Int. J. Nonlinear Anal. Appl. 3 (2012) No. 2, 44-48
ISSN: 2008-6822 (electronic)
http://www.ijnaa.semnan.ac.ir

The Convexity of the Integral Operator on the Class $B(\mu, \alpha)$

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(Communicated by M. Eshaghi Gordji)

Abstract

In this paper, we study the convexity of the integral operator $\int_0^z \prod_{i=1}^n \left(te^{f_i(t)}\right)^{\gamma_i} dt$ where the functions f_i, $i \in \{1, 2, ..., n\}$ satisfy the condition

$$\left| f_i'(z) \left(\frac{z}{f_i(z)}\right)^{\mu_i} - 1 \right| < 1 - \alpha_i, \qquad i \in \{1, 2, ..., n\}.$$

Keywords: Analytic Functions, Integral Operator, Starlike Functions, Convex Functions.
2010 MSC: Primary 39B82; Secondary 39B52.

1. Introduction and preliminaries

Let A denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disk

$$U = \{z \in \mathbb{C} : |z| < 1\}$$

and satisfy the following usual normalization condition

$$f(0) = f'(0) - 1 = 0.$$

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Received: October 2011 *Revised:* November 2011

We denote by S the subclass of A consisting of functions $f(z)$ which are univalent in U.
A function $f \in A$ is a starlike function by the order α, $0 \leq \alpha < 1$ if it satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in U).$$

We denote this class with $S^*(\alpha)$.
A function $f \in A$ is a convex function by the order α, $0 \leq \alpha < 1$ if it satisfies

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in U)$$

We denote this class with $K(\alpha)$.
A function $f \in A$ is in the class $\mathcal{R}(\alpha)$ if and only if

$$Re\left(f'(z)\right) > \alpha, \qquad (z \in U).$$

It is well known that $K(\alpha) \subset S^*(\alpha) \subset S$.
Frasin and Jahangiri [1] defined the family $B(\mu, \alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ so that it consists of functions $f \in A$ satisfying the condition

$$\left| f'(z)\left(\frac{z}{f(z)}\right)^{\mu} - 1 \right| < 1 - \alpha, \qquad (z \in U) \tag{1.1}$$

The family $B(\mu, \alpha)$ is a comprehensive class of analytic functions which includes various classes of analytic univalent functions as well as some very well-known ones. For example, $B(1, \alpha) = S^*(\alpha)$ and $B(0, \alpha) = R(\alpha)$. Frasin and Darus in [2] (see also [3], [4]) has been introduced another interesting subclass $B(2, \alpha) = B(\alpha)$.
In this paper, we will obtain the convexity of the following integral operator

$$F_{\gamma_1, \gamma_2, \ldots, \gamma_n}(z) = \int_0^z \prod_{i=1}^{n} \left(te^{f_i(t)}\right)^{\gamma_i} dt \tag{1.2}$$

where the functions $f_i(t)$ are in $B(\mu_i, \alpha_i)$ for $i \in \{1, 2, ..., n\}$.
In the case $n = 1$ we obatin the integral operator introduced by Frasin and Ahmad in paper [5].

Lemma 1.1. *(General Schwarz Lemma) [6] Let the function f be regular in the disk $U_R = \{z \in \mathbb{C} : |z| < R\}$, with $|f(z)| < M$ for fixed M. If f has one zero with multiplicity order bigger than m for $z = 0$, then*

$$|f(z)| \leq \frac{M}{R^m}|z|^m \qquad (z \in U_R).$$

The equality can hold only if

$$f(z) = e^{i\theta}\frac{M}{R^m}z^m,$$

where θ is constant.

