Int. J. Nonlinear Anal. Appl. 11 (2020) No. 1, 37-45
ISSN: 2008-6822 (electronic)
http://dx.doi.org/10.22075/ijnaa.2019.16022.1841

Finite time blow up of solutions of the Kirchhoff-type equation with variable exponents

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(Communicated by Madjid Eshaghi Gordji)

Abstract

In this work, we investigate the following Kirchhoff-type equation with variable exponent nonlinearities

$$u_{tt} - M\left(\|\nabla u\|^2\right) \triangle u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u.$$

We proved the blow up of solutions in finite time by using modified energy functional method.

Keywords: Blow up, Kirchhoff-type equation, Variable exponent.
2010 MSC: 35B44, 35L72.

1. Introduction

In this article, we investigate the following Kirchhoff-type equation with variable exponent nonlinearities

$$\begin{cases} u_{tt} - M\left(\|\nabla u\|^2\right) \triangle u + |u_t|^{p(x)-2} u_t = |u|^{q(x)-2} u, & (x,t) \in \Omega \times (0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with smooth boundary $\partial\Omega$ in R^n $(n \geq 1)$ and $M(s) = \alpha + \beta s^{\gamma}$, $\alpha, \beta \geq 0,\ \gamma \geq 1$. The variable exponents $p(.)$ and $q(.)$ are given as measurable functions on Ω satisfying

$$2 \leq p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ \leq q^* \tag{1.2}$$

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Received: December 2017 *Revised:* November 2018

where

$$\begin{aligned} p^{-} &= ess\inf_{x\in\Omega} p\left(x\right), \quad p^{+} = ess\sup_{x\in\Omega} p\left(x\right), \\ q^{-} &= ess\inf_{x\in\Omega} q\left(x\right), \quad q^{+} = ess\sup_{x\in\Omega} q\left(x\right), \end{aligned}$$

and

$$q^{*} = \begin{cases} \infty, & \text{if } n = 1, 2, \\ \frac{2n}{n-2}, & \text{if } n \geq 3. \end{cases}$$

This type of problems is a generalization of a model introduced by Kirchhoff [5].

The following Kirchhoff type equation

$$u_{tt} - M\left(\|\nabla u\|^{2}\right) \triangle u + g\left(u_{t}\right) = f\left(u\right), \tag{1.3}$$

have been discussed by many authors. For $g\left(u_{t}\right) = u_{t}$, the global existence and nonexistence results can be found in [12, 16]; for $g\left(u_{t}\right) = \left|u_{t}\right|^{p} u_{t},\ p > 0$, the main results of existence and nonexistence are in [1, 9].

When $M\left(s\right) \equiv 1$, (1.3) becomes the classical wave equation

$$u_{tt} - \triangle u + g\left(u_{t}\right) = f\left(u\right).$$

In [4, 7, 8, 10, 15], the authors studied existence and blow up of solutions.

Recently, In [11], Messaoudi et al. studied local existence and blow up of the solutions for the following wave equation with variable exponent nonlinearities

$$u_{tt} - \triangle u + \left|u_{t}\right|^{p(x)-2} u_{t} = \left|u\right|^{q(x)-2} u.$$

Motivated by the above studies, in this paper, we consider the blow up of the solution (1.1) under some conditions.

The outline of this paper is as follows. In section 2, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}\left(\Omega\right)$ and $W^{1,p(x)}\left(\Omega\right)$. In section 3, the blow up results will be proved.

2. Preliminaries

In this part, we state some results about the variable exponent Lebesgue and Sobolev spaces $L^{p(x)}\left(\Omega\right)$ and $W^{1,p(x)}\left(\Omega\right)$ (see [2, 3, 6, 14]). Also, $\|\cdot\|$ and $\|\cdot\|_{p}$ denote the usual $L^{2}(\Omega)$ norm and $L^{p}(\Omega)$ norm, respectively.

