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Some results on second transpose of a dual valued derivation

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Abstract

Let A be a Banach algebra and X be an arbitrary Banach A-module. In this paper, we study the second transpose of derivations with value in dual Banach A-module X^*. Indeed, for a continuous derivation $D : A \longrightarrow X^*$ we obtain a necessary and sufficient condition such that the bounded linear map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ to be a derivation, where Λ is composition of restriction and canonical injection maps. This characterization generalizes some well known results in [2].

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1. Introduction

The second transpose of a derivation from a Banach algebra A into dual module A^* has been discussed in some papers, see for example [1], [4], [6], [9] and [10].

Dales, Rodriguez and Velasco in [6] studied the second transpose of a A^*-valued derivation $D : A \longrightarrow A^*$ and obtained conditions under which the second transpose $D'' : A^{**} \longrightarrow A^{***}$ is a derivation. Indeed, it is shown that D'' is a derivation if and only if $D''(A^{**}) \cdot A^{**} \subseteq A^*$ [6, Theorem 7.1]. Furthermore, Eshaghi Gordji and Filali in [8] and [7] obtained some results on this subject and studied weak amenability of second dual of Banach algebras. Also in [10], the authors investigated the second transpose of a derivation D on a Banach algebra A with value in X^*, where X is an arbitrary Banach A-module. Indeed, they obtained a necessary and sufficient condition under which $D'' : A^{**} \longrightarrow X^{***}$ be a derivation and generalized some results in [6].

Weak amenability of the second dual of Banach algebras has been studied with a different approach by Ghahramani, Loy and Willis in [9]. They considered some conditions under which the map $\Lambda \circ D'' : A^{**} \longrightarrow A^{***}$ to be a derivation, where Λ is composition of restriction and canonical injection maps as defined in Section 2. Indeed, they claimed that $\Lambda \circ D''$ is a derivation in the case

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where A is a left ideal of A^{**} or D is weakly compact, see also [3]. They used the fact that for each $F, G \in A^{**}$, the equality $\Lambda(D''(F) \cdot G) = (\Lambda \circ D''(F)) \cdot G$ holds on A^{**} if and only if it holds just on A.

Recently, the authors in [2] obtained a necessary and sufficient condition such that the map $\Lambda \circ D'' : A^{**} \longrightarrow A^{***}$ be a derivation as follows:

Theorem 1.1. *([2, Theorem 3.1]) Let A be a Banach algebra and $D : A \longrightarrow A^*$ be a continuous derivation. Then the following are equivalent:*

(i) $\Lambda \circ D'' : A^{**} \longrightarrow A^{***}$ is a derivation.
(ii) For every $F, G \in A^{**}$, $\Lambda(D''(F) \cdot G) = (\Lambda \circ D''(F)) \cdot G$ on A and

$$\Lambda \circ D''(A^{**}) \subseteq WAP(A).$$

The present paper organized as follows. In Section 2, we recall some standard notations and define some basic concepts that we shall need. In Section 3, we consider the situation when $\Lambda \circ D''$ is itself a derivation which is an extension of some results in [2]. Indeed, for an arbitrary Banach A-module X and continuous derivation $D : A \longrightarrow X^*$, we obtain a necessary and sufficient condition under which the map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ to be a derivation. As a consequence, in the case where D is a weakly compact derivation, we obtain a simple condition equivalent to the map $\Lambda \circ D''$ be a derivation.

2. Preliminaries

In this section, we recall some standard notations and define some basic concepts about transpose of a bounded bilinear map and Banach modules, for more details see [3], [6], [9] and [10].

Throughout the paper, X denotes a Banach space and X^* , X^{**} and X^{***} are first, second and third dual of X, respectively. The canonical embedding map $\kappa_X : X \longrightarrow X^{**}$ is defined by

$$\kappa_X(x)(f) = f(x) \qquad (x \in X, f \in X^*).$$

We also use the notation $\kappa_X(x) = \widehat{x}$. Furthermore, we consider the restriction map $R : X^{***} \longrightarrow X^*$ by the equation

$$R(\psi)(x) = \psi(\widehat{x}) \qquad (\psi \in X^{***}, x \in X).$$

Definition 2.1. *Let X be a Banach space. We define the bounded linear map $\Lambda : X^{***} \longrightarrow X^{***}$ by*

