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Quasilinear elliptic systems in perturbed form

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Abstract

In this paper, we consider the boundary value problem of a quasilinear elliptic system in degenerate form with data belongs to the dual of Sobolev Spaces. The existence result is proved by means of Young measures and mild monotonicity assumptions.

Keywords: Quasilinear elliptic system, Weak solution, Young measure.
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1. Introduction

The present paper is concerned with the following boundary value system

$$-\operatorname{div}\big(\sigma(x, Du) + \phi(u)\big) = f \quad \text{in } \Omega; \tag{1.1}$$

$$u = 0 \qquad \text{on } \partial\Omega, \tag{1.2}$$

where Ω is a bounded open set of $\mathbb{R}^n$, $(n \geq 2)$, $u : \Omega \to \mathbb{R}^m$, $m \in \mathbb{N}$, is a vector-valued function. By $\mathbb{M}^{m\times n}$ we denote the space of $m \times n$ matrices equipped with the inner product $F : G = F_{ij}G_{ij}$, with conventional summation.

In [9] the following quasilinear elliptic system was considered:

$$\begin{aligned} -\operatorname{div}\sigma(x, u, Du) &= f \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{aligned} \tag{1.3}$$

where f belongs to the dual space $W^{-1,p'}(\Omega; \mathbb{R}^m)$ of $W_0^{1,p}(\Omega; \mathbb{R}^m)$. The author proved the existence of weak solutions under weak monotonicity assumptions on the stress tensor $\sigma : \Omega \times \mathbb{R}^m \times \mathbb{M}^{m\times n} \to$

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$\mathbb{M}^{m\times n}$ and by the theory of Young measures. When the right hand side in (1.3) is equal to $v(x) + f(x,u) + \operatorname{div} g(x,u)$, the existence of a weak solution under classical regularity, growth and coercivity conditions for σ, but with only very mild monotonicity assumptions, was proved in [2].

By the same theory (i.e. of Young measures), we have established in [3] the existence result for a generalized p-Laplacian system of the form

$$-\operatorname{div}(\Phi(Du - \Theta(u)) = f$$

supplemented with Dirichlet condition $u = 0$ on $\partial\Omega$, where $\Phi(F) = |F|^{p-2}F$ for $F \in \mathbb{M}^{m\times n}$ and Θ satisfies some Lipschitz continuity condition. Second-order estimates are established for solutions to the p-Laplace system with right hand in $L^2(\Omega)$ and local estimates for local solutions are provided in [6].

In the scalar case and f belongs to $H^{-1}(\Omega)$, uniqueness in the class of weak solution in $H_0^1(\Omega)$ was proved in [1] if $\phi \equiv 0$ and $\sigma \equiv a(x,u)\nabla u$, and then in [16], where ϕ is still assumed to be in $C(\mathbb{R},\mathbb{R}^n)$ and f belongs to $L^1(\Omega)$. Di Nardo and Perrotta [14] considered the problem (1.1) and fixed some structural conditions on σ and ϕ to prove uniqueness result when $f \in L^1(\Omega)$. For two lower order terms, we refer to [15] where the existence result is obtained as limit of approximations. See also [5, 8].

A large number of papers was devoted to the study of the existence for solutions of elliptic problem of the type (1.3) under classical monotone operator methods developed by [4, 12, 13, 17]. These works employ the standard theory of monotone operator on the Sobolev space $W^{1,p}(\Omega)$.

The difficulty that arises in our problem (1.1)-(1.2) is that we can't use such theory, because we assume only W in (H3)(b) (see Section 2) to be convex, but if it is strictly convex, then σ becomes strict monotone and the standard method may apply. Moreover, we assume that σ is strictly quasimonotone (see (H3)(d) in Section 2) which allows to proceed the proof differently to [2] and [9]. The presence of the lower term $\phi(u)$ in (1.1)-(1.2) is an addition difficulty besides previous ones.

In the present paper, a slightly different notions of monotonicity and quasimonotonicity are used. Moreover, we use another condition namely strict quasimontone instead of p-quasimonotone used in [9] and we proceed the proof differently by using Lemma 5.2.

2. Assumptions and main result

Let Ω be an open bounded set of $\mathbb{R}^n$ $(n \geq 2)$. The functions σ and ϕ are assumed to satisfy the following conditions:

(H0) The function $\phi : \mathbb{R}^m \to \mathbb{M}^{m\times n}$ is linear and continuous and there exists a constant α_0 such that

$$|\phi(u)| \leq \alpha_0.$$

(H1) $\sigma : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{M}^{m\times n}$ is a Carathéodory function, i.e. measurable w.r.t $x \in \Omega$ and continuous w.r.t $F \in \mathbb{M}^{m\times n}$.

(H2) There exist $\alpha > \alpha_0 > 0$, $d_1(x) \in L^{p'}(\Omega)$ and $d_2(x) \in L^1(\Omega)$ such that

$$|\sigma(x,F)| \leq d_1(x) + |F|^{p-1}$$

$$\sigma(x,F) : F \geq \alpha|F|^p - d_2(x)$$

for any $F \in \mathbb{M}^{m\times n}$.

