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Ranking all units with non-radial models in DEA

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Abstract

Data envelopment analysis (DEA) helps the managers to separate and classify the efficient and inefficient units in a homogenous group. DEA is a set of methods inferred from mathematics and other sciences in which the branch of unit ranking can be significantly effective in improving managerial decisions. Although this branch in DEA is considered still young, it has proved its ability in solving some problems like production planning, resource allocation, inventory control, etc. The managers who care about their results quality cannot be indifferent to units ranking. In this article, to rank the units which are under-evaluated, firstly the decision-making unit (DMU) is removed from the production possibility set (PPS), and then the new PPS is produced. The unit under evaluation is inside or outside of the new PPS. Therefore, to benchmark the under-evaluation DMU to new frontiers, two models are solved. If the removed unit is outside of the new PPS, the first model is feasible, and the second model is infeasible. If the removed unit is inside or on the frontier of the new PPS, both models are feasible. The method presented in this article for ranking the under-evaluation units has these characteristics: 1- this model can distinguish extreme and non-extreme efficient units and inefficient units. 2- Also, the presented models for ranking DMUs can be changed into a linear model. 3- This method shows stability in changing small or near-zero data. 4- It does not assign a false ranking. The presented methods in this article are able to distinguish the set of extreme and non-extreme efficient and inefficient units as well as being able to overcome the common problems in ranking. In this article, suggested models are introduced in 3.1 which are able to rank all under evaluation units except non-extreme efficient units, this problem is solved in 3.2, in other words in 3.2 all DMUs are ranked

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1. Introduction

Economical resources are various examples of work, capital, etc., which are applied in producing goods and services. Resource limitations made the managers find a method for optimum use of

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these factors. In this regard, one of the important factors is efficiency assessment. To obtain an efficiency assessment, the production function is required. DEA is a nonparametric method which identifies an envelopment approach using some observation. The shape of this approach is named experimental production function and this envelopment approach is named efficient frontier. DEA was first based in the article CCR by Charnes- Cooper and Roads [1, 2]. They generalized Farrels [3] primary analysis, which was in one output-multi input mode to multi input-multi output mode. Then Charnes- Cooper and Banker [4] were able to establish the model BCC by recognizing the return to scale method and modifying the CCR model. Having efficiency evaluation models developed and managers need to distinguish efficient units increased, ranking models were formed. Evaluation and assessment of the performance level of people and efficient units are a subject that has been paying attention since many years ago. Resource limitations and unlimited needs and wills have made a human being plan and manage resources in order to succeed in affairs; since human beings want to assure that they achieve their maximum results and goals of available resources. There is at least one efficient unit among the units and its efficiency score by data envelopment analysis equals 1. Now this question is raised that if there are several DMUs whose efficiencies are 1 (100% efficient), which unit performs better? In other words, which unit is better among the efficient units and how can the efficient units be ranked? Different methods have been presented for efficient units each of which uses a particular character as a criterion for ranking, for instance the following models can be mentioned:

In (1986) Sexton et al. Suggested [5] method. This method computes the efficiency index of DMUs for n times and summarizes the related results of cross-efficiency index of all DMUs in a matrix by the achieved weights of solving each problem. Each row of this matrix has cross-efficiency index of a DMU. Sexton et al suggested an average of efficiency index of each DMU as the efficiency ranking. Although performing this method seems to be simple, it may face major problems in practice. The biggest problem of this method is when the DEA models have alternative optimal solution. It is necessary to notice the point that some strategies and techniques (whether efficient or inefficient) are effective in(1993) another method for ranking the efficient units was suggested by Andersen and Petersen [6] which evaluated the efficient units by comparing the under-evaluation unit with a linear composition of other units (except the under-evaluation unit). This method is known as AP model. In Mehrabian [7] et al presented a model to tackle the problems of AP which is known as MAJ model. Also, in Saati [8] et al presented modified-MAJ model to tackle the infeasibility problem of MAJ. In (2002) another model was introduced by Jahanshahloo [9] et al to tackle AP and MAJ problems. It is known as ranking by using the norm one. Hosseinzadeh-lotfi and jahanshahloo [9-13] et al proposed several methods for ranking the units under evaluation. Memariani [7, 8] et al They defined DEA ranking by modifying linear models of DMUs by eliminating the data column program in the matrix.Therefore, many different methods have been presented for ranking. In this section, first we have an overview of the method S-SBM and SBM, which were introduced by Tone [14] to evaluate and rank efficient units. Also, there is a brief look at J-SBM which was suggested by Chien-Ming Chen[15].Then, the suggested model is presented. As you know, the models suggested by Charnes et al were named radial models. Although these models have many advantages in analyzing DMUs, they have two major disadvantages. Firstly, some DMUs may be recognized as low efficient, which is because of the presence of positive quantitative variables. Secondly, many of DMUs may be recognized as efficient. Subsequently, differentiating efficient units cannot be done using classical DEA models. A solution for solving the second problem is using the super-efficient model; however, this model still has the first problem. To solve the first problem, a model which is based on helping variables was presented. Tone [14] presented the S-SBM to solve the second problem. This model is a super-efficient model. In super-efficient models, the under-evaluation DMU is removed from (PPS) and the efficiency level of DMU is calculated in the new (PPS). SBM characteristic is that its efficiency level is a function of helping input and output variables. One of the significant advantages of this model is that the model SBM identifies the Pareto efficiency of a reference point for the under-evaluation DMU. When the models SBM and S-SBM are implemented together, different issues happen: firstly, on the contrary, to standard super-efficient model, S-SBM can be just used for super-efficient scores, but not for SBM scores. Secondly, the model S-SBM may result in reference points with low efficiency. Thirdly, the scores of SBM and S-SBM for same DMUs may be discontinuous with perturbation to their inputs and outputs. The S-SBM model may overestimate super-efficiency score. In other words, it may give a false ranking. For more information, see the Joint-SBM, which was presented by Chein-Ming Chen [15]. The article is designed as follow, in section 2 required models are introduced. In section 3 the suggested models are introduced in the other words, in 3.1 the suggested model which is able to rank all units except non-extreme efficient units. In 3.2 the inability of the model in 3.1 is solved in other words, the suggested is able to rank all units including non-extreme efficient units. It is necessary to mention the suggest models is less complicated than the model in section 3.1 and J-SBM model. Section 4 analyses the results of section 3.1 and 3.2 by solving 2 numerical examples. Section 5 deals with some conclusion and judgment.