Let $p : \Omega \to [1, \infty]$ be a measurable function, where Ω is a bounded domain of R^{n}. We define the variable exponent Lebesgue space by

$$L^{p(x)}\left(\Omega\right) = \left\{u : \Omega \to R,\ u \text{ is measurable and } \rho_{p(.)}\left(\lambda u\right) \ < \infty, \quad \text{for some } \lambda > 0\right\}$$

where

$$\rho_{p(.)}\left(u\right) = \int_{\Omega} \left|u\right|^{p(x)} dx.$$

Also endowed with the Luxemburg norm

$$\left\|u\right\|_{p(x)} = \inf\left\{\lambda > 0 : \int_{\Omega} \left|\frac{u}{\lambda}\right|^{p(x)} dx \leq 1\right\},$$

$L^{p(x)}(\Omega)$ is a Banach space.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \left\{ u \in L^{p(x)}(\Omega) : \nabla u \text{ exists and } |\nabla u| \in L^{p(x)}(\Omega) \right\}.$$

Variable exponent Sobolev space is a Banach space with respect to the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

The space $W_0^{1,p(x)}(\Omega)$ is defined as the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$ with respect to the norm $\|u\|_{1,p(x)}$. For $u \in W_0^{1,p(x)}(\Omega)$, we can define an equivalent norm

$$\|u\|_{1,p(x)} = \|\nabla u\|_{p(x)}.$$

Let the variable exponents $p(.)$ and $q(.)$ satisfy the log-Hölder continuity condition:

$$|p(x) - p(y)| \leq \frac{A}{\log \frac{1}{|x-y|}}, \text{ for all } x, y \in \Omega \text{ with } |x-y| < \delta, \tag{2.1}$$

where $A > 0$ and $0 < \delta < 1$.

Lemma 2.1. *(Poincare inequality) Let Ω be a bounded domain of R^n and $p(.)$ satisfies log-Hölder condition, then*

$$\|u\|_{p(x)} \leq c \|\nabla u\|_{p(x)}, \text{ for all } u \in W_0^{1,p(x)}(\Omega),$$

where $c = c(p^-, p^+, |\Omega|) > 0$.

Lemma 2.2. *Let $p(.) \in C(\overline{\Omega})$ and $q : \Omega \to [1, \infty)$ be a measurable function and satisfy*

$$\underset{x\in\overline{\Omega}}{essinf}(p^*(x) - q(x)) > 0.$$

Then the Sobolev embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is continuous and compact. Where

$$p^*(x) = \begin{cases} \frac{np^-}{n-p^-}, & \text{if } p^- < n \\ \infty, & \text{if } p^- \geq n. \end{cases}$$

Next, we state the local existence theorem of problem (1.1), that can be obtained by combining arguments in [11, 13].

Theorem 2.3. *(Local existence). Assume that (1.2) and (2.1) hold, and that $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, then there exists a unique solution u of (1.1) satisfying*

$$u \in C\left([0,T); H_0^1(\Omega)\right), \quad u_t \in C\left([0,T); L^2(\Omega)\right) \cap L^{p(.)}(\Omega \times (0,T)).$$

3. Blow up

In this part, we will proved the blow up of the solution for problem (1.1). Firstly, we give following lemma:

Lemma 3.1. *[11] If $q:\Omega\rightarrow[1,\infty)$ is a measurable function and*

$$2\leq q^{-}\leq q\left(x\right)\leq q^{+}<\frac{2n}{n-2};\quad n\geq 3 \tag{3.1}$$

holds. Then, we have following inequalities:

i)

$$\rho_{q(.)}^{\frac{s}{q^{-}}}\left(u\right)\leq c\left(\left\Vert \nabla u\right\Vert^{2}+\rho_{q(.)}\left(u\right)\right), \tag{3.2}$$

ii)

$$\left\Vert u\right\Vert_{q^{-}}^{s}\leq c\left(\left\Vert \nabla u\right\Vert^{2}+\left\Vert u\right\Vert_{q^{-}}^{q^{-}}\right), \tag{3.3}$$

iii)

$$\rho_{q(.)}^{\frac{s}{q^{-}}}\left(u\right)\leq c\left(\left\vert H\left(t\right)\right\vert+\left\Vert u_{t}\right\Vert^{2}+\rho_{q(.)}\left(u\right)\right), \tag{3.4}$$

iv)

$$\left\Vert u\right\Vert_{q^{-}}^{s}\leq c\left(\left\vert H\left(t\right)\right\vert+\left\Vert u_{t}\right\Vert^{2}+\left\Vert u\right\Vert_{q^{-}}^{q^{-}}\right), \tag{3.5}$$

v)

$$c\left\Vert u\right\Vert_{q^{-}}^{q^{-}}\leq\rho_{q(.)}\left(u\right) \tag{3.6}$$

for any $u\in H_{0}^{1}\left(\Omega\right)$ and $2\leq s\leq q^{-}$. Where $c>1$ a positive constant and $H\left(t\right)=-E\left(t\right)$.