(H3) σ satisfies one of the following conditions:

(a) For any $x \in \Omega$, $F \mapsto \sigma(x,F)$ is C^1 and monotone, i.e.

$$\big(\sigma(x,F)-\sigma(x,G)\big) : (F-G) \geq 0$$

for any $x \in \Omega$ and $F, G \in \mathbb{M}^{m\times n}$.

(b) There exists a function $W : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ such that $\sigma(x,F) = \frac{\partial W}{\partial F}(x,F)$ and $F \mapsto W(x,F)$ is convex and C^1.

(c) σ is stricIty monotone, i.e. σ is monotone and

$$\big(\sigma(x,F)-\sigma(x,G)\big) : (F-G) = 0 \Rightarrow F = G.$$

(d) σ is strictly quasimonotone, i.e. there exists a constant $\alpha_1 > 0$ such that

$$\int_\Omega \big(\sigma(x,Du)-\sigma(x,Dv)\big) : (Du-Dv)dx \geq \alpha_1 \int_\Omega |Du-Dv|^p dx.$$

Our main result can be stated as follows:

Theorem 2.1. *If σ and ϕ satisfy the conditions (H0)-(H3), then problem* (1.1)-(1.2) *has a weak solution for every $f \in W^{-1,p'}(\Omega;\mathbb{R}^m)$.*

Example 2.2. *As example of problem to which the present result can be applied, we give:*

$$-div\,(|Du|^{p-2}Du + Du) = f,$$

with $f \in W^{-1,p'}(\Omega;\mathbb{R}^m)$. The conditions (H3)(a), (c) and (d) are obvious by direct calculations. For the condition (b), one can take the potential $W = \frac{1}{p}|F|^p + \frac{1}{2}|F|^2$.

3. A review on Young measures

In the following $\mathcal{C}_0(\mathbb{R}^m)$ denotes the closure of the space of continuous functions on $\mathbb{R}^m$ with compact support with respect to the $\|.\|_\infty$-norm. Its dual space can be identified with $\mathcal{M}(\mathbb{R}^m)$, the space of signed Radon measures with finite mass. The related duality pairing is given by

$$\langle \nu, f\rangle = \int_{\mathbb{R}^m} f(\lambda)d\nu(\lambda).$$

Note that $id(\lambda) = \lambda$, thus $\langle \nu, id\rangle = \int_{\mathbb{R}^m} \lambda d\nu(\lambda)$.

Definition 3.1. *Assume that the sequence $\{w_j\}_{j\geq 1}$ is bounded in $L^\infty(\Omega;\mathbb{R}^m)$. Then there exists a subsequence $\{w_k\}_k$ and a Borel probability measure ν_x on $\mathbb{R}^m$ for a.e. $x \in \Omega$, such that for almost each $g \in \mathcal{C}(\mathbb{R}^m)$ we have*

$$g(w_k) \rightharpoonup^* \overline{g} \quad \text{weakly in } L^\infty(\Omega),$$

where

$$\overline{g}(x) = \int_{\mathbb{R}^m} g(\lambda)d\nu_x(\lambda).$$

We call $\nu = \{\nu_x\}_{x\in\Omega}$ the family of Young measures associated with the subsequence $\{w_k\}_k$.

The fundamental theorem on Young measures may be stated in the following lemma:

Lemma 3.2. *[7] Let $\Omega \subset \mathbb{R}^n$ be Lebesgue measurable (not necessarily bounded) and $w_j : \Omega \to \mathbb{R}^m$, $j = 1, .., $ be a sequence of Lebesgue measurable functions. Then there exists a subsequence w_k and a family $\{\nu_x\}_{x\in\Omega}$ of non-negative Radon measures on $\mathbb{R}^m$, such that*

(i) $\|\nu_x\|_{\mathcal{M}(\mathbb{R}^m)} := \int_{\mathbb{R}^m} d\nu_x \leq 1$ *for almost* $x \in \Omega$.

(ii) $\varphi(w_k) \rightharpoonup^* \overline{\varphi}$ *weakly in* $L^\infty(\Omega)$ *for all* $\varphi \in \mathcal{C}_0(\mathbb{R}^m)$, *where* $\overline{\varphi}(x) = \langle \nu_x, \varphi \rangle$.

(iii) If for all $R > 0$

$$\lim_{L\to\infty} \sup_k \big| \{x \in \Omega \cap B_R(0) : \ |w_k(x)| \geq L\} \big| = 0, \tag{3.1}$$

then $\|\nu_x\| = 1$ for a.e. $x \in \Omega$, and for all measurable $\Omega' \subset \Omega$ there holds $\varphi(w_k) \rightharpoonup \overline{\varphi} = \langle \nu_x, \varphi \rangle$ weakly in $L^1(\Omega')$ for a continuous function φ provided the sequence $\varphi(w_k)$ is weakly precompact in $L^1(\Omega')$.