$$\Lambda(\psi) = \kappa_{X^*} \circ R(\psi) \qquad (\psi \in X^{***}).$$

We recall that the weak topology on X is denoted by $\sigma(X, X^*)$, the weak-star topology on X^* is $\sigma(X^*, X)$, and the weak-star topology on X^{**} is $\sigma(X^{**}, X^*)$. For a bounded linear map $T : X \longrightarrow Y$, the transpose $T' : Y^* \longrightarrow X^*$ of T is defined by

$$T'(f)(x) = f(T(x)) \qquad (f \in Y^*, x \in X),$$

which is a bounded linear operator and weak*-weak* continuous map.

Definition 2.2. *Let X, Y and Z be Banach spaces and $\pi : X \times Y \longrightarrow Z$ be a bounded bilinear map. The flip map $\pi^r : Y \times X \longrightarrow Z$ is regarded by the equation*

$$\pi^r(y, x) = \pi(x, y) \qquad (x \in X, y \in Y).$$

We also define the transpose of $\pi^* : Z^* \times X \longrightarrow Y^*$ by

$$\pi^*(z^*, x)(y) := z^*(\pi(x, y)) \quad (x \in X, y \in Y, z^* \in Z^*).$$

We note that the maps $\pi^{**} : Y^{**} \times Z^* \longrightarrow X^{**}$ and $\pi^{***} : X^{**} \times Y^{**} \longrightarrow Z^{**}$ can be defined, in a similar way. It is east to see that for each $y^{**} \in Y^{**}$ the map $\pi^{***}(\cdot, y^{**}) : X^{**} \longrightarrow Z^{**}$ is weak*-continuous and for each $x^{**} \in X^{**}$ the map $\pi^{***}(x^{**}, \cdot) : Y^{**} \longrightarrow Z^{**}$ is weak*-continuous. Moreover, if $x^{**} \in X^{**}$, $y^{**} \in Y^{**}$ and (x_α) and (y_β) are nets in X and Y such that $x^{**} = w^* - \lim \widehat{x_\alpha}$ and $y^{**} = w^* - \lim \widehat{y_\beta}$, then it is straightforward to check that

$$\pi^{***}(x^{**}, y^{**}) = w^* - \lim_\alpha \lim_\beta \pi\widehat{(x_\alpha, y_\beta)}, \tag{2.1}$$

and

$$\pi^{r***r}(x^{**}, y^{**}) = w^* - \lim_\beta \lim_\alpha \pi\widehat{(x_\alpha, y_\beta)}. \tag{2.2}$$

Remark 2.3. *Suppose that A is a Banach algebra with the multiplication map $m : A \times A \longrightarrow A$. In this case, it is easy to see that (A^{**}, m^{***}) and (A^{**}, m^{r***r}) are Banach algebras. The maps m^{***} and m^{r***r} are called first and second Arens product and denoted by $\Box$ and $\Diamond$, respectively.*

Let A be a Banach algebra and X be a Banach A-module with the left module action $\pi_1 : A \times X \longrightarrow X$ and the right module action $\pi_2 : X \times A \longrightarrow X$. Similar to [10], we denote this module structure by the triple (π_1, X, π_2). It is straightforward to check that X^* is a Banach A-module where the left and right module actions are π_2^{r*r} and π_1^*, respectively. Moreover, X^{**} is a Banach $(A^{**}, \Box)$-module with the left and right module actions π_1^{***} and π_2^{***}. By this fact, we conclude that the triple $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$ is a Banach $(A^{**}, \Box)$-module.

3. Main results

Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. A bounded linear map $D : A \longrightarrow X$ is a derivation if

$$D(ab) = \pi_1(a, D(b)) + \pi_2(D(a), b) \qquad (a, b \in A).$$

Now, consider the A-module structure $(\pi_2^{r*r}, X^*, \pi_1^*)$ and suppose that $D : A \longrightarrow X^*$ is a derivation. In this section, we obtain a necessary and sufficient condition such that the map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ be a derivation by regarding the A^{**}-module structure $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$.

We commence this section with the following elementary lemma.

