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Fixed point theorems for new J-type mappings in modular spaces

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Abstract

The aim of this article is to introduce ρ-altering J-type mappings in modular spaces and prove some fixed point theorems for ρ-altering and ρ-altering J-type mappings in modular spaces. We also furnish illustrative examples to express relationship between these mappings. As a consequence, the results are applied to the existence of solution of an integral equation arising from an ODE system.

Keywords: Modular space, ρ-Altering mapping, ρ-Asymptotically regular mapping, ρ-J-type mapping, ρ-altering J-type mapping, Integral equation.
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1. Introduction

The theory of fixed point is a very extensive field which was received attention from 19 century. The fixed point theorem states the existences of fixed points under suitable conditions. The famous Brouwer fixed point theorem was given 1912 that is only true in finite dimensional spaces and in an infinite dimensional Banach space was given by Schauder in 1930. Tychonoff generalized that theorem to locally convex topological vector spaces. Moreover, fixed point theory has been developed through different spaces such as topological vector space, menger space, modular space, etc. Our aim is on modular spaces developed by Nakano [?] in 1950 and were modified by Musielak and Orlicz [?] in 1959. It is easy to check that every normed space is modular space. Since then the problem of finding weaker conditions implies existence of fixed points for mappings has been the subject of study by many authors. Eventhough a metric is not defined many problems in fixed point theory can be formulated in modular spaces.

On the other hand, contraction and nonexpansive mappings have been extended and several intresting

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results are proved. In 1984, M. S. Khan, M. Swaleh and S. Sessa in [?] extend the research of metric fixed point theory to new category by introducing a control function that they called it an altering distance function. So that is satisfies the following in a quality:

$$\psi(d(T(x), T(y))) \leq k\psi(d(x, y)) \qquad (x, y \in X),$$

where $k \in (0, 1)$ and $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is monotone increasing and continuous and $\psi(t) = 0$ if and only if $t = 0$.
Here, we porove some fixed point theorems for ρ-altering mappings in modular spaces with a relatively new notation of these mappings in metric spaces. However, there are many nonlinear mappings which are more general. In section 3, we introduce ρ-altering J-type mappings in modular spaces and demonstrate some theorems in these spaces. Recently, J. Garcia-Falset at el. [?] introduced a center points and the new class of nonlinear mappings that is called J-type map. Let B is a closed bounded and convex subset of a Banach space $(X, \|.\|)$. $T : B \to X$ is a J-type Mapping if T is a continuous and for each $x \in B$, $\|T(x) - y_0\| \leqslant \|x - y_0\|$ for some $y_0 \in X$. In this case y_0 is a center for mapping T. We extend these notations to modular spaces. Several consequences are also obtained and some approximated examples are as well provided . In section 4, we give an application to solve an integral equation.

2. Priliminaries

We start with base definitions which will be used throughout in this paper.

Definition 2.1. *Let X be an arbitrary vector space over $K(= \mathbb{R}$ or $\mathbb{C})$*
a) A function $\rho : X \to [0, \infty]$ is called modular if

(i) $\rho(x) = 0$ if and only if $x = 0$,
(ii) $\rho(\alpha x) = \rho(x)$ for $\alpha \in K$ with $|\alpha| = 1$ and for all $x \in X$,
(iii) $\rho(\alpha x + \beta y) \leq \rho(x) + \rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$ and for all $x, y \in X$.

If (iii) is replaced by

(iii)' $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$ if $\alpha, \beta \geq 0, \alpha + \beta = 1$ and for all $x, y \in X$

then the modular ρ is called convex modular.
b) A modular ρ defines a corresponding modular space, that is, the space X_ρ given by

$$X_\rho = \{x \in X : \rho(\alpha x) \to 0 \text{ as } \alpha \to 0\}.$$

It is clear that every convex modular is modular, but the converse is not true.