2. Preliminaries

Consider n number of $DMUj$ with coordinate (x_i, y_j), $j = 1, \cdots, n$.
Consider the DMUs PPS is as No.1

$$T_v = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \sum_{j=1}^{n} \lambda_j x_j \leq x, \quad \sum_{j=1}^{n} \lambda_j y_j \geq y, \quad \sum_{j=1}^{n} \lambda_j = 1, \quad \lambda_j \geq 0 \quad j = 1, \cdots, n \right\} \tag{2.1}$$

which was first introduced by Banker et al.

2.1. Consider the additive model [16].

$$\begin{aligned} \textbf{model (1)} \quad & z_p^* = \max \left(\sum_{i=1}^{m} s_i^- + \sum_{r=1}^{s} s_i^- \right) \\ & \sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ip} \quad i = 1, \cdots, m \\ & \sum_{j=1}^{n} \lambda_j y_{rj} + s_r^+ = y_{rp} \quad r = 1, \cdots, s \\ & \sum_{j=1}^{n} \lambda_j = 1 \\ & \lambda_j, s_i^-, s_r^+ \geq 0, \quad j = 1, \cdots, m, \quad r = 1, \cdots, s \end{aligned} \tag{2.2}$$

Definition 2.1. *Pareto efficiency SBM efficiency DMU is Pareto efficient if and only if we have model* (2.2) *in optimality [14]*

$$s_i^{-*} = 0, \quad i = 1, \cdots, m \tag{2.3}$$

$$s_r^{+*} = 0, \quad i = 1, \cdots, s \tag{2.4}$$

Definition 2.2. *DMUP is Pareto efficient in SBM if and only if* $z_p^* = 0$
To rank DMU_p $p \in \{1, \cdots, n\}$ *first it is omitted from the observation there for* T_v *is defined as below*

$$T_v' = \left\{ (x, y) \middle| \sum_{j \neq p} \lambda_j x_j \leq x, \middle| \sum_{j \neq p} \lambda_j y_j \geq y \sum_{j \neq p} \lambda_j = 1, \quad \lambda_j \geq 0, \quad j \neq p \right\} \tag{2.5}$$

Definition 2.3. DMU_p *is an extreme efficient unit if only if* $DMU_p \notin T_v'$.

3. Proposed model

3.1. Ranking all units except extreme efficient unit

To imagine the DMU on PS fthe Prontier, we have to consider two modes below, whether the under-evaluation unit is inside or outside of PPS after removing from DMU and drawing the new PPS creates two different modes. Mode 1- We have to reduce the inputs to the Pthunit and enhance the outputs of Pth unit. This mode occurs when the under-evaluation unit is inside PPS. In other words, this region is a set of points which are overcome by that under-evaluation unit. Mode 2- We have to enhance the inputs to the Pth unit and reduce the outputs of Pth unit. This mode occurs when the under-evaluation unit is outside PPS. In other words, this region is a set of points which dominate all the points in the region DMU_p after removing DMU_p and re-drawing the region by technological principles. Regarding the SBM model with free slacks, the necessary and sufficient condition to benchmark the under-evaluation unit on the frontier is that the production of slacks is non-negative. Consequently, we can whether to reduce inputs or enhance outputs (from the inside toward the frontier) or whether increase inputs or reduce outputs (from outside toward the frontier). We designed (3.1) with free slacks by this model.

$$\begin{aligned}
\textbf{model (2)} \quad & z = \min \frac{1 - \frac{1}{m} \sum_{i=1}^{m} \frac{s_i^-}{x_{ip}}}{1 + \frac{1}{s} \sum_{r=1}^{s} \frac{s_r^+}{y_{rp}}} \\
& \sum_{j \neq p} \lambda_j x_{ij} = x_{ip} - s_i^- \quad i = 1, \cdots, m \\
& \sum_{j \neq p} \lambda_j y_{rj} = y_{rp} + s_r^+ \quad r = 1, \cdots, s \\
& \sum_{j \neq p} \lambda_j = 1 \qquad\qquad j = 1, \cdots, n, \;\; j \neq p \\
& \lambda_j \geq 0, \quad j = 1, \cdots, n, \quad s_i^-, s_r^+ \text{ unristricted} \quad i = 1, \cdots, m, \quad r = 1, \cdots, s
\end{aligned} \tag{3.1}$$

Now to imagine the benchmark on technology frontier it is needed to:

Case1) If the first mode, this mode occurs when under-evaluation unit is outside or on the frontier of PPS. In other words, this unit is efficient. Obviously, if it is a non-extreme efficient unit, the feasible region does not change and if it is an extreme efficient unit, the input number must be enhanced, and the outputs must be reduced to arrive to the frontier. We achieve the model (3.1). The model (3.1) has been created as below. In the model (3.6)

$$s_i \leq 0, \quad s_r \leq 0 \quad \forall\, i, r \tag{3.2}$$

Case 2) The input numbers of Pth unit is reduced (reduced or stayed unchanged) and the output number of Pth unit is enhanced (enhanced or stayed unchanged). This mode occurs when the under-evaluation unit is inside or on the weak frontier of technology so

$$s_i \geq 0, \quad s_r \geq 0 \quad \forall\, i, r \tag{3.3}$$

As it is noted, in the last constraint in the model (3.7), $s_i^- \times s_r^+ \geq 0$ is seen. When the product is non-negative, then model (3.1) is changed to model (3.5)

$$s_i \times s_r \geq 0 \Rightarrow \begin{cases} case1. & \Rightarrow \quad s_i \leq 0, \;\; s_r \leq 0 \\ case2. & \Rightarrow \quad s_i \geq 0, \;\; s_r \geq 0 \end{cases} \tag{3.4}$$

$$\begin{aligned} \textbf{model (3)} \quad & \rho_p^* = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{ip}}}{1 + \frac{1}{S}\sum_{r=1}^{s} \frac{s_r^+}{y_{rp}}} \\ & \sum_{j \neq p} \lambda_j x_{ij} + s_i^- = x_{ip} \\ & \sum_{j \neq p} \lambda_j y_{rj} - s_r^+ = y_{rp} \\ & \sum_{j \neq p} \lambda_j = 1 \\ & s_i^- \times s_r^+ \geq 0 \quad i = 1, \cdots, m, \quad r = 1, \cdots, s \end{aligned} \tag{3.5}$$