The following lemmas are considered as the applications of the fundamental theorem on Young measures (i.e. Lemma 3.2), which will be needed in the sequel.

Lemma 3.3 ([10]). *If $|\Omega| < \infty$ and ν_x is the Young measure generated by the (whole) sequence w_j then there holds*

$$w_j \to w \text{ in measure } \Leftrightarrow \ \nu_x = \delta_{w(x)} \quad \text{for a.e. } x \in \Omega.$$

Lemma 3.4 ([10]). *Let $\psi : \Omega \times \mathbb{M}^{m\times n} \to \mathbb{R}$ be a Carathéodory function and $u_k : \Omega \to \mathbb{R}^m$ a sequence of measurable functions such that Du_k generates the Young measure ν_x. Then*

$$\liminf_{k\to\infty} \int_\Omega \psi\big(x, Du_k(x)\big) dx \geq \int_\Omega \int_{\mathbb{M}^{m\times n}} \psi(x, \lambda) d\nu_x(\lambda) dx,$$

provided that the negative part $(\psi(x, Du_k(x))^-$ is equiintegrable.

4. Galerkin approximation

Let $V_1 \subset V_2 \subset .. \subset W_0^{1,p}(\Omega; \mathbb{R}^m)$ be a sequence of finite dimensional subspaces with the property that $\underset{i\in\mathbb{N}}{\cup} V_i$ is dense in $W_0^{1,p}(\Omega; \mathbb{R}^m)$. We define the operator

$$\begin{aligned} T :\ & W_0^{1,p}(\Omega; \mathbb{R}^m) \to W^{-1,p'}(\Omega; \mathbb{R}^m) \\ & u \mapsto \Big(w \mapsto \int_\Omega \big(\sigma(x, Du) : Dw + \phi(u) : Dw \big) dx - \langle f, w \rangle \Big), \end{aligned}$$

where $\langle ., . \rangle$ denotes the dual pairing of $W^{-1,p'}(\Omega; \mathbb{R}^m)$ and $W_0^{1,p}(\Omega; \mathbb{R}^m)$.

Lemma 4.1. *For arbitrary $u \in W_0^{1,p}(\Omega; \mathbb{R}^m)$, the functional $T(u)$ is linear and bounded.*

Proof . $T(u)$ is trivially linear. We have

$$\int_\Omega |\sigma(x, Du)|^{p'} dx \leq \int_\Omega \big(|d_1(x)|^{p'} + |Du|^p \big) dx < \infty,$$

by the growth condition in (H2). It follows from the Hölder inequality that for each $w \in W_0^{1,p}(\Omega; \mathbb{R}^m)$

$$\begin{aligned} |\langle T(u), w \rangle| &= \Big| \int_\Omega \big(\sigma(x, Du) : Dw + \phi(u) : Dw \big) dx - \langle f, w \rangle \Big| \\ &\leq \| |\sigma(x, Du)| \|_{p'} \|Dw\|_p + \alpha_0 \|Dw\|_1 + \|f\|_{-1,p'} \|w\|_{1,p} \\ &\leq c \|Dw\|_p, \end{aligned}$$

where we have used Poincaré's inequality and $1 < p$. Thus $T(u)$ is bounded. □

Lemma 4.2. *The restriction of T to a finite dimensional linear subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ is continuous.*

Proof . Let V be a finite subspace of $W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that the dimension of V is equal to r and $(e_i)_{i=1}^r$ a basis of V. Let $(u_k = a_k^i e_i)$ be a sequence in V which converges to $u = a^i e_i$ in V (with the standard summation convention). Hence the sequence (a_k) converges to a in $\mathbb{R}^r$. This implies $u_k \to u$ and $Du_k \to Du$ almost everywhere. On the other hand, $\|u_k\|_p$ and $\|Du_k\|_p$ are bounded by a constant C. Indeed, we have

$$\int_\Omega |u_k - u|^p dx \to 0 \quad \text{and} \quad \int_\Omega |Du_k - Du|^p dx \to 0,$$

then there exists a subsequence of (u_k) still denoted by (u_k) and $g_1, g_2 \in L^1(\Omega)$ such that $|u_k-u|^p \leq g_1$ and $|Du_k - Du|^p \leq g_2$. By using the fact that

$$(a+b)^p \leq 2^{p-1}(|a|^p + |b|^p),$$

we obtain

$$\begin{aligned} |u_k|^p = |u_k - u + u|^p &\leq 2^{p-1}\big(|u_k - u|^p + |u|^p\big) \\ &\leq 2^{p-1}\big(g_1 + |u|^p\big). \end{aligned}$$