Example 2.2. *Let $X = \mathbb{R}$ and $\rho : X \to [0, \infty]$ be defined as following*

$$\rho(x) = \begin{cases} 0 & x = 0 \\ 1 & x \neq 0 \end{cases},$$

It is obviously ρ is modular and if we set $\beta = \alpha = \frac{1}{2}, y \neq 0$ and $x = 0$, then ρ is not convex modular.

Remark 2.3. *Note that for every $x \in X$, the function $g(t) = \rho(tx)$, is an increasing function on $(0, \infty)$.*

For this, let $0 < a < b$, from the property (iii) in the above definition for $y = 0$ we have

$$g(a) = \rho(ax) = \rho(\frac{a}{b}(bx)) = \rho(\frac{a}{b}(bx)) + \rho(\frac{b-a}{b}(y)) \leq \rho(bx) = g(b). \tag{2.1}$$

Definition 2.4. *Let X_ρ be a modular space.*

(a) A sequence $\{x_n\} \subseteq X_\rho$ is said to be

(i) ρ-convergent to x if $\rho(x_n - x) \to 0$ as $n \to \infty$,

(ii) ρ-Cauchy if $\rho(x_n - x_m) \to 0$ as $n \to \infty$.

(b) X_ρ is a ρ-complete if any ρ-Cauchy sequence is ρ-convergent.

(c) A subset $B \subseteq X_\rho$ is called ρ-closed if for any sequence $\{x_n\} \subseteq B$, which is ρ-convergent to $x \in X_\rho$, then $x \in B$.

(d) A subset $B \subseteq X_\rho$ is called ρ-bounded if diam$\rho(B) < \infty$, where diam$\rho(B) = \sup\{\rho(x-y) | x, y \in B\}$ is called the ρ-diameter of B.

(e) ρ is said to satisfy the Δ_2-condition if $\rho(2x_n) \to 0$ whenever $\rho(x_n) \to 0$ as $n \to \infty$.

(f) We say that ρ has Fatou property if $\rho(x - y) \leq \liminf \rho(x_n - y_n)$ whenever $\rho(x_n - x) \to 0$ and $\rho(y_n - y) \to 0$ as $n \to \infty$.

In modular spaces, it is not necessary that the limit of sequence $\{x_n\}$ be unique. The details follow.

Remark 2.5. *Let (X_ρ, ρ) be a modular space which ρ satisfies in Δ_2-condition. Then the limit is unique.*

3. Fixed point theorems of ρ-altering Mappings

Throughout this paper we assume that (X_ρ, ρ) is a modular space. As usual we will denote by $\mathbb{R}^+$ nonnegative real numbers. In this section, we prove fixed point theorems for class of contractive mappings in modular spaces. We begin by recalling some basic definition and theorem.

Remark 3.1. *$\psi : \mathbb{R}^+ \to \mathbb{R}^+$ is an increasing, sub linear and continuous function such that $\psi(t) = 0$ if and only if $t = 0$.*

Definition 3.2. *Let X_ρ be a modular space and $T : X_\rho \to X_\rho$ be a mapping. The mapping T is called ρ-altering mapping if there exists $0 < k < 1$ such that,*

$$\psi(\rho(T(x) - T(y))) \leq k\psi(\rho(x - y)) \qquad (x, y \in X_\rho)$$

where ψ satisfies in the remarks **??**.

Theorem 3.3. *Let X_ρ be a ρ-complete modular space where ρ satisfies the Δ_2-condition. If B is a ρ-closed and ρ-bounded subset of X_ρ and $T : B \to B$ is an ρ-altering mapping, then T has a unique fixed point.*

Proof . We consider the sequence $\{x_n\} \subseteq B$ as follows $x_{n+1} = T(x_n) \quad (n \geq 0)$. If there exists $n \in \mathbb{N}$ such that $x_n = x_{n+1}$ then $x_n = u$ is a fixed point of T.
Otherwise, if for all positive integer n, $x_n \neq x_{n+1}$ we have

$$\psi(\rho(x_{n+m} - x_m)) = \psi(\rho(T(x_{n+m-1}) - T(x_{m-1}))) \leq k\psi(\rho(x_{n+m-1} - x_{m-1})).$$