Two modes can be expressed geometrically as below

Case1) If the first mode, this mode occurs when under-evaluation unit is outside or on the frontier of PPS. In other words, this unit is efficient. Obviously, if it is a non-extreme efficient unit, the feasible region does not change and if it is an extreme efficient unit, the input number must be enhanced, and the outputs must be reduced to arrive to the frontier. We achieve the model (3.6). The model (3.6) has been created as below. In the model (3.6), $s_i \leq 0, \; s_r \leq 0, \quad \forall\, i, r$
Defining $\tilde{s}_i = -s_i$ and $\tilde{s}_r = -s$ variable changing occurs which can be seen, as model (3.6)

$$\begin{aligned} \textbf{model (4)} \quad & \rho^* = \min \frac{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{\tilde{s}_i^-}{x_{ip}}}{1 - \frac{1}{S}\sum_{r=1}^{s} \frac{\tilde{s}_r^+}{y_{rp}}} \\ & \sum_{j \neq p} \lambda_j x_{ij} = x_{ip} + \tilde{s}_i^- \quad \forall i \\ & \sum_{j \neq p} \lambda_j y_{rj} = y_{rp} - \tilde{s}_r^+ \quad \forall r \\ & \sum_{j \neq p} \lambda_j = 1 \qquad\qquad j = 1, \cdots, n, \;\; j \neq p \\ & \lambda_j, \tilde{s}_i^-, \tilde{s}_r^+ \geq 0 \quad \forall\, i, r, j \end{aligned} \tag{3.6}$$

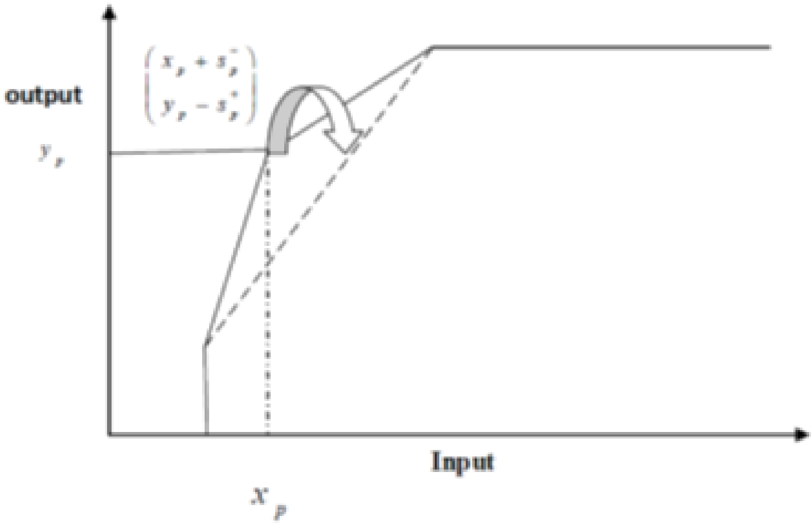

Figure 1: for case (1).

Therefore, the model (3.5) is changed into the model (3.6) using this change of variable. To have a new benchmark on the frontier, we modify the model (3.6) and achieve the model (3.7).

$$
\begin{aligned}
\textbf{model (5)} \quad & \rho_2^* = \min \frac{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{\tilde{s}_i^-}{x_{ip}}}{1 - \frac{1}{S}\sum_{r=1}^{s} \frac{\tilde{s}_r^+}{y_{rp}}} \\
& \sum_{j \neq p} \lambda_j x_{ij} \leq x_{ip} + \tilde{s}_i^- \quad \forall i \\
& \sum_{j \neq p} \lambda_j y_{rj} \geq y_{rp} - \tilde{s}_r^+ \quad \forall r \\
& \sum_{j \neq p} \lambda_j = 1 \\
& \lambda_j, \tilde{s}_i^-, \tilde{s}_r^+ \geq 0 \quad \forall\, i, r, j
\end{aligned} \tag{3.7}
$$

Case2) If the second mode occurs, this mode occurs when under-evaluation unit is inside PPS. In other words, this unit is inefficient and the model (3.5) changes and becomes the same with SBM as below. The second case in model (3.5) is changed to model (3.8)

$$
\begin{aligned}
\textbf{model (6)}\quad & \rho_p^* = \min \frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{ip}}}{1 + \frac{1}{S}\sum_{r=1}^{s} \frac{s_r^+}{y_{rp}}} \\
& \sum_{j \neq p} \lambda_j x_{ij} + s_i^- = x_{ip} \\
& \sum_{j \neq p} \lambda_j y_{rj} - s_r^+ = y_{rp} \\
& \sum_{j \neq p} \lambda_j = 1 \\
& s_i^- \geq 0,\ s_r^+ \geq 0 \quad i = 1, \cdots, m, \quad r = 1, \cdots, s
\end{aligned}
\tag{3.8}
$$

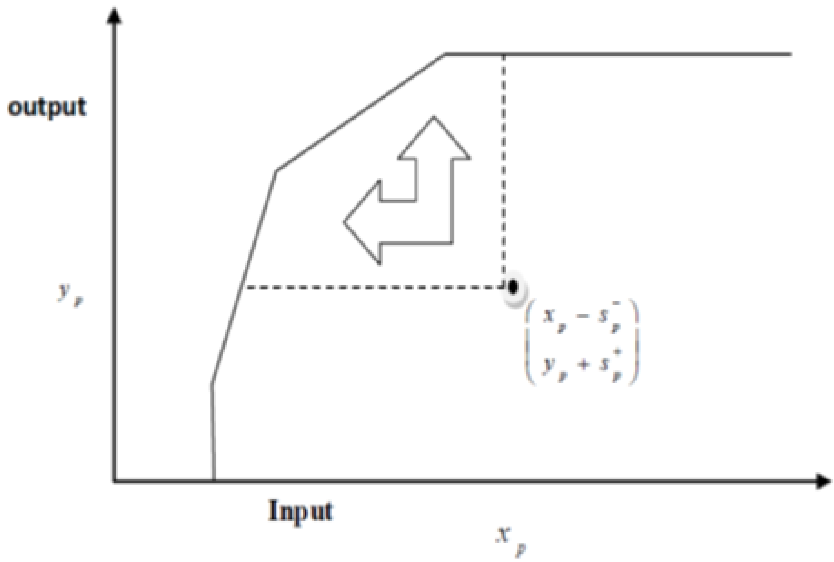

Figure 2: for case (2).