Now, we state and prove our main result:

Theorem 3.2. *Let the assumptions of Theorem 3 be satisfied and assume that*

$$E\left(0\right)<0.$$

Then the solution (1.1) blows up in finite time.

Proof . Multiplying u_t on two sides of the problem (1.1) and integrate over the domain Ω, we have

$$\frac{d}{dt}\left[\frac{1}{2}\left\Vert u_{t}\right\Vert^{2}+\frac{1}{2}\left\Vert \nabla u\right\Vert^{2}+\frac{1}{2\left(\gamma+1\right)}\left\Vert \nabla u\right\Vert^{2(\gamma+1)}-\int_{\Omega}\frac{1}{q\left(x\right)}\left\vert u\right\vert^{q(x)}dx\right]=-\int_{\Omega}\frac{1}{p\left(x\right)}\left\vert u_{t}\right\vert^{p(x)}dx,$$

$$E'\left(t\right)=-\int_{\Omega}\left\vert u_{t}\right\vert^{p(x)}dx, \tag{3.7}$$

where

$$E\left(t\right)=\frac{1}{2}\left\Vert u_{t}\right\Vert^{2}+\frac{1}{2}\left\Vert \nabla u\right\Vert^{2}+\frac{1}{2\left(\gamma+1\right)}\left\Vert \nabla u\right\Vert^{2(\gamma+1)}-\int_{\Omega}\frac{1}{q\left(x\right)}\left\vert u\right\vert^{q(x)}dx \tag{3.8}$$

Set
$$H(t) = -E(t)$$
then $E(0) < 0$ and (3.7) gives $H(t) \geq H(0) > 0$. Also, by the definition $H(t)$, we have
$$\begin{aligned} H(t) &= -\frac{1}{2}\|u_t\|^2 - \frac{1}{2}\|\nabla u\|^2 - \frac{1}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} + \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ &\leq \int_\Omega \frac{1}{q(x)}|u|^{q(x)}dx \\ &\leq \frac{1}{q^-}\rho_{q(.)}(u). \end{aligned} \tag{3.9}$$
Define
$$\Psi(t) = H^{1-\sigma}(t) + \varepsilon\int_\Omega uu_t dx, \tag{3.10}$$
where ε small to be chosen later and
$$0 < \sigma \leq \min\left\{\frac{q^- - p^+}{(p^+-1)q^-}, \frac{q^- - 2}{2q^-}\right\}. \tag{3.11}$$

By taking a derivative of (3.10) and using Eq. (1.1), we obtain
$$\begin{aligned} \Psi'(t) &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\int_\Omega \left(u_t^2 + uu_{tt}\right)dx \\ &= (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon\|u_t\|^2 - \varepsilon\|\nabla u\|^2 \\ &\quad -\varepsilon\|\nabla u\|^{2(\gamma+1)} + \varepsilon\int_\Omega |u|^{q(.)}dx - \varepsilon\int_\Omega uu_t|u_t|^{p(.)-2}dx. \end{aligned} \tag{3.12}$$
By using the definition of the $H(t)$, it follows that
$$\begin{aligned} -\varepsilon q^-(1-\xi)H(t) &= \frac{\varepsilon q^-(1-\xi)}{2}\|u_t\|^2 + \frac{\varepsilon q^-(1-\xi)}{2}\|\nabla u\|^2 \\ &\quad + \frac{\varepsilon q^-(1-\xi)}{2(\gamma+1)}\|\nabla u\|^{2(\gamma+1)} - \varepsilon q^-(1-\xi)\int_\Omega \frac{1}{q(x)}|u|^{q(.)}dx, \end{aligned} \tag{3.13}$$
where $0 < \xi < 1$.