Therefore from ρ-boundedness of B

$$\psi(\rho(x_{n+m} - x_m)) \leq k^m\psi(\rho(x_n - x_0)) \to 0.$$

Then $\{x_n\}$ is a ρ-Cauchy sequence in B and so there exist $z \in B$ such that $\rho(x_n - z) \to 0$. We clime that z is a fixed point of T.

$$\begin{aligned}\psi(\rho(\frac{T(z)-z}{2})) &\leq \psi(\rho(T(z) - T(x_{n-1}))) + \psi(\rho(T(x_{n-1}) - z)) \\ &\leq k\psi(\rho(x_{n-1} - z) + \psi(\rho(x_n - z)) \to 0\end{aligned}$$

Finally, for uniqueness, let z and w be two fixed points. Then

$$\psi(\rho(z - w)) = \psi(\rho(T(z) - T(w))) \leq k\psi(\rho(z - w))$$

which implies that $\psi(\rho(z - w)) = 0$ and then $z = w$. □ The boundedness property can be omitted.

Theorem 3.4. *Let X_ρ be a ρ-complete modular space where ρ satisfies the Δ_2-condition and B is a ρ-closed subset of X_ρ. If $T : B \to B$ is a $(\rho - c)$-altering mapping, i.e., there exists $c, k, l \in \mathbb{R}^+$ with $c > l$ and $0 < k < 1$ such that,*

$$\psi(\rho(c(Tx - Ty))) \leq k\psi(\rho(l(x - y))) \qquad (x, y \in X_\rho),$$

then T has a unique fixed point.

Proof . Let $\alpha \in \mathbb{R}^+$ be the conjugate of $\frac{c}{l}$; that is, $\frac{l}{c} + \frac{1}{\alpha} = 1$. Let $x \in B$ and $(T^n x)_{n\in\mathbb{N}}$ be a sequence in B. For each integer $n \geq 1$,

$$\begin{aligned}\psi(\rho(c(T^{n+1}x - T^n x))) &\leq k\psi(\rho(l(T^n x - T^{n-1}x))) \\ &\leq k\psi(\rho(c(T^n x - T^{n-1}x))) \\ &\leq k^2\psi(\rho(l(T^{n-1}x - T^{n-2}x))).\end{aligned}$$

By induction,

$$\psi(\rho(c(T^{n+1}x - T^n x))) \leq k^n\psi(\rho(l(Tx - x))). \tag{3.1}$$

Taking the limit as $n \to \infty$ yields

$$\lim_n \psi(\rho(c(T^{n+1}x - T^n x))) = 0.$$

We now show that $(T^n x)_{n\in\mathbb{N}}$ is ρ-Cauchy sequence. Assume to the contrary that there exists an $\varepsilon > 0$ and subsequences $\{m(t)\}$ and $\{n(t)\}$ such that $m(t) < n(t) < m(t + 1)$ with

$$\psi(\rho(c(T^{m(t)}x - T^{n(t)}x))) \geq \varepsilon \quad \text{and} \quad \psi(\rho(c(T^{m(t)}x - T^{n(t)-1}x))) < \varepsilon. \tag{3.2}$$

Moreover,

$$\begin{aligned}\rho(l(T^{m(t)-1}x - T^{n(t)-1}x)) &= \rho(l(T^{m(t)-1}x - T^{m(t)}x + T^{m(t)}x - T^{n(t)-1}x))\\ &= \rho(\alpha\tfrac{l}{\alpha}(T^{m(t)-1}x - T^{m(t)}x) + \tfrac{lc}{c}(T^{m(t)}x - T^{n(t)-1}x))\\ &\le \rho(\alpha l(T^{m(t)-1}x - T^{m(t)}x)) + \rho(c(T^{m(t)}x - T^{n(t)-1}x)).\end{aligned}$$