Similarly

$$|Du_k|^p \leq 2^{p-1}\big(g_2 + |Du|^p\big).$$

The continuity condition (H0) and (H1) allow to deduce that $\sigma(x, Du_k) : Dw \to \sigma(x, Du) : Dw$ and $\phi(u_k) : Dw \to \phi(u) : Dw$ almost everywhere. Furthermore, $\big(\sigma(x, Du_k) : Dw\big)$ and $\big(\phi(u_k) : Dw\big)$ are equiintegrable sequences by (H2). Hence, for all $w \in W_0^{1,p}(\Omega;\mathbb{R}^m)$

$$\begin{aligned} \|T(u_k) - T(u)\|_{-1,p'} &= \sup_{\|w\|_{1,p}=1} \big|\langle T(u_k) - T(u), w\rangle\big| \\ &\leq c\big(\|\sigma(x, Du_k) - \sigma(x, Du)\|_{p'} + \|\phi(u_k) - \phi(u)\|_{p'}\big) \\ &\leq c. \end{aligned}$$

□

Now, we fix some k and assume that dim $V_k = r$. Then we define the map

$$\Theta : \mathbb{R}^r \to \mathbb{R}^r, \qquad \begin{pmatrix} a^1 \\ a^2 \\ . \\ . \\ a^r \end{pmatrix} \mapsto \begin{pmatrix} \langle T(a^i e_i), e_1\rangle \\ \langle T(a^i e_i), e_2\rangle \\ . \\ . \\ \langle T(a^i e_i), e_r\rangle \end{pmatrix}.$$

Lemma 4.3. *Θ is continuous and $\Theta(a).a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$, where the dote . denotes the inner product of two vectors in $\mathbb{R}^r$.*

Proof . The continuity of Θ can be deduced from that of T restricted to V_k. Take $a \in \mathbb{R}^r$ and consider $u = a^i e_i \in V_k$. On the one hand, we have $\Theta(a).a = \langle T(u), u\rangle$ and $\|a\|_{\mathbb{R}^r} \to \infty$ is equivalent to $\|u\|_{1,p} \to \infty$. On the other hand, since $1 < p$ then there exists $\beta = \frac{\alpha}{2\alpha_0} > 0$ such that $\int_\Omega |Du|dx \leq \beta \int_\Omega |Du|^p dx$. Therefore

$$\begin{aligned}\Theta(a).a &= \langle T(a^i e_i), a^i e_i\rangle \\ &= \langle T(u), u\rangle \\ &= \int_\Omega \sigma(x, Du) : Du + \phi(u) : Du\, dx - \langle f, u\rangle \\ &\geq \int_\Omega \big(\alpha|Du|^p - d_2(x)\big)dx - \alpha_0 \int_\Omega |Du|dx - \|f\|_{-1,p'}\|u\|_{1,p} \\ &\geq \frac{\alpha}{2}\|u\|_{1,p}^p - c - \|f\|_{-1,p'}\|u\|_{1,p} \longrightarrow \infty\end{aligned}$$

as $\|u\|_{1,p} \to \infty$. □

The properties of Θ allow the construction of the Galerkin approximations:

Lemma 4.4. *For all $k \in \mathbb{N}$ there exists $u_k \in V_k$ such that*

$$\langle T(u_k), w\rangle = 0 \quad \textit{for all } w \in V_k. \tag{4.1}$$

Proof . We have by Lemma 4.3, $\Theta(a).a \to \infty$ as $\|a\|_{\mathbb{R}^r} \to \infty$. Then there exists $R > 0$ such that for all $a \in \partial B_R(0) \subset \mathbb{R}^r$ we have $\Theta(a).a > 0$. The usual topological argument [11] implies that $\Theta(x) = 0$ has a solution $x \in B_R(0)$. Hence, for all k there exists $u_k \in V_k$ such that $\langle T(u_k), w\rangle = 0$ for all $k \in \mathbb{N}$. □

5. Identification of limits by Young measures

In this section, first we give the Young measure generated by the gradient of sequences defined by the Galerkin method. Then we give some lemmas which permits the construction of the proof of the main theorem. The following lemma describes the limit points of gradient sequences by means of the Young measures. The proof of the following lemma is similar to that in [3, Lemma 4.1], but for completeness of the present paper we present its proof.

Lemma 5.1. *(i) If the sequence $\{Du_k\}_k$ is bounded in $L^p(\Omega; \mathbb{M}^{m\times n})$, then there is a Young measure ν_x generated by $\{Du_k\}_k$ satisfying $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})} = 1$ and the weak L^1-limit of $\{Du_k\}$ is $\langle \nu_x, id\rangle = \int_{\mathbb{M}^{m\times n}} \lambda d\nu_x(\lambda)$.*
(ii) For almost every $x \in \Omega$, ν_x satisfies $\langle \nu_x, id\rangle = Du(x)$ for a.e. $x \in \Omega$.