Add and subtract (3.13) into (3.12), we obtain
$$\begin{aligned} \Psi'(t) &\geq (1-\sigma)H^{-\sigma}(t)H'(t) + \varepsilon q^-(1-\xi)H(t) \\ &\quad + \varepsilon\left(\frac{q^-(1-\xi)}{2} + 1\right)\|u_t\|^2 + \varepsilon\left(\frac{q^-(1-\xi)}{2} - 1\right)\|\nabla u\|^2 \\ &\quad + \varepsilon\left(\frac{q^-(1-\xi)}{2} - 1\right)\|\nabla u\|^{2(\gamma+1)} + \varepsilon\xi\int_\Omega |u|^{q(.)}dx - \varepsilon\int_\Omega uu_t|u_t|^{p(.)-2}dx. \end{aligned} \tag{3.14}$$
Then, for ξ small enough, we get
$$\begin{aligned} \Psi'(t) &\geq \varepsilon\beta\left[H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + \rho_{q(.)}(u)\right] \\ &\quad + (1-\sigma)H^{-\sigma}(t)H'(t) - \varepsilon\int_\Omega uu_t|u_t|^{p(.)-2}dx \end{aligned} \tag{3.15}$$

where

$$\beta = \min\left\{q^-\left(1-\xi\right),\ \varepsilon\xi,\ \frac{q^-\left(1-\xi\right)}{2}-1,\ \frac{q^-\left(1-\xi\right)}{2}+1\right\} > 0$$

and

$$\rho_{q(.)}\left(u\right) = \int\limits_{\Omega} \left|u\right|^{q(.)} dx.$$

In order to estimate the last term in (3.15), we make use of the following Young inequality

$$XY \leq \frac{\delta^k X^k}{k} + \frac{\delta^{-l} Y^l}{l},$$

where $X, Y \geq 0,\ \delta > 0,\ k, l \in R^+$ such that $\frac{1}{k}+\frac{1}{l}=1$. Consequently, applying the previous we have

$$\begin{aligned}
\int_{\Omega} u\left|u_t\right|^{p(.)-1} dx &\leq \int\limits_{\Omega} \frac{1}{p\left(x\right)}\delta^{p(x)}\left|u\right|^{p(x)} dx + \int\limits_{\Omega} \frac{p\left(x\right)-1}{p\left(x\right)}\delta^{-\frac{p(x)}{p(x)-1}}\left|u_t\right|^{p(x)} dx \\
&\leq \frac{1}{p^-}\int\limits_{\Omega} \delta^{p(x)}\left|u\right|^{p(x)} dx + \frac{p^+-1}{p^+}\int\limits_{\Omega} \delta^{-\frac{p(x)}{p(x)-1}}\left|u_t\right|^{p(x)} dx \qquad (3.16)
\end{aligned}$$

where δ is constant depending on the time t and specified later. Inserting estimate (3.16) into (3.15), we get

$$\begin{aligned}
\Psi'\left(t\right) \geq\ & \varepsilon\beta\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\rho_{q(.)}\left(u\right)\right] \\
&+\left(1-\sigma\right)H^{-\sigma}\left(t\right)H'\left(t\right)-\varepsilon\frac{1}{p^-}\int\limits_{\Omega}\delta^{p(x)}\left|u\right|^{p(x)} dx \\
&-\varepsilon\frac{p^+-1}{p^+}\int\limits_{\Omega}\delta^{-\frac{p(x)}{p(x)-1}}\left|u_t\right|^{p(x)} dx \qquad (3.17)
\end{aligned}$$

Therefore, by taking δ so that $\delta^{-\frac{p(x)}{p(x)-1}} = kH^{-\sigma}\left(t\right)$, where $k > 0$ is specified later, we obtain

