi-matrix game with $\rho, \nu \in (0, 1)$ and the pair of mixed strategy $(x^*, y^*) \in S_I \times S_J$ be Nash equilibrium strategy of G. Then $(x^*, y^*) \in S_I \times S_J$ is the Pareto Nash equilibrium strategy of the fuzzy bi-matrix game $\tilde{G}$.*

Proof . Let $(x^*, y^*) \in S_I \times S_J$ be the Nash equilibrium strategy of the parametric bi-matrix game $\Gamma(\rho, \nu)$, which $\rho, \nu \in (0, 1)$. By Definition (4.1) for $x \in S_I$, we obtain

$$(1 - \rho)x^T(A - H)y^* + \rho x^T(C + K)y^* \leq (1 - \rho)x^{*T}(A - H)y^* + \rho x^{*T}(C + K)y^*, \qquad (4.2)$$

and for $y \in S_J$ we obtain

$$(1 - \nu)x^{*T}(B - Z)y + \nu x^{*T}(D + W)y \leq (1 - \nu)x^{*T}(B - Z)y^* + \nu x^{*T}(D + W)y^*. \qquad (4.3)$$

Let there exists $\hat{x} \in S_I$ such that $x^{*T}\tilde{A}y^* \preceq \hat{x}^T\tilde{A}y^*$. From Definition (3.1), it followes that

$$(x^{*T}A_0^L y^*, x^{*T}A_0^U y^*) \leq (\hat{x}^T A_0^L y^*, \hat{x}^T A_0^U y^*).$$

But $x^{*T}A_0^L y^* = \hat{x}^T A_0^L y^*, x^{*T}A_0^U y^* = \hat{x}^T A_0^U y^*$ do not occur simultaneously. Then we have

$$(1 - \rho)x^{*T}A_0^L y^* + \rho x^{*T}A_0^U y^* < (1 - \rho)\hat{x}^T A_0^L y^* + \rho\hat{x}^T A_0^U y^*,$$

and consequently

$$(1 - \rho)x^{*T}(A - H)y^* + \rho x^{*T}(C + K)y^* < (1 - \rho)\hat{x}^T(A - H)y^* + \rho\hat{x}^T(C + K)y^*,$$

This is a contradiction (4.2). The condition (ii) can de proved, similarly. $\square$

Theorem 4.4. *Let the pair $(x^*, y^*) \in S_I \times S_J$ be the Nash equilibrium strategy of crisp prametric bi-matrix game $G(\rho, \nu)$ with $\rho, \nu \in [0, 1]$. Then $(x^*, y^*) \in S_I \times S_J$ is the weak Pareto Nash equilibrium strategy of fuzzy bi-matrix game $\tilde{G}$.*

The following corollary is direct result of Theorem (4.3) and Theorem (4.4).

Corollary 4.5. *A fuzzy bi-matrix game $\tilde{G}$ satisfies the following properties:*

   (i) *There exsist at least one Pareto Nash equilibrium strategy of fuzzy game $\tilde{G}$,*

   (ii) *There exsist at least one weak Pareto Nash equilibrium strategy of fuzzy game $\tilde{G}$.*

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