By using the Δ_2-condition, $\lim_{m(t)\to\infty}\psi(\rho(\alpha l(T^{m(t)-1}x - T^{m(t)}x))) = 0$. Therefore

$$\begin{aligned}\varepsilon &\le \psi(\rho(c(T^{m(t)}x - T^{n(t)}x)))\\ &\le k\psi(\rho(l(T^{m(t)-1}x - T^{n(t)-1}x))) \le k\varepsilon,\end{aligned}$$

which is a contradiction. Therefore by Δ_2-condition $(T^n x)_{n\in\mathbb{N}}$ is ρ-Cauchy. Since X_ρ is a ρ-complete and B is ρ-closed, there exists a $z \in B$ such that $\rho(c(T^n x - z)) \to 0$ as $n \to \infty$.

We now prove that z is a fixed point of T. For all $n \in \mathbb{N}$,

$$\begin{aligned}\psi(\rho(\frac{c(Tz-z)}{2})) &\le \psi(\rho(c(Tz - T^n x))) + \psi(\rho(c(T^n x - z))\\ &\le k\psi(\rho(l(T^{n-1}x - z)) + \psi(\rho(c(T^n x - z))),\end{aligned}$$

as $n \longrightarrow \infty$ we have $\psi(\rho(\frac{c(Tz-z)}{2})) = 0$. The uniqueness of z is easily proved, if z and w are two fixed points, then

$\psi(\rho(c(z-w))) = \psi(\rho(c(Tz-Tw))) \le k\psi(\rho(c(z-w)))$, and this implies that $\psi(\rho(c(z-w))) = 0$. Thus $z = w$ and this completes the proof. □ Now, we consider the following inequality,

$$\psi(\rho(Tx - Ty)) \le k\psi(m(x,y)), \tag{3.3}$$

where $m(x,y) = \max\{\rho(x-y), \rho(x-Tx), \rho(y-Ty), \rho(x-Ty), \rho(y-Tx)\}$, $0 < k < 1$ and ψ satisfies the remark **??**.

Recall the self-mapping T on X_ρ is a ρ-asymptotically regular, if for all $x \in X_\rho$

$$\rho(T^{n+1}x - T^n x) \to 0,$$

as $n \to \infty$. Next, we give a fixed point theorem by last inequality.

Theorem 3.5. *Let X_ρ be a ρ-complete modular space which ρ satisfies the Δ_2-condition and B is a ρ-closed subset of X_ρ. Let T is a decreasing and ρ-asymptotically self-mapping on B which satisfies in (* **??** *). Then T has a unique fixed point.*

Proof . We consider the sequence $\{T^n(x)\} \subseteq B$. $\{T^n(x)\}$ is ρ-Cauchy sequence.

$$\begin{aligned}\psi(\rho(T^{n+m}(x) - T^n(x))) \le\; & k\psi(\max\{\rho(T^{n+m-1}(x) - T^{n-1}(x)), \rho(T^{n+m}(x) - T^{n+m-1}(x)),\\ & \rho(T^n(x) - T^{n-1}(x)), \rho(T^{n+m}(x) - T^{n-1}(x)),\\ & \rho(T^n(y) - T^{n+m-1}(x))\}).\end{aligned}$$

Because of T is ρ-asymptotically regular mapping, then

$$\begin{aligned}\lim(\psi(\rho(T^{n+m}(x) - T^n(x)))) &\le \lim k\psi(\rho(T^{n+m-1}(x) - T^{n-1}(x)))\\ &\le \ldots\\ &\le \lim k^m\psi(\rho(T^n(x) - x)) = 0.\end{aligned}$$

Then there exists $z \in B$ such that $\rho(T^n(x) - z) \to 0$. By using a similar technique as in the proof of theorem (**??**), z is a fixed point of T. Now, let z, w are fixed points of T.

$$\begin{aligned} \psi(\rho(z-w)) &= \psi(\rho(T(z) - T(w)) \\ &\leq k\psi(\rho(z-w)). \end{aligned}$$

Hence $z = w$ and so T has a unique fixed point. □

4. Center and ρ-J-type Mappings

We begin this chapter by providing a definition of center point in modular space.