Lemma 3.1. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. If $T : A \longrightarrow X^*$ is a bounded linear map, then*

$$\pi_2^{***r*r}(A^{**}, \Lambda(T''(A^{**}))) \subseteq X^*.$$

Proof . Using a classical theorem in functional analysis, we have $(X^{**}, wk^*)^* = X^*$, see [5, Chapter V, Theorem 1.3]. By this fact, suppose that $x^{**} \in X^{**}$ and (x^{**}_α) is a net in X^{**} such that $x^{**}_\alpha \longrightarrow x^{**}$ in $\sigma(X^{**}, X^*)$. Therefore, for each $F, G \in A^{**}$, we have

$$\begin{aligned}\lim_\alpha \pi_2^{***r*r}(F, \Lambda(T''(G)))(x^{**}_\alpha) &= \lim_\alpha \pi_2^{***r*}(\Lambda(T''(G)), F)(x^{**}_\alpha)\\ &= \lim_\alpha \Lambda(T''(G))(\pi_2^{***}(x^{**}_\alpha, F))\\ &= \lim_\alpha (\pi_2^{***}(x^{**}_\alpha, F))(R(T''(G)))\\ &= \lim_\alpha x^{**}_\alpha(\pi_2^{**}(F, R(T''(G)))).\end{aligned}$$

Since $x^{**}_\alpha \longrightarrow x^{**}$ in $\sigma(X^{**}, X^*)$, by the above equation we conclude that

$$\begin{aligned}\lim_\alpha \pi_2^{***r*r}(F, \Lambda(T''(G)))(x^{**}_\alpha) &= x^{**}(\pi_2^{**}(F, R(T''(G))))\\ &= \pi_2^{***r*r}(F, \Lambda(T''(G)))(x^{**}).\end{aligned}$$

This follows that $\pi_2^{***r*r}(F, \Lambda(T''(G))) \in X^*$ as required. □

Theorem 3.2. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. If $D : A \longrightarrow X^*$ is a continuous derivation, then for each $F, G \in A^{**}$,*

$$\Lambda \circ D''(F\Box G) = \Lambda(\pi_1^{****}(D''(F), G)) + \Lambda(\pi_2^{r*r***}(F, D''(G))).$$

Proof . Let $F, G \in A^{**}$ and (a_α) and (b_β) be two nets in A such that

$$F = w^* - \lim_\alpha \widehat{a_\alpha}, \qquad G = w^* - \lim_\beta \widehat{b_\beta}.$$

Thus,

$$\begin{aligned}D''(F\Box G) &= w^* - \lim_\alpha \lim_\beta D''(\widehat{a_\alpha}\widehat{b_\beta})\\ &= w^* - \lim_\alpha \lim_\beta \widehat{D(a_\alpha b_\beta)}\\ &= w^* - \lim_\alpha \lim_\beta \pi_2^{r*r}(a_\alpha, \widehat{D(b_\beta)) + \pi_1^*}(D(a_\alpha), b_\beta).\end{aligned} \tag{3.1}$$

Since D'' is a weak*-weak* continuous map, so we have

$$D''(\widehat{a_\alpha}) \longrightarrow D''(F), \qquad D''(\widehat{b_\beta}) \longrightarrow D''(G),$$

where limits are taken in $\sigma(A^{**}, A^*)$. Now, by applying the equations (3.1) and (2.1), we conclude that

$$D''(F\Box G) = \pi_1^{****}(D''(F), G) + \pi_2^{r*r***}(F, D''(G)).$$

Hence, it follows that

$$\Lambda \circ D''(F\Box G) = \Lambda(\pi_1^{****}(D''(F), G)) + \Lambda(\pi_2^{r*r***}(F, D''(G))).$$

□

We recall that in throughout this section, X^* is regarded as a A-module with the structure $(\pi_2^{r*r}, X^*, \pi_1^*)$ and moreover X^{***} is a A^{**}-module with the structure $(\pi_2^{***r*r}, X^{***}, \pi_1^{****})$.

The following theorem gives a necessary and sufficient condition such that the map $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ to be a derivation.