2. Main Results

Theorem 2.1. *Let* $f_i(z) \in B(\mu_i, \alpha_i)$*,* $\mu_i \geq 0$*,* $0 \leq \alpha_i < 1$ *for all* $i \in \{1, 2, ..., n\}$*. If* $|f_i(z)| \leq M_i, (M_i \geq 1, z \in U)$ *for all* $i \in \{1, 2, ..., n\}$ *then the integral operator* $F_{\gamma_1, \gamma_2, ..., \gamma_n}(z)$ *defined by (1.2) is in* $K(\delta)$ *where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i| \left((2 - \alpha_i) M_i^{\mu_i} + 1\right)$$

and

$$\sum_{i=1}^{n} |\gamma_i| \left((2 - \alpha_i) M_i^{\mu_i} + 1\right) \leq 1, \qquad \gamma_i \in \mathbb{C}$$

for all $i = 1, 2, ..., n$*.*

Proof . From (1.2) we obtain:

$$F'_{\gamma_1, \gamma_2, ..., \gamma_n}(z) = \prod_{i=1}^{n} \left(z e^{f_i(z)}\right)^{\gamma_i}$$

and

$$F''_{\gamma_1, \gamma_2, ..., \gamma_n}(z) = \gamma_i \sum_{i=1}^{n} \left(z e^{f_i(z)}\right)^{\gamma_i - 1} \left(e^{f_i(z)} + z f'_i(z) e^{f_i(z)}\right) \prod_{\substack{k=1 \\ k \neq i}}^{n} \left(z e^{f_k(z)}\right)^{\gamma_k}.$$

After the calculus we obtain that:

$$\frac{F''_{\gamma_1, \gamma_2, ..., \gamma_n}(z)}{F'_{\gamma_1, \gamma_2, ..., \gamma_n}(z)} = \sum_{i=1}^{n} \gamma_i \left(\frac{1}{z} + f'_i(z)\right) \tag{2.1}$$

Multiply the relation (2.1) with z and applying modulus, we obtain:

$$\left|\frac{z F''_{\gamma_1, \gamma_2, ..., \gamma_n}(z)}{F'_{\gamma_1, \gamma_2, ..., \gamma_n}(z)}\right| = \sum_{i=1}^{n} |\gamma_i| \left(|z f'_i(z) + 1|\right)$$

$$\leq \sum_{i=1}^{n} |\gamma_i| \left(\left|f'_i(z) \left(\frac{z}{f_i(z)}\right)^{\mu_i}\right| \left|\left(\frac{f_i(z)}{z}\right)^{\mu_i}\right| |z| + 1\right) \tag{2.2}$$

Applying the General Schwarz lemma for the functions f_i, we have

$$|f_i(z)| \leq M_i |z| \qquad (z \in U; i \in \{1, 2, ..., n\})$$

Therefore, from (2.2), we obtain

$$\left|\frac{z F''_{\gamma_1, \gamma_2, ..., \gamma_n}(z)}{F'_{\gamma_1, \gamma_2, ..., \gamma_n}(z)}\right| \leq \sum_{i=1}^{n} |\gamma_i| \left(\left(\left|f'_i(z)\left(\frac{z}{f_i(z)}\right)^{\mu_i} - 1\right| + 1\right) m_i^{\mu_i} + 1\right) \tag{2.3}$$

From (1.1) and (2.3), we have

$$\left|\frac{z F''_{\gamma_1, \gamma_2, ..., \gamma_n}(z)}{F'_{\gamma_1, \gamma_2, ..., \gamma_n}(z)}\right| \leq \sum_{i=1}^{n} |\gamma_i| \left((2 - \alpha_i) M_i^{\mu_i} + 1\right) = 1 - \delta.$$

So

$$\left|\frac{zF''_{\gamma_1,\gamma_2\,,\ldots,\gamma_n}(z)}{F'_{\gamma_1,\gamma_2\,,\ldots,\gamma_n}(z)}\right| \leq 1$$

which imply that the integral operator $F_{\gamma_1,\gamma_2\,,\ldots,\gamma_n}(z)$ is in $K(\delta)$. $\square$

Letting $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \alpha$, $\mu_1 = \mu_2 = \ldots = \mu_n = \mu$ and $M_1 = M_2 = \ldots = M_n = M$ in Theorem 2.1, we have

