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Strong convergence results for fixed points of nearly weak uniformly L-Lipschitzian mappings of I-Dominated mappings

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(Communicated by Madjid Eshaghi Gordji)
(Dedicated to dearest mother, Mrs F. M. Mogbademu on her 75th birthday.)

Abstract

In this paper, we prove strong convergence results for a modified Mann iterative process for a new class of I- nearly weak uniformly $L-$ Lipschitzian mappings in a real Banach space. The class of I- nearly weak uniformly $L-$ Lipschitzian mappings is an interesting generalization of the class of nearly weak uniformly $L-$ Lipschitzian mappings which inturn is a generalization of the class of nearly uniformly $L-$ Lipschitzian mappings which inturn generalises uniformly $L-$ Lipschitzian mappings. Our theorems include some very recent results in fixed point theory and applications, in the context of nearly uniformly $L-$ Lipschitzian mappings.

Keywords: Modified Mann iteration process, Banach space, Fixed point, $I-$ nearly weak uniformly L-Lipschitzian.
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1. Introduction and Preliminaries

Let X be an arbitrary real Banach space with the dual X^*. We denote by J the normalized duality mapping from X into

$$J(x) = \{f \in \mathrm{X}^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\},$$

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for all $x \in X$, and $\langle ., . \rangle$ denotes the generalized duality pairing between elements of X and X^*.

Let K be a nonempty subset of real Banach space X. Let $T : K \to K$ be a mapping.

Definition 1.1. *T is called asymptotically nonexpansive if for each $x, y \in K$*

$$\|T^n x - T^n y\| \leq k\|x - y\| \leq k_n\|x - y\|^2, \forall n \geq 1,$$

where $k_n \subset [1, \infty)$ with $lim_{n\to\infty} k_n = 1$.

Definition 1.2. *T is called asymptotically pseudocontractive with the sequence $k_n \subset [1, \infty)$ if and only if $lim_{n\to\infty} k_n = 1$, and for all $x, y \in K$, there exists $j(x - y) \in J(x - y)$ such that*

$$< T^n x - T^n y, j(x - y) > \leq k_n\|x - y\|^2, \forall n \geq 1.$$

Every asymptotically nonexpansive mappings is seen to be asymptotically pseudocontractive. However, the converse may not be true in the natural sense [9].

Definition 1.3. *T is called uniformly $L-$ Lipschitzian if, for any $x, y \in K$, there exists a constant $L > 1$ such that*

$$\|T^n x - T^n y\| \leq L\|x - y\|, \forall n \geq 1.$$

The theory of fixed point of uniformly $L-$ Lipschitzian mappings has an interesting role in the applied nonlinear functional analysis. Therefore, it is of interest to study these mappings in the theory of fixed point and its applications [1-24].

In recent years, several researchers have given much attention to these classes of nonlinear mappings with T satisfying certain conditions: for results on these, see e.g., [1], [2], [3], [7], [9], [12], [17] and [19].

In 2005, Sahu [19] introduced the following class of nonlinear maps which he called nearly Lipschitzian mappings.

Let K be a nonempty subset of a Banach space X and fix a sequence $\{u_n\}_{n\geq 1} \subset [0, \infty)$ with $lim_{n\to\infty} u_n = 0$. $T : K \to K$ is called nearly Lipschitzian with respect to $\{u_n\}$ if for each $n \in N$, there exists a constant $k_n \geq 1$ such that

$$\|T^n x - T^n y\| \leq k_n(\|x - y\| + u_n), \quad \forall \;\; x, y \in K. \tag{1.1}$$

The infimum $\eta(T^n)$ of constants k_n in (1.1) is called nearly Lipschitz constant of the mapping T.