Theorem 3.3. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. If $D : A \longrightarrow X^*$ is a continuous derivation, then the following are equivalent:*

(i) $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ *is a derivation.*

(ii) For each $F, G \in A^{**}$ *we have*

$$\Lambda(\pi_1^{****}(D''(F), G)) = \pi_1^{****}(\Lambda \circ D''(F), G).$$

Proof . For each $F, G \in A^{**}$, we first prove that

$$\Lambda(\pi_2^{r*r***}(F, D''(G))) = \pi_2^{***r*r}(F, \Lambda(D''(G))).$$

By Lemma 3.1, it is suffices to show that the above equality holds on X.
Let (a_α) and (b_β) be nets in A such that

$$F = w^* - \lim_\alpha \widehat{a_\alpha}, \qquad G = w^* - \lim_\beta \widehat{b_\beta},$$

and take an arbitrary element $x \in X$. Then

$$\begin{aligned}
\Lambda(\pi_2^{r*r***}(F, D''(G)))(\widehat{x}) &= \pi_2^{r*r***}(F, D''(G))(\widehat{x}) \\
&= \lim_\alpha \lim_\beta \pi_2^{r*r}\widehat{(a_\alpha, D}(b_\beta))(\widehat{x}) \\
&= \lim_\alpha \lim_\beta \pi_2^{r*r}(a_\alpha, D(b_\beta))(x) \\
&= \lim_\alpha \lim_\beta \pi_2^{r*}(D(b_\beta), a_\alpha)(x) \\
&= \lim_\alpha \lim_\beta D(b_\beta)\pi_2(x, a_\alpha) \\
&= \lim_\alpha \pi_2\widehat{(x, a_\alpha})(R(D''(G))).
\end{aligned}$$

So, we conclude that

$$\begin{aligned}
\Lambda(\pi_2^{r*r***}(F, D''(G)))(\widehat{x}) &= \pi_2^{***}(\widehat{x}, F)(R(D''(G))) \\
&= \Lambda(D''(G))(\pi_2^{***}(\widehat{x}, F)) \\
&= \Lambda(D''(G))(\pi_2^{***r}(F, \widehat{x})) \\
&= (\pi_2^{***r*}(\Lambda(D''(G)), F))(\widehat{x}) \\
&= (\pi_2^{***r*r}(F, \Lambda(D''(G))))(\widehat{x}).
\end{aligned}$$

It follows that

$$\Lambda(\pi_2^{r*r***}(F, D''(G))) = \pi_2^{***r*r}(F, \Lambda(D''(G))).$$

Now, by applying Theorem 3.2, we conclude that

$$\Lambda \circ D''(F \Box G) = \Lambda(\pi_1^{****}(D''(F), G)) + \pi_2^{***r*r}(F, \Lambda(D''(G))).$$

Thus $\Lambda \circ D''$ is a derivation if and only if for each $F, G \in A^{**}$,

$$\Lambda(\pi_1^{****}(D''(F), G)) = \pi_1^{****}(\Lambda \circ D''(F), G),$$

and this complete the proof. □

Definition 3.4. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. We define the set $\mathcal{W}$ by*

$$\mathcal{W} = \{f \in X^* : \text{the map } x^{**} \longrightarrow \langle \pi_1^{***}(G, x^{**}), f\rangle \text{ is weak}^* \text{ continuous on } X^{**}, \text{for each } G \in \mathcal{A}^{**}\}.$$

In particular, let A be a Banach algebra, $X = A$ and $\pi_1 = \pi_2 = m$, where m is the multiplication map A. It is easy to see that $\mathcal{W}$ is exactly the set of all almost periodic functionals on A, see [11].

By this fact, the following theorem may be regarded as a generalization of [2, Theorem 3.1].

Theorem 3.5. *Let A be a Banach algebra and (π_1, X, π_2) be a Banach A-module. If $D : A \longrightarrow X^*$ is a continuous derivation, then the following are equivalent:*

(i) $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ *is a derivation.*

(ii) For each $F, G \in A^{**}$, the equality $\Lambda(\pi_1^{****}(D''(F), G)) = \pi_1^{****}(\Lambda \circ D''(F), G)$ holds on X and moreover

$$\Lambda \circ D''(A^{**}) \subseteq \mathcal{W}.$$

Proof .(i)⇒(ii) Suppose that $x^{**} \in X^{**}$ and (x_α^{**}) is a net in X^{**} such that $x_\alpha^{**} \longrightarrow x^{**}$ in $\sigma(X^{**}, X^*)$. Then for each $F, G \in A^{**}$, we have,

$$\lim_\alpha \Lambda \circ D''(F)(\pi_1^{***}(G, x_\alpha^{**})) = \lim_\alpha \pi_1^{****}(\Lambda \circ D''(F), G)(x_\alpha^{**}).$$