Now consider the suggested model which is achieved out of the models (3.7) and (3.8)

$$\begin{aligned}
\textbf{model (5)} \quad & \rho_2^* = \min \frac{1+\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{ip}}}{1-\frac{1}{S}\sum_{r=1}^{s}\frac{s_r^+}{y_{rp}}} \\
& \sum_{j\neq p}\lambda_j x_{ij} \le x_{ip}+s_i^- \\
& \sum_{j\neq p}\lambda_j y_{rj} \ge y_{rp}-s_r^+ \\
& \sum_{j\neq p}\lambda_j = 1 \\
& \lambda_j, s_i^-, s_r^+ \ge 0 \quad \forall\, i,r,j
\end{aligned} \tag{3.9}$$

$$\begin{aligned}
\textbf{model (6)} \quad & \rho_1^* = \min \frac{1-\frac{1}{m}\sum_{i=1}^{m}\frac{t_i^-}{x_{ip}}}{1+\frac{1}{S}\sum_{r=1}^{s}\frac{t_r^+}{y_{rp}}} \\
& \sum_{j\neq p}\lambda_j x_{ij} = x_{ip}-t_i^- \quad \forall\, i \\
& \sum_{j\neq p}\lambda_j y_{rj} = y_{rp}+t_r^+ \quad \forall\, i \\
& \sum_{j\neq p}\lambda_j = 1 \\
& \lambda_j, t_i^-, t_r^+ \ge 0 \quad \forall i,r,j
\end{aligned} \tag{3.10}$$

Theorem 3.1. *the model* (3.9) *is always feasible.*

Proof . suppse $(x_q, y_p) \in T$ we put $q \neq p$ and considering s_i^- and s_r^+ we define as we below ($s_i^- = \max\{x_{iq} - x_{ip}, 0\} \quad \forall\ i, s_r^+ = \max\{y_{rp} - y_{rq}, 0\} \quad \forall\ r$). Therefore, the model (3.9) is always feasible. □

Theorem 3.2. *DMU_p is extreme if only if $\rho_2^* > 1$.*
Suppose DMU_p is extreme-efficient unit, we show $\rho_2^* > 1$. Proved by contradiction. suppose $\rho_2^* \le 1$ therefore

$$\begin{aligned}
& \rho_2^* = \frac{1+\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{ip}}}{1-\frac{1}{S}\sum_{r=1}^{s}\frac{s_r^+}{y_{rp}}} \le 1 \Rightarrow 1+\frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{ip}} \le 1-\frac{1}{S}\sum_{r=1}^{s}\frac{s_r^+}{y_{rp}} \\
& \Rightarrow \frac{1}{m}\sum_{i=1}^{m}\frac{s_i^-}{x_{ip}}+\frac{1}{S}\sum_{r=1}^{s}\frac{s_r^+}{y_{rp}} \le 0 \Rightarrow (s_i^{-*}, s_r^{+*}) = (0,0) \\
& \Rightarrow \sum_{j\neq p}\lambda_j x_j \le x_p, \quad \sum_{j\neq p}\lambda_j y_j \ge y_p \Rightarrow (x_p, y_p) \in T_v'
\end{aligned} \tag{3.11}$$

Which contradicts definition No3. The condition is sufficient, suppose that $\rho_2^* > 1$ we show that DMU_p is efficient. Using the supposition $\rho_2^* > 1$ we show that DMU_p is an extreme efficient unit. Proof by contradiction. Suppose that DMU_p is not a non-extreme efficient unit. Two case can be seen

Case1) *DMU_p is inefficient. Therefore, producing $(\bar{x}, \bar{y})$ is possible in which:*

$$\exists\,(\bar{x},\bar{y}) \in T_v;\quad \begin{pmatrix} -\bar{x} \\ \bar{y} \end{pmatrix} \gneqq \begin{pmatrix} -x_p \\ y_p \end{pmatrix} \Rightarrow \exists\,\bar{\lambda} \geq 0,\quad \left(\bar{x} = \textstyle\sum_{j=1}^{n} \bar{\lambda}_j x_j \leq x_p + 0,\ \ \bar{y} = \sum_{j=1}^{n} \bar{\lambda}_j = 1\right) \tag{3.12}$$

Therefore $(\bar{\lambda}, s^- = 0,\ s^+ = 0)$ is a feasible answer for the model (3.9). $\bar{\rho}_2 = 1 < \rho_2^*$ which is a contradiction

Case 2) *DMU_p is an extreme efficient unit.*

$$\exists\ \bar{\lambda} \geq 0,\quad \bar{\lambda}_p = 0,\quad \bar{x} = \sum_{j=1}^{n} \bar{\lambda}_j y_j \geq y_p - 0,\quad \sum_{j=1}^{n} \bar{\lambda}_j = 1 \tag{3.13}$$

Therefore $(\bar{\lambda}, s^- = 0, s^+ = 0)$ is a feasible answer for the model (3.9). $\bar{\rho}_2 = 1 < \rho_2^*$ which is a contradiction.

Theorem 3.3. *$\rho_2^* > 1$ if only if the model (3.10) is infeasible.*
the condition is necessary .in other words if $\rho_2^* > 1$ we show the model (3.10) is infeasible. proof by contradiction: suppose that $(\bar{\lambda}, \bar{t_i}, \bar{t_r})$ is feasible solution for model (3.10) therefore

$$\left\{\left(\textstyle\sum_{j\neq p} \bar{\lambda}_j x_{ij} = x_{ip} - t_i^- \leq x_{ip},\ \ \forall\, i\right), \left(\sum_{j\neq p} \bar{\lambda}_j y_{rj} = y_{rp} + t_r^+ \geq x_{rp},\ \ \forall\, r\right), \left(\sum_{j\neq p} \bar{\lambda}_j = 1\right)\right\} \tag{3.14}$$

Therefore $(\bar{\lambda}, s_i^- = 0, s_r^+ = 0)$ is the feasible solution to model (3.9) and consequently $\bar{\rho}_2 = 1$ that is a contradiction to optimality. Therefore $(\bar{\lambda}, s_i^- = 0, s_r^+ = 0)$ is a feasible solution for the model (3.9) $\bar{\rho}_2 > 1$ which is a contradiction.
The sentence $\bar{\rho}_2^* > 1$. Proof by contradiction that $\bar{\rho}_2^* \leq 1$ therefore

$$\begin{aligned} &\frac{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{ip}}}{1 - \frac{1}{S}\sum_{r=1}^{s} \frac{s_r^+}{y_{rp}}} \leq 1 \Rightarrow s_i^{*-} = 0,\quad s_i^{*+} = 0 \quad \forall\, i, r \\ &\Rightarrow \left(\textstyle\sum_{j\neq p} \lambda_j^* x_{ij} \leq x_{ip},\ \ \forall\, i,\quad \sum_{j\neq p} \lambda_j^* y_{rj} \geq y_{rp}\ \ \forall\, r\right) \end{aligned} \tag{3.15}$$

$$\Rightarrow \left(t_i^- = x_{ip} - \sum_{j\neq p} \lambda_j^* x_{ij} \leq x_{ip} \quad \forall\, i, t_r^+ = y_{rp} - \sum_{j\neq p} \lambda_j^* y_{rj} \geq y_{rp} \quad \forall\, r\right) \tag{3.16}$$

as a result, $(\lambda^*, t_i^-, t_r^+)$ is the feasible solution to model (3.10) and this contradictory to the feasibility of model (3.10).

Theorem 3.4. *$\rho_2^* = \rho_1^* = 1$ if only if DMU_p is a non-extreme efficiency.*

Proof . Proof by contradiction, suppose it is not a non- extreme efficiency, then we have two modes below.