Proof . (i) By Lemma 4.3 we have $\langle T(u), u\rangle \to \infty$ as $\|u\|_{1,p} \to \infty$. Hence, there exists $R > 0$ with the property that $\langle T(u), u\rangle > 1$ whenever $\|u\|_{1,p} > R$. Then for the sequence of the Galerkin approximations $u_k \in V_k$ which satisfy $\langle T(u_k), u_k\rangle = 0$, we have

$$\|u_k\|_{1,p} \leq R \quad \text{for all } k. \tag{5.1}$$

We deduce the existence of a constant $c \geq 0$ such that for any $R > 0$,

$$\begin{aligned} c \geq \int_\Omega |Du_k|^p dx &\geq \int_{\{x\in\Omega\cap B_R(0):\,|Du_k|\geq L\}} |Du_k|^p dx \\ &\geq L^p \big|\{x \in \Omega \cap B_R(0) :\, |Du_k| \geq L\}\big|. \end{aligned}$$

Therefore

$$\sup_{k\in\mathbb{N}}\big|\{x\in\Omega\cap B_R(0):\,|Du_k|\geq L\}\big|\leq\frac{c}{L^p}\to 0 \text{ as } L\to\infty.$$

Due to Lemma 3.2(iii), we have $\|\nu_x\|_{\mathcal{M}(\mathbb{M}^{m\times n})}=1$. Notice that the existence of ν_x is guaranteed by (5.1). On the other hand, since the space $L^p(\Omega;\mathbb{M}^{m\times n})$ is reflexive, there exists a subsequence (still denoted by $\{Du_k\}$) weakly convergent in $L^p\subset L^1$ and by taking $\varphi\equiv id$ in Lemma 3.2(iii), it follows that

$$Du_k\rightharpoonup\langle\nu_x,id\rangle=\int_{\mathbb{M}^{m\times n}}\lambda d\nu_x(\lambda) \text{ weakly in } L^1(\Omega;\mathbb{M}^{m\times n}).$$

(ii) Since $u_k\rightharpoonup u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ and $u_k\to u$ in $L^p(\Omega;\mathbb{R}^m)$, we have $Du_k\rightharpoonup Du$ in $L^p(\Omega;\mathbb{M}^{m\times n})$. Therefore

$$Du_k\rightharpoonup Du\quad\text{in } L^1(\Omega;\mathbb{M}^{m\times n}).$$

By vertue of the property (i), we can infer that $\langle\nu_x,id\rangle=Du(x)$ for a.e. $x\in\Omega$. □

The following lemma is the key ingredient in the passage to the limit in the approximating equations.

Lemma 5.2. *If σ satisfy (H1)-(H3) and $\{Du_k\}$ generates the Young measure ν_x, then the following inequality holds:*

$$\liminf_{k\to\infty}\int_\Omega\big(\sigma(x,Du_k)-\sigma(x,Du)\big):(Du_k-Du)dx\leq 0. \tag{5.2}$$

Proof . Let us consider the sequence

$$\begin{aligned}I_k &:=\big(\sigma(x,Du_k)-\sigma(x,Du)\big):(Du_k-Du)\\ &=\sigma(x,Du_k):(Du_k-Du)-\sigma(x,Du):(Du_k-Du)\\ &=:I_{k,1}+I_{k,2}.\end{aligned}$$

Remark that since $\sigma\in L^{p'}(\Omega)$ we deduce by Lemma 3.2

$$\begin{aligned}\liminf_{k\to\infty}\int_\Omega\sigma(x,Du):(Du_k-Du)dx &=\int_\Omega\int_{\mathbb{M}^{m\times n}}\sigma(x,Du):(\lambda-Du)dx\\ &=\int_\Omega\sigma(x,Du):\Big(\int_{\mathbb{M}^{m\times n}}\lambda d\nu_x(\lambda)-Du\Big)dx=0.\end{aligned}$$

Using the Mazur's theorem (see e.g. [18, Theorem 2, p120]) there exists a sequence $(v_k)\subset W_0^{1,p}(\Omega;\mathbb{R}^m)$ such that $v_k\to u$ in $W_0^{1,p}(\Omega;\mathbb{R}^m)$ where each v_k is a convex linear combination of $\{u_1,..,u_k\}$. Thus $v_k\in V_k$. Taking u_k-v_k as test function in (4.1), we get

$$\int_\Omega\sigma(x,Du_k):(Du_k-Dv_k)dx=\langle f,u_k-v_k\rangle-\int_\Omega\phi(u_k):(Du_k-Dv_k)dx.$$

Notice that since ϕ is linear and continuous and (u_k) is bounded then $\phi(u_k)$ is bounded. By Hölder's inequality we have

$$\begin{aligned}&\Big|\langle f,u_k-v_k\rangle-\int_\Omega\phi(u_k):(Du_k-Dv_k)dx\Big|\\ &\qquad\leq\|f\|_{-1,p'}\|u_k-v_k\|_{1,p}+c_1\|Du_k-Dv_k\|_1\longrightarrow 0\end{aligned}$$

by definition of v_k, $1 < p$ and

$$\|Du_k - Dv_k\|_p \leq \|Du_k - Du\|_p + \|Dv_k - Du\|_p \longrightarrow 0 \quad \text{as } k \to \infty.$$