$$\begin{aligned}
\Psi'\left(t\right) \geq\ & \varepsilon\beta\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\rho_{q(.)}\left(u\right)\right] \\
&+\left(1-\sigma\right)H^{-\sigma}\left(t\right)H'\left(t\right)-\varepsilon\frac{1}{p^-}\int\limits_{\Omega}k^{1-p(x)}H^{\sigma(p(x)-1)}\left(t\right)\left|u\right|^{p(x)} dx \\
&-\varepsilon\frac{p^+-1}{p^+}\int\limits_{\Omega}kH^{-\sigma}\left(t\right)\left|u_t\right|^{p(x)} dx \\
\geq\ & \varepsilon\beta\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\rho_{q(.)}\left(u\right)\right] \\
&+\left(1-\sigma\right)H^{-\sigma}\left(t\right)H'\left(t\right)-\varepsilon\frac{k^{1-p^-}}{p^-}H^{\sigma\left(p^+-1\right)}\left(t\right)\int\limits_{\Omega}\left|u\right|^{p(x)} dx \\
&-\varepsilon\left(\frac{p^+-1}{p^+}\right)kH^{-\sigma}\left(t\right)\int\limits_{\Omega}\left|u_t\right|^{p(x)} dx \\
\geq\ & \varepsilon\beta\left[H\left(t\right)+\left\|u_t\right\|^2+\left\|\nabla u\right\|^2+\left\|\nabla u\right\|^{2(\gamma+1)}+\rho_{q(.)}\left(u\right)\right] \\
&+\left[\left(1-\sigma\right)-\varepsilon\left(\frac{p^+-1}{p^+}\right)k\right]H^{-\sigma}\left(t\right)H'\left(t\right)-\varepsilon\frac{k^{1-p^-}}{p^-}H^{\sigma\left(p^+-1\right)}\left(t\right)\int\limits_{\Omega}\left|u\right|^{p(x)} dx. \qquad (3.18)
\end{aligned}$$

By using (3.6) and (3.9), we get

$$\begin{aligned} H^{\sigma\left(p^{+}-1\right)}(t)\int_{\Omega}|u|^{p(x)}\,dx &\leq H^{\sigma\left(p^{+}-1\right)}(t)\left[\int_{\Omega_{-}}|u|^{p^{-}}\,dx+\int_{\Omega_{+}}|u|^{p^{+}}\,dx\right]\\ &\leq H^{\sigma\left(p^{+}-1\right)}(t)\,c\left[\left(\int_{\Omega_{-}}|u|^{q^{-}}\,dx\right)^{\frac{p^{-}}{q^{-}}}+\left(\int_{\Omega_{+}}|u|^{q^{-}}\,dx\right)^{\frac{p^{+}}{q^{-}}}\right]\\ &= H^{\sigma\left(p^{+}-1\right)}(t)\,c\left[\|u\|_{q^{-}}^{p^{-}}+\|u\|_{q^{-}}^{p^{+}}\right]\\ &\leq c\left(\frac{1}{q^{-}}\rho_{q(.)}(u)\right)^{\sigma\left(p^{+}-1\right)}\left[\left(\rho_{q(.)}(u)\right)^{\frac{p^{-}}{q^{-}}}+\left(\rho_{q(.)}(u)\right)^{\frac{p^{+}}{q^{-}}}\right]\\ &= c_{1}\left[\left(\rho_{q(.)}(u)\right)^{\frac{p^{-}}{q^{-}}+\sigma\left(p^{+}-1\right)}+\left(\rho_{q(.)}(u)\right)^{\frac{p^{+}}{q^{-}}+\sigma\left(p^{+}-1\right)}\right] \end{aligned} \tag{3.19}$$

where $\Omega_{-}=\{x\in\Omega:|u|<1\}$ and $\Omega_{+}=\{x\in\Omega:|u|\geq 1\}$.

We then use Lemma 4 and (3.11), for

$$s=p^{-}+\sigma q^{-}\left(p^{+}-1\right)\leq q^{-}$$

and

$$s=p^{+}+\sigma q^{-}\left(p^{+}-1\right)\leq q^{-},$$

to deduce, from (3.19),

$$H^{\sigma\left(p^{+}-1\right)}(t)\int_{\Omega}|u|^{p(x)}\,dx\leq c_{1}\left[\|\nabla u\|^{2}+\rho_{q(.)}(u)\right]. \tag{3.20}$$

Thus, inserting estimate (3.20) into (3.18), we have

$$\begin{aligned} \Psi'(t) &\geq \varepsilon\left(\beta-\frac{k^{1-p^{-}}}{p^{-}}c_{1}\right)\left[H(t)+\|u_{t}\|^{2}+\|\nabla u\|^{2}+\|\nabla u\|^{2(\gamma+1)}+\rho_{q(.)}(u)\right]\\ &\quad+\left[(1-\sigma)-\varepsilon\left(\frac{p^{+}-1}{p^{+}}\right)k\right]H^{-\sigma}(t)\,H'(t). \end{aligned} \tag{3.21}$$