Case1) DMU_p is an extreme efficiency, therefore, according to the theorem 3.2 we have $\rho_2^* > 1$ and this is a contradiction.

Case 2) Dumps is inefficient,

$$\exists \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} \in T_V; \quad \begin{bmatrix} -\bar{x} \\ \bar{y} \end{bmatrix} \gneqq, \begin{bmatrix} -x_p \\ y_p \end{bmatrix} \in T_V \Rightarrow \left\{ \exists\, \bar{\lambda} \geq 0 \middle| \sum_{j \neq p} \bar{\lambda}_j x_j \leq x_p,\ \sum_{j \neq p} \bar{\lambda}_j y_j \geq y_{rp},\ \sum_{j \neq p} \bar{\lambda}_j = 1 \right\} \tag{3.17}$$

In other side, because DMU_p is inefficient, then we put $\bar{\lambda}_p = 0$. So we put $(\lambda^*, \bar{t_i}, \bar{t_r})$ as the feasible solution for the model (3.10) therefore, we have $\bar{\rho}_1 < 1$ that is a contradiction to optimality $\rho_1^* = 1$.
Inverse proof: the proposition is that DMU_p is non-extreme efficiency. We show $\rho_1^* = \rho_2^* = 1$.
Proof by contradiction: $\rho_1^* \neq 1 \vee \rho_2^* \neq 1$
If $\rho_1^* > 1$ then the model (3.9) is not feasible which is in contradiction to the proposition the theorem
If $\rho_1^* < 1$ so we have

$$\frac{1 - \frac{1}{m}\sum_{i=1}^{m} \frac{t_i^-}{x_{ip}}}{1 + \frac{1}{S}\sum_{r=1}^{s} \frac{t_r^+}{y_{rp}}} < 1 \Rightarrow (t_i^{-*}, t_r^{+*}) \neq 0 \Rightarrow \left\{ \begin{array}{ll} \sum_{j \neq p} \lambda_j^* x_{ij} = x_{ip} - t_i^{-*} \leq x_{ip} & \forall\, i \\ \sum_{j \neq p} \lambda_j^* y_{rj} = y_{rp} + t_r^{+*} \leq y_{rp} & \forall\, r \end{array} \right. \tag{3.18}$$

We put $\bar{\lambda} = (\lambda_1^*, \cdots, \lambda_{p-1}^*, 0, \lambda_{p+1}^*, \cdots, \lambda_n^*)$ therefore

$$\left(\bar{x}_i = x_{ip} - t_i^{*-} \lneqq x_{ip} \quad \forall\, i,\ \bar{y}_r = y_{rp} + t_r^{*+} \gneqq y_{rp} \ \ \forall\, r \right)$$

which is in contradiction to the pareto-efficiency of DMU_p. $\square$
Regarding the subject and theorems above ,if DMU_p is an efficient ,then its ranking is ρ_2^* and if DMU_p is an inefficient ,then its ranking is ρ_1^*. The suggestion models in section 3.1 are able to rank all the units except non-extreme efficient DMUs . in order to rank all DMUs models and solving the mentioned problem.

3.2. Full ranking

In this section, a method is represented which ranks the whole DMUs including extreme and non-extreme efficient units and inefficient units. The units which are used to rank the units are non-radial. Therefore, the common problems seen in ranking including infeasibility and instability and false ranking are not seen in these models. Also, this model can rank the non-extreme efficient units. Regarding.

$$\begin{aligned} \text{The ranking of extreme efficient units} &> \text{the ranking of non-extreme efficient units} \\ &> \text{the ranking of inefficient units} \end{aligned} \tag{3.19}$$

The sign " > " is used here as being better. It is supposed that the rank of extreme efficient DMUs is better than non-extreme efficient DMUs and then non-extreme efficient DMUs is better than inefficient DMUs all units are ranked.

Firstly, the model (3.10) is solved for all the models to distinguish the efficient and inefficient units. If the DMU_p is inefficient, then $\rho_1^* < 1$. So, for inefficient units, the achieved efficiency amount shows the ranking. In other words, if the DMU_j is inefficient, the ρ_j^* is the DMU_j ranking. Consequently, solving the model (3.10), the inefficient models are ranked, and the efficient models are distinguished. If the DMU is efficient, $\rho_j = 1$ the set W is defined

$$w = \left\{ |\rho_j^* = 1 \right\}. \tag{3.20}$$

Initial stage

The initial stage of the algorithm is considered for ranking the extreme efficient units. The set W includes all the efficient units, either extreme or non-extreme. Therefore, to rank the extreme efficient units, the model ρ^0 is solved as below for all the members of W. Therefore, suppose that $p \in w$. Therefore, the model (3.9) is solved for efficient unit DMU_p.

$$\begin{aligned}
&\rho^0 = \min \frac{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{ip}}}{1 - \frac{1}{S}\sum_{r=1}^{s} \frac{s_r^+}{y_{rp}}} \\
&\sum_{j\neq p} \lambda_j x_{ij} \leq x_{ip} + s_i^- \quad i = 1, \cdots, m \\
&\sum_{j\neq p} \lambda_j y_{rj} \geq y_{rp} - s_r^+ \quad r = 1, \cdots, s \\
&\sum_{j\neq p} \lambda_j = 1 \\
&\lambda_j, s_i^-, s_r^+ \geq 0, \quad i = 1, \cdots, m, \quad j = 1, \cdots, n,\ j \neq p \quad r = 1, \cdots, s
\end{aligned} \tag{3.21}$$

If the DMU_p is an extreme efficient, then it is placed outside of T_{new} after being be removed from observations, it would $\rho_p^0 > 1$ in this case and if the DMU_p in inefficient or non-extreme efficient, then $\rho_p^0 = 1$ consider the set E^0 as number (3.22) .

$$E^0 = \left\{ j | \rho_j^0 > 1 \right\}. \tag{3.22}$$

So, for each $j \in E^0$, ρ_j^0 shows the rank of DMU_j. In other words, solving the model (3.9) for efficient units, the extreme models are ranked.