Thus

$$\int_\Omega \sigma(x, Du_k) : (Du_k - Dv_k)dx \longrightarrow 0 \quad \text{as } k \to \infty.$$

Using this fact and the construction of v_k to deduce that

$$\begin{aligned}
&\liminf_{k\to\infty} \int_\Omega I_k dx \\
&\quad = \liminf_{k\to\infty} \int_\Omega I_{k,1} dx \\
&\quad = \liminf_{k\to\infty} \int_\Omega \sigma(x, Du_k) : (Du_k - Du)dx \\
&\quad = \liminf_{k\to\infty} \Big(\int_\Omega \sigma(x, Du_k) : (Du_k - Dv_k)dx + \int_\Omega \sigma(x, Du_k) : (Dv_k - Du)dx \Big) \\
&\quad = \liminf_{k\to\infty} \int_\Omega \sigma(x, Du_k) : (Dv_k - Du)dx \\
&\quad \leq \liminf_{k\to\infty} \big\| |\sigma(x, Du_k)| \big\|_{p'} \|Dv_k - Du\|_p = 0.
\end{aligned}$$

Therefore

$$\liminf_{k\to\infty} \int_\Omega \big(\sigma(x, Du_k) - \sigma(x, Du)\big) : (Du_k - Du)dx \leq 0.$$

□

Lemma 5.3. *Suppose* (5.2) *holds, then for almost every* $x \in \Omega$

$$\big(\sigma(x, \lambda) - \sigma(x, Du)\big) : (\lambda - Du) = 0 \text{ on } \operatorname{supp} \nu_x.$$

Proof . From Lemma 5.2 we may deduce the following intermediary result:

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \big(\sigma(x, \lambda) - \sigma(x, Du)\big) : (\lambda - Du)d\nu_x(\lambda)dx \leq 0.$$

Indeed, by Lemma 5.2 we have

$$\liminf_{k\to\infty} \int_\Omega \sigma(x, Du_k) : (Du_k - Du)dx \leq 0.$$

Since $\{\sigma(x, Du_k) : (Du_k - Du)\}$ is equiintegrable, it follows by Lemma 3.4 that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, \lambda) : (\lambda - Du)dx \leq \liminf_{k\to\infty} \int_\Omega \sigma(x, Du_k) : (Du_k - Du)dx \leq 0.$$

We have that

$$\int_\Omega \int_{\mathbb{M}^{m\times n}} \sigma(x, Du) : (\lambda - Du)dx = 0.$$

Put these results into consideration, we deduce the intermediary result. Now, the monotonicity of σ implies that the integral of our intermediary result is non-negative, thus must vanish with respect to the product measure $d\nu_x(\lambda) \otimes dx$. Consequently, for almost every $x \in \Omega$

$$\big(\sigma(x, \lambda) - \sigma(x, Du)\big) : (\lambda - Du) = 0 \text{ on } \operatorname{supp} \nu_x.$$

□

6. Proof of Theorem 2.1

Before we give the proof, it is useful to note that, the cases (H3)(c) and (d) permit to deduce

$$Du_k \to Du \text{ in measure on } \Omega. \tag{6.1}$$

However, this property does not satisfy in the other cases (a) and (b). Let start with the easiest case:

Case (c): By the strict monotonicity of σ and Lemma 5.3, we deduce that $\operatorname{supp}\nu_x = \{Du(x)\}$ which implies $\nu_x = \delta_{Du(x)}$ for a.e. $x \in \Omega$. We infer from Lemma 3.3 that $Du_k \to Du$ in measure on Ω.

Case (d): Remark that for a positive constant c

$$\int_\Omega |Du_k - Du|^p dx \leq c \int_\Omega \big(\sigma(x, Du_k) - \sigma(x, Du)\big) : (Du_k - Du) dx.$$

Passing to the limit inf and using Lemma 5.2, we infer that

$$\lim_{k\to\infty} \int_\Omega |Du_k - Du|^p dx = 0.$$

This implies $Du_k \to Du$ in measure on Ω.

Case (a): We prove that the identity

$$\sigma(x, \lambda) : \mu = \sigma(x, Du) : \mu + \big(\nabla\sigma(x, Du)\mu\big) : (Du - \lambda) \tag{6.2}$$

holds on $\operatorname{supp}\nu_x$, for every $\mu \in \mathbb{M}^{m\times n}$. Here ∇ denotes the derivative with respect to the second variable of σ. On the one hand, the monotonicity of σ implies

$$\begin{aligned}0 &\leq \big(\sigma(x, \lambda) - \sigma(x, Du + \tau\mu)\big) : (\lambda - Du - \tau\mu)\\ &= \sigma(x, \lambda) : (\lambda - Du) - \sigma(x, \lambda) : \tau\mu - \sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu),\end{aligned}$$

for each $\tau \in \mathbb{R}$. On the other hand, Lemma 5.3 permits to write

$$0 \leq \sigma(x, Du) : (\lambda - Du) - \sigma(x, \lambda) : \tau\mu - \sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu).$$