It follows from Theorem 3.3, that

$$\begin{aligned} \lim_\alpha \Lambda \circ D''(F)(\pi_1^{***}(G, x_\alpha^{**})) &= \lim_\alpha \Lambda(\pi_1^{****}(D''(F), G))(x_\alpha^{**}) \\ &= \lim_\alpha x_\alpha^{**}(R(\pi_1^{****}(D''(F), G))). \end{aligned}$$

Since $x_\alpha^{**} \longrightarrow x^{**}$ in $\sigma(X^{**}, X^*)$, it follows from the above equation that

$$\begin{aligned} \lim_\alpha \Lambda \circ D''(F)(\pi_1^{***}(G, x_\alpha^{**})) &= x^{**}(R(\pi_1^{****}(D''(F), G))) \\ &= \Lambda(\pi_1^{****}(D''(F), G))(x^{**}) \\ &= \pi_1^{****}(\Lambda \circ D''(F), G)(x^{**}) \\ &= \Lambda \circ D''(F)(\pi_1^{***}(G, x^{**})). \end{aligned}$$

This complete the proof.

(ii)⇒(i) It suffices to show that for each $F, G \in A^{**}$, we have

$$\pi_1^{****}(\Lambda \circ D''(F), G) \in X^*.$$

Suppose that $x^{**} \in X^{**}$ and (x_α^{**}) is a net in X^{**} such that $x_\alpha^{**} \longrightarrow x^{**}$ in $\sigma(X^{**}, X^*)$. By the assumption we have $\Lambda \circ D''(A^{**}) \subseteq \mathcal{W}$ and so

$$\begin{aligned} \lim_\alpha \pi_1^{****}(\Lambda \circ D''(F), G)(x_\alpha^{**}) &= \lim_\alpha \Lambda \circ D''(F)(\pi_1^{***}(G, x_\alpha^{**})) \\ &= \Lambda \circ D''(F)(\pi_1^{***}(G, x^{**})) \\ &= \pi_1^{****}(\Lambda \circ D''(F), G)(x^{**}), \end{aligned}$$

and this complete the proof. □

As a consequence, we have the following characterization for weakly compact derivations.

Corollary 3.6. *Let A be a Banach algebra, (π_1, X, π_2) be a Banach A-module and $D : A \longrightarrow X^*$ be a weakly compact derivation. Then $\Lambda \circ D'' : A^{**} \longrightarrow X^{***}$ is a derivation if and only if $D''(A^{**}) \subseteq \mathcal{W}$.*

Proof . First, suppose that $\Lambda \circ D''$ is a derivation. By Theorem 3.5, we conclude that $\Lambda \circ D''(A^{**}) \subseteq \mathcal{W}$. Since D is weakly compact, so by [5, Theorem 5.5] we have $D''(A^{**}) \subseteq X^*$. Hence

$$D''(A^{**}) \subseteq \Lambda \circ D''(A^{**}) \subseteq \mathcal{W}.$$

Conversely, let $D''(A^{**}) \subseteq \mathcal{W}$. By Theorem 3.5, it suffices to show that for each $F, G \in A^{**}$, the equation

$$\Lambda(\pi_1^{****}(D''(F), G)) = \pi_1^{****}(\Lambda \circ D''(F), G),$$

holds on X. By assumption, we can put $D''(F) = \widehat{f}$, where $f \in X^*$. Now, for an arbitrary element $x \in X$, we have

$$\begin{aligned}
\Lambda(\pi_1^{****}(D''(F), G))(\widehat{x}) &= \pi_1^{****}(D''(F), G)(\widehat{x}) \\
&= D''(F)(\pi_1^{***}(G, \widehat{x})) \\
&= \widehat{f}(\pi_1^{***}(G, \widehat{x})) \\
&= (\pi_1^{****}(\widehat{f}, G))(\widehat{x}) \\
&= \pi_1^{****}(\Lambda \circ D''(F), G)(\widehat{x}),
\end{aligned}$$

as required. □

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