Corollary 2.2. *Let $f_i(z)$ be in the class $B(\mu,\alpha)$, $\mu \geq 0$, $0 \leq \alpha < 1$ for all $i = 1,2,...,n$. If $|f_i(z)| \leq M\,(M \geq 1; z \in U)$ then the integral operator $F_{\gamma_1,\gamma_2\,,\ldots,\gamma_n}(z)$ defined by (1.2) is in $K(\delta)$ where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i|\left((2-\alpha)\,M^{\mu}+1\right)$$

and

$$\sum_{i=1}^{n} |\gamma_i|\left((2-\alpha)\,M^{\mu}+1\right) \leq 1, \gamma_i \in \mathbb{C}$$

for all $i = 1,2,...,n$.

Letting $\delta = 0$ in Theorem 2.1, we have

Corollary 2.3. *Let $f_i(z) \in B(\mu_i,\alpha_i)$, $\mu_i \geq 0$, $0 \leq \alpha_i < 1$ for all $i = 1,2,...,n$. If $|f_i(z)| \leq M_i, (M_i \geq 1, z \in U)$ for all $i \in \{1,2,...,n\}$ then the integral operator $F_{\gamma_1,\gamma_2\,,\ldots,\gamma_n}(z)$ defined by (1.2) is convex function in U where*

$$\sum_{i=1}^{n} |\gamma_i|\left((2-\alpha_i)\,M_i^{\mu_i}+1\right) = 1, \qquad \gamma_i \in \mathbb{C}$$

for all $i = 1,2,...,n$.

Letting $\mu_1 = \mu_2 = \ldots = \mu_n = 0$ in Theorem 2.1, we have

Corollary 2.4. *Let $f_i(z)$ be in the class $R(\alpha_i)$, $0 \leq \alpha_i < 1$ for all $i = 1,2,...,n$. Then the integral operator $F_{\gamma_1,\gamma_2\,,\ldots,\gamma_n}(z)$ defined by (1.2) is in $K(\delta)$ where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i|\,(3-\alpha_i)$$

and

$$\sum_{i=1}^{n} |\gamma_i|\,(3-\alpha_i) \leq 1, \gamma_i \in \mathbb{C}$$

for all $i = 1,2,...,n$.

Letting $\mu_1 = \mu_2 = \ldots = \mu_n = 1$ in Theorem 2.1, we have

Corollary 2.5. *Let* $f_i(z)$ *be in the class* $S^*(\alpha)$, $0 \leq \alpha_i < 1$ *for all* $i = 1, 2, ..., n$. *If* $|f_i(z)| \leq M_i, (M_i \geq 1, z \in U)$ *for all* $i \in \{1, 2, ..., n\}$ *then the integral operator* $F_{\gamma_1, \gamma_2, ..., \gamma_n}(z)$ *defined by* (1.2) *is in* $K(\delta)$ *where*

$$\delta = 1 - \sum_{i=1}^{n} |\gamma_i| \left((2 - \alpha_i) M_i + 1 \right)$$

and

$$\sum_{i=1}^{n} |\gamma_i| \left((2 - \alpha_i) M_i + 1 \right) \leq 1, \gamma_i \in \mathbb{C}$$

for all $i = 1, 2, ..., n$.

Letting $\delta = 0$, $\alpha_1 = \alpha_2 = ... = \alpha_n = 0$, $n = 1$ in Corollary 2.4, we have

Corollary 2.6. *Let* $f(z) \in A$ *be a regular function in* U. *Then the integral operator*

$$F(z) = \int_0^z \left(te^{f(t)} \right)^{\gamma} dt$$

is convex in U *where*

$$|\gamma| = \frac{1}{3}, \qquad \gamma \in \mathbb{C}.$$

3. Acknowledgment

This work was partially supported by the strategic project POSDRU 107/1.5/S/77265, inside POSDRU Romania 2007–2013 co–financed by the European Social Fund-Investing in People.

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