Therefore

$$-\sigma(x, \lambda) : \tau\mu \geq -\sigma(x, Du) : (\lambda - Du) + \sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu).$$

Note that

$$\begin{aligned}&\sigma(x, Du + \tau\mu) : (\lambda - Du - \tau\mu)\\ &\quad= \sigma(x, Du + \tau\mu) : (\lambda - Du) - \sigma(x, Du + \tau\mu) : \tau\mu\\ &\quad= \sigma(x, Du) : (\lambda - Du) + \nabla\sigma(x, Du)\tau\mu : (\lambda - Du) - \sigma(x, Du) : \tau\mu\\ &\qquad\quad - \nabla\sigma(x, Du)\tau\mu : \tau\mu + o(\tau)\\ &\quad= \sigma(x, Du) : (\lambda - Du) + \tau\big[\big(\nabla\sigma(x, Du)\mu\big) : (\lambda - Du) - \sigma(x, Du) : \mu\big] + o(\tau).\end{aligned}$$

It follows that

$$-\sigma(x, \lambda) : \tau\mu \geq \tau\big[\big(\nabla\sigma(x, Du)\mu\big) : (\lambda - Du) - \sigma(x, Du) : \mu\big] + o(\tau).$$

Since τ is arbitrary in $\mathbb{R}$, the above inequality implies (6.2).
The equation (6.2) together with the equiintegrability of the sequence $\sigma(x, Du_k)$ allow to deduce the weak L^1-limit $\overline{\sigma}$ of $\sigma(x, Du_k)$ as follows:

$$\begin{aligned}\overline{\sigma} &= \int_{\operatorname{supp}\nu_x} \sigma(x,\lambda)d\nu_x(\lambda)\\ &= \int_{\operatorname{supp}\nu_x} \sigma(x,Du)d\nu_x(\lambda) + \big(\nabla\sigma(x,Du)\big)^t : \underbrace{\int_{\operatorname{supp}\nu_x} (Du-\lambda)d\nu_x(\lambda)}_{=0}\\ &= \sigma(x,Du).\end{aligned}$$

Case (b): Let show that for almost every $x \in \Omega$, $\operatorname{supp}\nu_x \subset K_x$ where

$$K_x = \big\{\lambda \in \mathbb{M}^{m\times n} : W(x,\lambda) = W(x,Du) + \sigma(x,Du) : (\lambda - Du)\big\}.$$

If $\lambda \in \operatorname{supp}\nu_x$, by Lemma 5.3 it follows that

$$(1-\tau) : \big(\sigma(x,\lambda) - \sigma(x,Du)\big) : (\lambda - Du) = 0 \text{ for all } \tau \in [0,1]. \tag{6.3}$$

Due to the monotonicity of σ, we have for $\tau \in [0,1]$

$$(1-\tau) : \big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,\lambda)\big) : (Du-\lambda) \geq 0. \tag{6.4}$$

Therefore, by subtracting (6.3) from (6.4), we get

$$(1-\tau) : \big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : (Du-\lambda) \geq 0. \tag{6.5}$$

The monotonicity of σ allows again to write

$$\big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : \tau(\lambda-Du) \geq 0,$$

and since $\tau \in [0,1]$, we have then

$$\big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : (1-\tau)(\lambda-Du) \geq 0.$$

From the last inequality and Eq. (6.5) we deduce

$$\big(\sigma(x,Du+\tau(\lambda-Du)) - \sigma(x,Du)\big) : (\lambda-Du) = 0, \tag{6.6}$$

for $\tau \in [0,1]$ and $\lambda \in \operatorname{supp}\nu_x$. Therefore

$$\sigma(x,Du+\tau(\lambda-Du)) : (\lambda-Du) = \sigma(x,Du) : (\lambda-Du).$$

Integrate this equality over $[0,1]$ and using the fact that $\sigma = \frac{\partial W}{\partial F}$ to deduce that

$$\begin{aligned}W(x,\lambda) &= W(x,Du) + \int_0^1 \sigma(x,Du+\tau(\lambda-Du)) : (\lambda-Du)d\tau\\ &= W(x,Du) + \sigma(x,Du) : (\lambda-Du).\end{aligned}$$

Consequently $\lambda \in K_x$, i.e. $\operatorname{supp}\nu_x \subset K_x$. By the convexity of W we can write

$$W(x,\lambda) \geq W(x,Du) + \sigma(x,Du) : (\lambda-Du) \quad \forall\lambda \in \mathbb{M}^{m\times n}. \tag{6.7}$$

Put $A(\lambda)$ (resp. $B(\lambda)$) the left (resp. the right) hand side of (6.7). Since $\lambda \mapsto A(\lambda)$ is C^1, it follows for $\tau \in \mathbb{R}$

$$\frac{A(\lambda+\tau\mu)-A(\lambda)}{\tau} \geq \frac{B(\lambda+\tau\mu)-B(\lambda)}{\tau} \quad \text{if } \tau > 0,$$
$$\frac{A(\lambda+\tau\mu)-A(\lambda)}{\tau} \leq \frac{B(\lambda+\tau\mu)-B(\lambda)}{\tau} \quad \text{if } \tau < 0.$$