The first stage of algorithm

The set E^1 is defined as below

$$E^1 = \{1, 2, \cdots, n\} - \left\{ j \in w | \rho_j^0 > 1 \right\} = \{1, \cdots, n\} - E^0 \tag{3.23}$$

T_v^1 is formed based on E^1 observation as seen in number (3.18).

$$T_v^1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \middle| \sum_{j\in E^1} \lambda_j x_j \leq x, \ \sum_{j\in E^1} \lambda_j y_j \geq y, \ \sum_{j\in E^1} \lambda_j = 1, \ \lambda_j \geq 0, \ j \in E^1 \right\} \tag{3.24}$$

The model ρ^1 is solved for the units $(E^1 \cap W)$ therefore, suppose $q \in (E^1 \cap W)$, card$(E^1 \cap W) \geq 2$ and as a result the ρ^1 is solved as in number (3.25).

$$\begin{aligned}
&\rho_q^1 = \min \frac{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{iq}}}{1 - \frac{1}{S}\sum_{r=1}^{s} \frac{s_r^+}{y_{rq}}} \\
&\sum_{(j \in E^1, j \neq q)} \lambda_j x_{ij} \leq x_{iq} + s_i^- \quad i = 1, \cdots, m \\
&\sum_{(j \in E^1, j \neq q)} \lambda_j y_{rj} \geq y_{rq} - s_r^+ \quad r = 1, \cdots, s \\
&\sum_{(j \in E^1, j \neq q)} \lambda_j = 1 \\
&\lambda_j, s_i^-, s_r^+ \geq 0, \quad i = 1, \cdots, m, \ \ r = 1, \cdots, s, \ \ j \in E^1, j \neq q
\end{aligned} \tag{3.25}$$

Consider a case that card $E^1 \cap W \leq 1$. This shows that $|E^1 \cap W| \leq 1$ which at most has one non-extreme efficient. Therefore, its ranking in the initial ranking is presented by P^0. In other words, $w - E^0$ has, at most, one efficient DMU whose ranking is determined by p^0 amount in the initial stage and the algorithm is finished in this stage and the ranking of all the units are determined. In the second repetition, the set E^2 is defined as below:

$$E^2 = \{1, \cdots, n\} - \{j \in w | \rho_j^0 > 1 \vee \rho_j^1 > 1\}$$

If card $|E^1 \cap W| \geq 2$ the new PPS is formed as seen in number (23) which is called T_v^2.

$$T_v^2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} | \sum_{j \in E^2} \lambda_j x_j \leq x, \ \ \sum_{j \in E^2} \lambda_j y_j \geq y, \ \ \sum_{j \in E^2} \lambda_j = 1, \ \ \lambda_j \geq 0, \ j \in E^2 \right\} \tag{3.26}$$

The model ρ^2 is solved to rank the units in the set $E^2 \cap W$. Therefore, suppose $k \in E^2 \cap W$.

$$\begin{aligned}
&\rho^2 = \min \frac{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{ik}}}{1 - \frac{1}{S}\sum_{r=1}^{s} \frac{s_r^+}{y_{rk}}} \\
&\sum_{(j \in E^2, j \neq k)} \lambda_j x_{ij} \leq x_{ik} + s_i^- \quad i = 1, \cdots, m \\
&\sum_{(j \in E^2, j \neq k)} \lambda_j y_{rj} \geq y_{rk} - s_r^+ \quad r = 1, \cdots, s \\
&\sum_{(j \in E^2, j \neq k)} \lambda_j = 1 \\
&\lambda_j, s_i^-, s_r^+ \geq 0, \quad i = 1, \cdots, m, \ \ r = 1, \cdots, s, \ \ j \in E^1, j \neq k
\end{aligned} \tag{3.27}$$

For each $j \in E^2 \cap W$, if $\rho_j^2 > 1$ then ρ_j^2 would be the DMU_j ranking in the set $E^2 \cap W$. Regarding $W \subseteq \{1, 2, \cdots, n\}$ therefore, the algorithm stops in a finite number of stages. Therefore, in the K^{th} repetition, the E^k set is defined as in number (3.22)

$$E^k = \{1, 2, \cdots, n\} - \{j \in w | \ \rho_j^0 > 1 \vee \rho_j^1 > 1 \vee \rho_j^2 > 1 \vee \cdots \vee \rho_j^{k-1} > 1\} \tag{3.28}$$

We rank the units in the set card$E^k \cap W$ using the model ρ^k. Therefore, the model ρ^k is solved as below for the model $p \in E^k \cap W$

Regarding the definition, No (3.29) is suggested.

$$\begin{aligned}
&\rho^k = \min \frac{1 + \frac{1}{m}\sum_{i=1}^{m} \frac{s_i^-}{x_{ip}}}{1 - \frac{1}{S}\sum_{r=1}^{s} \frac{s_r^+}{y_{rq}}} \\
&\sum_{(j\in E^k, j\neq p)} \lambda_j x_{ij} \leq x_{ip} + s_i^- \quad i = 1, \cdots, m, \quad k = 1, \cdots, t \\
&\sum_{(j\in E^k, j\neq p)} \lambda_j y_{rj} \geq y_{rp} - s_r^+ \quad r = 1, \cdots, s \\
&\sum_{(j\in E^k, j\neq p)} \lambda_j = 1 \\
&\lambda_j, s_i^-, s_r^+ \geq 0, \quad i = 1, \cdots, m, \quad r = 1, \cdots, s, \quad j \in E^k, j \neq p
\end{aligned} \tag{3.29}$$

Regarding the definition of E^k,

$$E^t \subseteq E^{t-1} \subseteq E^{t-2} \subseteq E^{t-3} \cdots \subseteq E^1 \tag{3.30}$$

Also we have rank $\left\{j \in w|\ \rho_j^k > 1\right\} <$ rank $\left\{j \in w|\ \rho_j^{k-1} > 1\right\}$, $k = t, t-1, t-2, \cdots, 2, 1$. In which, for $\rho_j^k > 1$ the ranking DMU_j is between the set

$$E^k \cap W. \tag{3.31}$$

4. Numerical example

We now provide 3 examples of this approach.

Example 4.1. *We consider ten DMUs with two inputs and one output. The data and adjust model* (3.10) *of the DMUs are shown in table 1.*

Table 1: We consider ten DMUs in a mode of 2 inputs and one output.

	A	B	C	D	E	F	G	H	K	P
I1	3	6	5	7	7	8	1	6	4	5
I2	2	3	0.5	1	0	0	5	3	1.25	2
O	1	1	1	1	1	1	1	1	1	1

According to the result of the SBM (MODEL6), DMU_A, DMU_C, DMU_E, DMU_G, and DMU_K, are efficient .
In table 2 for DMU_A, $\rho^*_{2A} > 1$ and the model (3.9) *are infeasible. this shows that DMU_A is placed out of the new PPS after being removed from production technology. therefore, this unit is pareto-efficient and ρ^*_{2A} shows the ranking. For DMU_B, $\rho^*_{2B} = 1$ and $\rho^*_{1B} < 1$ therefore, DMU_B is inefficient and ρ^*_{1B} shows the ranking. For DMU_C, $\rho^*_{2C} > 1$ and the model* (3.9) *are infeasible. This shows that DMU_C is placed outside of the new PPS after being removed from production technology. therefore, this units*

Table 2: We use the models (3.9) and (3.10) to rank these units. The results are as below.