Definition 4.1. *Let X_ρ be a modular space and B be a ρ-closed and ρ-bounded subset of X_ρ . A $y_0 \in X_\rho$ is called a center for mapping $T : B \to X_\rho$ if*

$$\rho(T(x) - y_0) \leq \rho(x - y_0) \qquad (x \in B).$$

Recall the definition of ρ-continuous mappings, that is a mapping $T : X_\rho \to X_\rho$ is said to be ρ-continuous if, $\rho(T(x_n) - T(x)) \to 0$ whenever $\rho(x_n - x) \to 0$.

Definition 4.2. *The mapping $T : B \to X_\rho$ is called ρ-J-type mapping whenever, it is ρ-continuous and it has some center $y_0 \in X_\rho$.*

We denote by $Z_\rho(T)$ the set of all centers of mapping T. It is obvious that if the center $y_0 \in B$, then y_0 is a fixed point of T. Several immediate consequences of the above definitions are considered the follow.

Remark 4.3.
- *(i) If a mapping $T : B \to B$ has a center $y_0 \in X_\rho$, then $T^n : B \to B$ has the same center y_0 for all $n \in \mathbb{N}$,*
- *(ii) If a mapping $T : B \to X_\rho$ has a center $y_0 \in X_\rho$, then the restriction of T to every subset of B does too,*
- *(iii) If a mapping $T : B \to X_\rho$ has a center $y_0 \in X_\rho$, then the translation mapping $\tilde{T} : B - \{y_0\} \to X_\rho$ with $\tilde{T}(x - y_0) = T(x) - y_0$ has the center 0_{X_ρ},*
- *(iv) Every fixed point of a ρ-altering mapping $T : B \to X_\rho$ is a center for this mapping.*

Note that it is not necessary $Z_\rho(T)$ be a ρ-closed subset of X_ρ. Next, we consider extra condition on ρ that implies $Z_\rho(T)$ is closed.

Remark 4.4. *Let (X_ρ, ρ) be a modular space such that $\rho(4x) = \rho(x)$ for all $x \in X_\rho$. Then $Z_\rho(T)$ is a ρ-closed subset of X_ρ.*

As we pointed out in the abstract, now we introduced the second class of mappings as follow.

Definition 4.5. *The mapping $T : B \to X_\rho$ is called ρ-altering J-type mapping if there exist $0 < k < 1$ and $y_0 \in X_\rho$ such that*

$$\psi(\rho(T(x) - y_0)) \leq k\psi(\rho(x - y_0)) \qquad (x, y \in X_\rho).$$

where B is ρ-closed subset of X_ρ and ψ satisfies in the remark **??**.

There is no clear relationship between ρ-altering and ρ-altering J-type mappings. To illustrate, let us state the following example.

Example 4.6. *Let $X_\rho = \mathbb{R}$ denote the set of all real numbers. Then:*

(1) Let $T : [0,1] \to X_\rho$ be a constant function $T(x) = 2$. Then T is ρ-altering and ρ-altering J-type mapping.
(2) Suppose that $T : [0,1] \to X_\rho$ is a constant function $T(x) = \frac{1}{2}$. Then T is ρ-altering mapping but it is not ρ-altering J-type mapping.
(3) Let $T : [0,1] \to X_\rho$ is giving by $T(x) = 2x$, it is easy to check that T is ρ-altering J-type but it is not ρ-altering mapping.
(4) Let $T : [0,1] \to X_\rho$ is giving by $T(x) = x+1$, it is not ρ-altering mapping and neither is ρ-altering J-type mapping.

The next theorem is inspired theorem (**??**).