Hence $D_\lambda A = D_\lambda B$. Therefore

$$\sigma(x,\lambda) = \sigma(x, Du) \;\text{ for all } \lambda \in \operatorname{supp}\nu_x \subset K_x.$$

We have then

$$\begin{aligned}\overline{\sigma} = \int_{\mathbb{M}^{m\times n}} \sigma(x,\lambda) d\nu_x(\lambda) &= \int_{\operatorname{supp}\nu_x} \sigma(x,Du) d\nu_x(\lambda)\\ &= \sigma(x,Du)\int_{\operatorname{supp}\nu_x} d\nu_x(\lambda) = \sigma(x,Du).\end{aligned}$$

Conclusion

For the cases (c) and (d), we have $Du_k \to Du$ in measure on Ω. Now, let $E_{k,\epsilon} = \big\{x : |u_k(x) - u(x)| \geq \epsilon\big\}$, then

$$\int_\Omega |u_k(x)-u(x)|^p dx \geq \int_{E_{k,\epsilon}} |u_k(x)-u(x)|^p dx \geq \epsilon^p \big|E_{k,\epsilon}\big|,$$

which implies

$$\big|E_{k,\epsilon}\big| \leq \frac{1}{\epsilon^p}\int_\Omega |u_k(x)-u(x)|^p dx \to 0 \quad \text{as } k \to 0.$$

Hence $u_k \to u$ in measure for $k \to \infty$. After extracting a suitable subsequence (if necessary), we can infer that $u_k \to u$ and $Du_k \to Du$ for almost every $x \in \Omega$. Then $\sigma(x, Du_k) \to \sigma(x, Du)$ and $\phi(u_k) \to \phi(u)$ almost everywhere, by continuity of σ and ϕ. Furthermore, we have $\sigma(x, Du_k) \to \sigma(x, Du)$ and $\phi(u_k) \to \phi(u)$ in measure. Since $\big(\sigma(x, Du_k) : Dw\big)$ and $\big(\phi(u_k) : Dw\big)$ are equiintegrable, it follows by Vitali's theorem that

$$\int_\Omega \big(\sigma(x,Du_k) - \sigma(x,Du)\big) : Dw dx \to 0 \quad \text{as } k \to \infty$$

and

$$\int_\Omega \big(\phi(u_k) - \phi(u)\big) : Dw dx \to 0 \quad \text{as } k \to \infty.$$

The proof of Theorem 2.1 follows for the cases (c) and (d).

Now, in the cases (a) and (b) we have $\overline{\sigma} = \sigma(x, Du)$. Since $L^p(\Omega; \mathbb{M}^{m\times n})$ is reflexive, the sequences $\{\sigma(x, Du_k)\}$ and $\{\phi(u_k)\}$ converges weakly in $L^{p'}(\Omega; \mathbb{M}^{m\times n})$ and their weak $L^{p'}$-limits are $\sigma(x, Du)$ and $\phi(u)$ (respectively). Hence

$$\int_\Omega \big[\big(\sigma(x,Du_k) - \sigma(x,Du)\big) : Dw + \big(\phi(u_k)-\phi(u)\big) : Dw\big] dx \to 0 \text{ as } k \to \infty.$$

Thus Theorem 2.1 follows also for the case (a). For the last case (b), we argue as follows: we consider the Carathéodory function

$$h(x,\lambda) = |\sigma(x,\lambda) - \overline{\sigma}(x)|, \;\; \lambda \in \mathbb{M}^{m\times n}.$$

Since $h_k(x) := h(x, Du_k)$ is equiintegrable, then

$$h_k \rightharpoonup \overline{h} \quad \text{weakly in } L^1(\Omega),$$

where $\overline{h}$ is given by

$$\begin{aligned}\overline{h}(x) &= \int_{\mathbb{M}^{m\times n}} |\sigma(x,\lambda) - \overline{\sigma}(x)| d\nu_x(\lambda) \\ &= \int_{\operatorname{supp}\nu_x} |\sigma(x,\lambda) - \overline{\sigma}(x)| d\nu_x(\lambda) = 0 \qquad (\text{since } \overline{\sigma} = \sigma(x, Du) = \sigma(x,\lambda)).\end{aligned}$$

Hence

$$\int_\Omega |\sigma(x, Du_k) - \sigma(x, Du)| dx \to 0 \quad (\text{since } h_k \geq 0).$$

Therefore, by Vitali's theorem

$$\int_\Omega \big[\big(\sigma(x, Du_k) - \sigma(x, Du)\big) : Dw + \big(\phi(u_k) - \phi(u)\big) : Dw\big] dx \to 0 \text{ as } k \to \infty.$$

This again accomplishes the proof of Theorem 2.1 in the case (b).

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