	ρ_2^*	S_1^-	S_1^-	S^+	ρ_1^*	t_1^-	t_2^-	t_1^+
A	1.067	0.4	0	0	Inf	⋯	⋯	⋯
B	1	0	0	0	0.5	1	2.5	0
C	1.080	0.8	0	0	Inf	⋯	⋯	⋯
D	1	0	0	0	0.5	0	1	0
E	1.071	1	0	0	Inf	⋯	⋯	⋯
F	1	0	0	0	0.927	1	0	0
G	2	2	0	0	Inf	⋯	⋯	⋯
H	1	0	0	0	0.5	1	2.5	0
K	1	0	0	0	1	0	0	0
P	1	0	0	0	0.625	0	1.5	0

is pareto-efficient and ρ_{1C}^* *shows the ranking .For* DMU_D, $\rho_{2D}^* = 1$ *and* $\rho_{1D}^* < 1$ *therefore this unit is inefficient and* ρ_{1D}^* *shows the ranking. For* DMU_E, $\rho_{2E}^* > 1$, *and the model* (3.9) *are infeasible. Tthis shows that* DMU_E *is placed outside of the new PPS after being removed from production technology. Therefor, this units is pareto-efficient and* ρ_{2E}^* *shows the ranking. For* DMU_F, $\rho_{2F}^* = 1$ *and* $\rho_{1F}^* < 1$ *therefore this unit is inefficient and* ρ_{1F}^* *shows the ranking. For* DMU_G, $\rho_{2G}^* > 1$, *and the model* (3.9) *are infeasible. This shows that* DMU_G *is placed outside of the new PPS after being removed from production technology. There for, this units is pareto-efficient and* ρ_{2G}^* *shows the ranking. For* DMU_H, $\rho_{2H}^* = 1$ *and* $\rho_{1H}^* < 1$ *therefore this unit is in efficient and* ρ_{1H}^* *shows the ranking. For* DMU_K, $\rho_{2K}^* = \rho_{1K}^* = 1$ *therefore,* DMU_K *is non-extreme efficiency unit. For* DMU_p, $\rho_{2p}^* = 1$ *and* $\rho_{1p}^* < 1$ *therefore this unit is inefficient and* ρ_{1p}^* *shows the ranking. Regarding the results achieved,* $E^1 = \{A, E, C, G\}$ *is the set of efficient units and* $E^2 = \{B, D, F, H\}$ *is the set of inefficient units. In short* $\rho_{2A}^* > 1, \rho_{2C}^* > 1, \rho_{2E}^* > 1, \rho_{2G}^* > 1$, *therefore* DMU_A, DMU_B, DMU_C, DMU_G *are extreme efficiency and also if* $\rho_{1K}^* = \rho_{2K}^* = 1$ *therefore* DMU_K *is a non-extreme efficient and other DMUs are inefficient. Therefore, Rank(G) > Rank(C) > Rank(E) > Rank(A). Therefore, rank G is better than rank C. Rank C is better than rank E and rank E is better than rank A.*

Example 4.2. *Numerical example 2 to implement the algorithm in the presence of non-extreme efficienct DMUs (more than one)*

Table 3: Consider 13 DMUs in the mode of one input-one output.

DMU	1	2	3	4	5	6	7	8	9	10	11	12	13
X	1	1	2	3	4	5	6	8	9	12	13	7	9
Y	1	2	4	6	8	9	10	12	13	13	13	3	6

Solving the model (3.10), *the efficient and inefficient units are determined and can be observed in.*

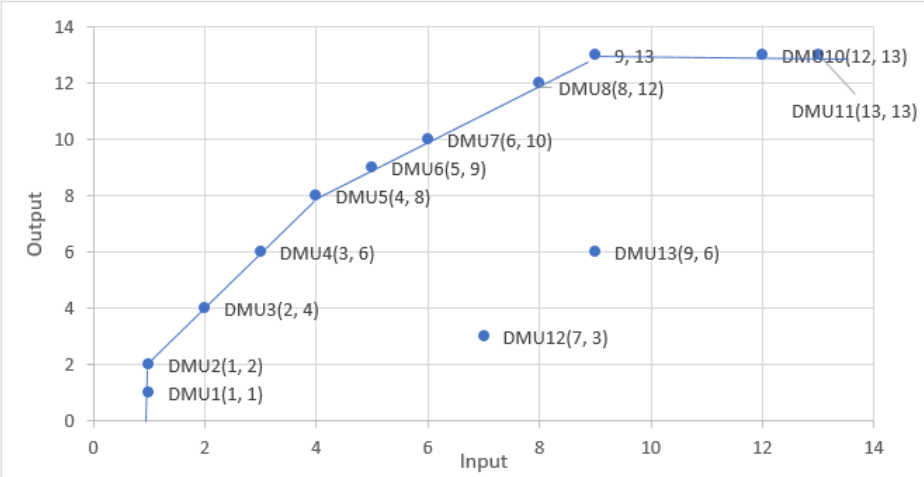

Figure 3: Illustration of table 3.

Table 4: The outcome of table 2, step (1)

DMU	1	2	3	4	5	6	7	8	9	10	11	12	13
S^-	0	0	0	0	0	0	0	0	0	3	4	3	5
S^+	0	0	0	0	0	0	0	0	0	0	0	5	2
ρ_j^*	0.5	1	1	1	1	1	1	1	1	0.75	0.69	0.21	0.33

Therefore, $w = [2, 3, 4, 5, 6, 7, 8, 9]$ *is the set of strong efficient units. The inefficient units are ranked solving the model SBM.*

$$\rho_{10}^{sbm} > \rho_{11}^{sbm} > \rho_1^{sbm} > \rho_{13}^{sbm} > \rho_{12}^{sbm}$$

Table 5: The outcome of table 2, algorithm 3.2, step (2)

Table 5: shows the results of the members of the set W.