Theorem 4.7. *Suppose that X_ρ and B satisfy the conditions of theorem (**??**) and ρ has the Δ_2-condition. If $T : B \to B$ is a ρ-altering J-type mapping, then T has a fixed point.*

Proof . Let $\{T^n(x)\}$ be a sequence in B. We have

$$\begin{aligned}\psi(\rho(\frac{T^{n+m}(x) - T^n(x)}{2})) &\leq \psi(\rho(T^{n+m}(x) - y_0)) + \psi(\rho(T^m(x) - y_0)) \\ &\leq k[\psi(\rho(T^{n+m-1}(x) - y_0)) + \psi(\rho(T^{m-1}(x) - y_0))] \\ &\leq ... \\ &\leq k^m[\psi(\rho(T^n(x) - y_0)) + \psi(\rho(x - y_0))] \to 0. \end{aligned} \tag{4.1}$$

Then there exists $z_0 \in B$ such that $\rho(T^n(x) - z_0) \to 0$. Therefor

$$\begin{aligned}\psi(\rho(\frac{T(z_0) - z_0}{4})) &\leq (k+1)\psi(\rho(\frac{z_0 - y_0}{2})) \\ &\leq \psi(\rho(T^n(x) - z_0)) + \psi(\rho(T^n(x) - y_0))] \\ &\leq k^n\psi(\rho(x - y_0)) \to 0. \end{aligned} \tag{4.2}$$

This implies that $T(z_0) = z_0$ and it completes the proof. □

5. Application

Integral equations comprise a very important and significant part of mathematical analysis and their applications to real world problems. This topic is now well developed with the help of several tools such as fixed point theory. We give here an application of our theorem to prove the existence of a solution of an integral equation.

For all positive number $a > 0$, we consider the integral equation

$$u(t) = \int_{X_\rho} \Phi(t, Tu(t))d(t) + kt, \qquad (t \in I = [0,a]), \tag{5.1}$$

such that the following conditions hold:

(i) $\Phi : I \times B \to B$ is linear in the second variable such that B is subset of X_ρ,
(ii) $\rho(\int_{X_\rho} \Phi(t,b)d(t)) \leq \rho(b)$ *for all* $b \in B$.
(ii) $k \in B$

Theorem 5.1. *Let X_ρ, ρ and B satisfy the conditions of theorem (**??**) and $T : C(I,B) \to C(I,B)$ be a ρ-altering mapping. Then there exist a solution $u(t)$ of the integral equation (**??**).*

Proof . Let $H : C(I,B) \to C(I,B)$ be a function such that

$$Hu(t) = \int_{X_\rho} \Phi(t, Tu(t))d(t) + kt, \qquad (t \in [0,a]), \tag{5.2}$$

and Φ satisfies in conditions of (**??**). Thus we have

$$\begin{aligned} Hu(t) - Hv(t) &= \int_{X_\rho} \Phi(t, Tu(t))d(t) - \int_{X_\rho} \Phi(t, Tv(t))d(t) \\ &= \int_{X_\rho} \Phi(t, Tu(t) - Tv(t))d(t). \end{aligned}$$

By the condition (ii) of (**??**);

$$\begin{aligned} \rho(Hu(t) - Hv(t)) &= \rho(\int_{X_\rho} \Phi(t, Tu(t) - Tv(t))d(t) \\ &\leq \rho(Tu(t) - Tv(t)). \end{aligned}$$

In special case, let $\rho_\alpha(u) = \sup_{t\in I} e^{-at}\rho(u(t))$. Thus, from the above inequality $\rho_\alpha(Hu - Hv) \leq \rho_\alpha(Tu - Tv)$, and, since T is a ρ-altering mapping, then

$$\rho_\alpha(Hu - Hv) \leq \rho_\alpha(Tu - Tv) \leq k\psi(\rho_\alpha(u - v)).$$

Then H has a fixed point of $C(I,B)$ and so the equation (**??**) has a solution. □

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