At this moment, choosing k large enough so that $\gamma=\beta-\frac{k^{1-p^{-}}}{p^{-}}c_{1}>0$, and picking ε small enough such that $(1-\sigma)-\varepsilon\left(\frac{p^{+}-1}{p^{+}}\right)k\geq 0$ and

$$\Psi(t)\geq\Psi(0)=H^{1-\sigma}(0)+\varepsilon\int_{\Omega}u_{0}u_{1}dx>0,\ \forall t\geq 0. \tag{3.22}$$

Consequently, (3.21) yields

$$\begin{aligned} \Psi'(t) &\geq \varepsilon\gamma\left[H(t)+\|u_{t}\|^{2}+\|\nabla u\|^{2}+\|\nabla u\|^{2(\gamma+1)}+\rho_{q(.)}(u)\right]\\ &\geq \varepsilon\gamma\left[H(t)+\|u_{t}\|^{2}+\|\nabla u\|^{2}+\|\nabla u\|^{2(\gamma+1)}+\|u\|_{q^{-}}^{q^{-}}\right], \end{aligned} \tag{3.23}$$

due to (3.6). Therefore we get

$$\Psi(t) \geq \Psi(0) > 0, \text{ for all } t \geq 0.$$

On the other hand, applying Hölder inequality, we obtain

$$\begin{aligned}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} &\leq \|u\|^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}} \\ &\leq C\left(\|u\|_{q^-}^{\frac{1}{1-\sigma}} \|u_t\|^{\frac{1}{1-\sigma}}\right).\end{aligned}$$

Young inequality gives

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u\|_{q^-}^{\frac{\mu}{1-\sigma}} + \|u_t\|^{\frac{\theta}{1-\sigma}}\right), \tag{3.24}$$

for $\frac{1}{\mu}+\frac{1}{\theta}=1$. We take $\theta = 2(1-\sigma)$, to obtain $\frac{\mu}{1-\sigma} = \frac{2}{1-2\sigma} \leq q^-$ by (3.11). Therefore, (3.24) becomes

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|u\|_{q^-}^s\right),$$

where $\frac{2}{1-2\sigma} \leq q^-$. By using (3.5), we get

$$\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}} \leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + H(t)\right).$$

Thus,

$$\begin{aligned}\Psi^{\frac{1}{1-\sigma}}(t) &= \left[H^{1-\sigma}(t) + \varepsilon\int_\Omega uu_t dx\right]^{\frac{1}{1-\sigma}} \\ &\leq 2^{\frac{\sigma}{1-\sigma}}\left(H(t) + \varepsilon^{\frac{1}{1-\sigma}}\left|\int_\Omega uu_t dx\right|^{\frac{1}{1-\sigma}}\right) \\ &\leq C\left(\|u_t\|^2 + \|u\|_{q^-}^{q^-} + H(t)\right) \\ &\leq C\left(H(t) + \|u_t\|^2 + \|\nabla u\|^2 + \|\nabla u\|^{2(\gamma+1)} + \|u\|_{q^-}^{q^-}\right)\end{aligned} \tag{3.25}$$

where

$$(a+b)^p \leq 2^{p-1}(a^p + b^p)$$

is used. By combining of (3.23) and (3.25), we arrive

$$\Psi'(t) \geq \xi\Psi^{\frac{1}{1-\sigma}}(t), \tag{3.26}$$

where ξ is a positive constant.

A simple integration of (3.26) over $(0,t)$ yields $\Psi^{\frac{\sigma}{1-\sigma}}(t) \geq \frac{1}{\Psi^{-\frac{\sigma}{1-\sigma}}(0) - \frac{\xi\sigma t}{1-\sigma}}$, which implies that the solution blows up in a finite time T^*, with

$$T^* \leq \frac{1-\sigma}{\xi\sigma\Psi^{\frac{\sigma}{1-\sigma}}(0)}.$$

This completes the proof of the theorem. □

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