DMU	2	3	4	5	6	7	8	9
S^-	0.33	0	0	0	0	0	0	0
S^+	0	0	0	0.5	0	0	0	0.75
ρ_j^{sbm}	1.33	1	1	1.066	1	1	1	1.061

Therefore, as the optimal response of p^0 *is* $E^2 = [2, 5, 9]$*, the units are extreme efficient and their ranking is as below: DMU9, DMU5, DMU6 are extreme efficient units and their ranking is as below:*

$$\rho_2^0 = 1.33, \quad \rho_5^0 = 1.066, \quad \rho_9^0 = 1.061, \quad \Rightarrow \rho_2^0 > \rho_5^0 > \rho_9^0 \tag{4.1}$$

Therefore, the extreme efficient units and inefficient units have been ranked to this stage. To rank the non-extreme efficient units, we form the set below:

$$E^2 = \{1, 2, \cdots, 13\} - \{j \in w | \rho_j^0 > 1\} = \{1, 3, 4, 6, 7, 8, 10, 11, 12, 1\} \tag{4.2}$$

$$E^1 \cap W = \{3,4,6,7,8\} = W - E^0, \quad Card(E^1 \cap W) \geq 2 \tag{4.3}$$

Therefore, the model p1 is solved by T1 for the members of $E^1 \cap W$. *T1 is the set of possible production with the outcome of variant scale on the observations E1.*
The optimal response of the model P1 is as in table 6:

Table 6: The optimal response of the model P1

DMU	3	4	6	7	8
S^-	0.2	0	0	0	0
S^-	0	0.33	0.33	0	1
S^-	1.01	1.058	1.038	1	1.09

Regarding the optimal response of the model P2, the units in the set $E^1 \cap W$ *are ranked as seen in number* $\rho_8^1 > \rho_4^1 > \rho_6^1 > \rho_3^1 > \rho_7^1$. *In the second repetition the set E2 is formed, as seen in number*

$$E^2 = \{1,2,\cdots,13\} - \{j \in w|\ \rho_j^0 > 1\} = \{1,7,10,11,12,13\}, \quad E^2 \cap W = \{7\}$$

because $|E^2 \cap W| \leq 1$ *therefore, the algorithm is finished. The ranking of DMU in P1 in the set* $E^2 \cap W$ *is determined and according to number* $rank\{j \in W|\ \rho_j^0 > 1\} > rank\{j \in W|\ \rho_j^1 > 1\}$. *The ranking of all the units can be seen in number*

$$\begin{aligned} &rank(2) > rank(5) > rank(9) > rank(8) > rank(4) > rank(6) > rank(3) > rank(7) \\ &> rank(10) > rank(11) > rank(1) > rank(13) > rank(12) \end{aligned} \tag{4.4}$$

Example 4.3. ***(Empirical example)*** *we want to rank 20 Iranian bank branches with our proposed method. In order to compare this method with AP and MAJ models, we also rank the DMUs of Example 4.1 by AP and MAJ models, the results of which are shown in columns 8, 9, 10, and 11. According to the results, the rankings of DMUs by the three methods are almost similar.*
Inputs and outputs and ranking by our proposed method and AP [6], MAJ [7], Jahanshahloo [10] ranking models.
According to the results of the table above, the proposed method performs better than the ideal points ranking method, and it also easily identifies both extreme efficient unit and non-extreme efficient unit. Other methods, however, are not capable of doing so at this stage. For example, the ranking of the DMU1, DMU4, DMU10, DMU15, DMU20 with the proposed method is equal to the AP method and MAJ methods.

Table 7: Data of the DMUs and their Russell efficiencies.

		Input Data			output Data		
Branch	Staff	Computer	Space(m2)	Deposite	loans	Charge	Russel
1	0.950	0.700	0.155	0.190	0.521	0.293	1.00
2	0.769	0.600	1.000	0.227	0.627	0.462	0.55
3	0.798	0.750	0.513	0.228	0.970	0.261	0.53
4	0.865	0.550	0.210	0.193	0.632	1.000	1.00
5	0.815	0.850	0.268	0.233	0.722	0.246	0.50
6	0.842	0.650	0.500	0.207	0.603	0.569	0.61
7	0.719	0.600	0.350	0.182	0.900	0.716	1.00
8	0.785	0.750	0.120	0.125	0.234	0.298	0.48
9	0.476	0.600	0.135	0.080	0.364	0.244	0.52
10	0.678	0.550	0.510	0.082	0.184	0.049	0.10
11	0.711	1.000	0.305	0.212	0.318	0.403	0.46
12	0.811	0.650	0.255	0.123	0.923	0.628	1.00
13	0.659	0.800	0.340	0.176	0.645	0.261	0.49
14	0.976	0.850	0.540	0.144	0.514	0.243	0.29
15	0.685	0.950	0.450	1.000	0.262	0.098	1.00
16	0.613	0.900	0.525	0.115	0.402	0.464	0.38
17	1.000	0.600	0.205	0.090	1.000	0.161	1
18	0.634	0.650	0.235	0.059	0.349	0.068	0.15
19	0.372	0.700	0.238	0.039	0.190	0.111	0.19
20	0.583	0.550	0.500	0.110	0.615	0.764	1.00

5. Conclusion

In this article, in 3.1, the suggested model was presented which is able to rank all the under-evaluation units except the non-extreme efficient units. Also, the suggested model in 3 is differentiated from the non-extreme efficient units and inefficient units. The model (3.9) is always feasible. To solve the problem in ranking the non-extreme efficient units in 3.2 a model was suggested that solves the problem in the previous section. It ranks all the units including extreme and non-extreme efficient units and inefficient units.In other words, as the ranking related to extreme efficient units is better than non-extreme efficient units and the ranking related to non-extreme efficient units is better than inefficient units, the model 6 presented in 3.1 must be solved first in order to determine the efficient and inefficient units. The model (3.10) achieved by modeling in the section 3.1 is SBM model. If the under-evaluation unit is inefficient, all the inefficient units are ranked easily. Consequently, the efficient units are determined. The models which are used for ranking the units are non-radial; therefore, they do not have some common problems in ranking such as infeasibility, inconsistency and false ranking. Moreover, this method is able to rank the non-extreme efficient units. Also, the suggested model includes the features below: firstly, the suggested model for ranking the DMUs can be transformed into non-linear models; secondly, it is consistent in changing the near-zero small data; thirdly, it does not produce false ranking.

Table 8: The real data of the article were obtained from reference [10] compiled Jahanshah et al.

Branch	MODEL	MODEL6	Proposed	Rank by ideal point	AP	MAJ
1	1.058	INF	7	2	7	7
2	1.00	0.74	12	9	10	13
3	1.053	INF	9	10	8	8
4	1.251	INF	2	1	2	2
5	1.00	0.51	15	12	9	11
6	1.00	0.61	11	8	14	16
7	1.103	INF	5	3	5	3
8	1.072	INF	6	14	12	9
9	1.264	INF	3	11	13	10
10	1.00	0.12	20	20	20	20
11	1.00	0.52	14	15	16	15
12	1.044	INF	8	4	6	6
13	1.00	0.57	13	13	11	12
14	1.00	0.29	17	17	18	19
15	1.384	INF	1	7	1	1
16	1.00	0.40	16	16	15	18
17	1.10	INF	10	5	3	5
18	1.00	0.22	19	19	17	14
19	1.093	INF	11	18	19	17
20	1.138	INF	4	6	4	4

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