A nearly Lipschitzian mapping T is said to be

(i) nearly contraction if $\eta(T^n) < 1$ for all $n \in N$;
(ii) nearly nonexpansive if $\eta(T^n) = 1$ for all $n \in N$;
(iii) nearly asymptotically nonexpansive if $\eta(T^n) \geq 1$ for all $n \in N$ and $\lim_{n\to\infty} \eta(T^n) = 1$;
(iv) nearly uniformly $L-$ Lipschitzian if $\eta(T^n) \leq L$ for all $n \in N$;
(v) nearly uniformly $k-$ contraction if $\eta(T^n) \leq k < 1$ for all $n \in N$.

A nearly Lipschitzian mapping T with sequence $\{u_n\}$ is said to be nearly uniformly $L-$ Lipschitzian if $k_n = L$ for all $n \in N$.

The class of nearly uniformly $L-$ Lipschitzian mappings is observed to be more general than the class of uniformly $L-$ Lipschitzian mappings.

For results on nearly uniformly $L-$ Lipschitzian mappings, see e.g., Kim et al., [9], Mogbademu [12,13].

In 2016, the author introduced see [14] the following concepts as an interesting generalization of the class of nearly uniformly $L-$ Lipschitzian mappings:

Let K be a subset of a Banach space X and let $T : K \to K$ be a mapping.

Definition 1.4. *$T : K \to K$ is called nearly weak uniformly L- Lipschitzian with a Lipschitz constant $L > 1$ if there exists a fix sequence $\{u_n\}_{n\geq 1} \subset [0, \infty)$ with $lim_{n\to\infty} u_n = 0$ such that*

$$\|T^n x - T^n y\| \leq L(\|x - y\| + u_n), \ \ \forall \ \ x \in K, \ y \in F(T), n \geq 1. \tag{1.2}$$

Definition 1.4 can be generalized as follows: Let $T, I : K \to K$ be two mappings. T is said to be dominated by I- nearly weak uniformly L-Lipschitzian with a Lipschitz constant $L > 1$, if $F(T) \cap F(I) \neq \emptyset$ and there exists a sequence $\{u_n\}_{n\geq 1} \subset [0, \infty)$ with $lim_{n\to\infty} u_n = 0$ such that

$$\|T^n x - T^n y\| \leq L(\|I^n x - I^n y\| + u_n), \ \ \forall \ \ x \in K, \ y \in F(T) \cap F(I), n \geq 1. \tag{1.3}$$

Remark 1.5. *It follows directly from definition above that, a nearly weak uniformly Lipschitzian mapping is an I- nearly weak uniformly Lipschitzian mapping. The converse of this claim may not be true in general sense. It is easy to see that, if I is an identity mapping, then, I- nearly weak uniformly Lipschitzian mapping concide with nearly weak uniformly Lipschitzian mapping.*

The interest of this paper is to prove strong convergence results for a new class of nonlinear mappings in a real Banach space in which the normalized duality map is norm-norm uniformly continuous on bounded sets. This space covers all uniformly smooth Banach spaces. In particular, it includes the Sobolev spaces $W^{m,p}(\Omega)$, L_p spaces, $1 < p < \infty$.
For our main purpose, we recall the following.

Definition 1.6. *[11] For arbitrary $x_1 \in K$, define the sequence $\{x_n\}_{n=1}^{\infty}$ in K by*

$$x_{n+1} = (1 - a_n)x_n + a_n T^n x_n, \ n \geq 1, \tag{1.4}$$

where $\{a_n\}_{n=1}^{\infty}$ is a sequence in $[0, 1]$.

We observe that the iteration process (1.4) is well known as the modified Mann iteration [11].

We shall need the following lemmas and proposition in our main theorem.

Lemma 1.7. *[4, 5] Let X be real Banach Space and $J : X \to 2^{X^*}$ be the normalized duality mapping. Then, for any $x, y \in X$*

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) >, \forall j(x + y) \in J(x + y).$$

Lemma 1.8. *[15] Let $\psi : [0, \infty) \to [0, \infty)$ be an increasing function with $\psi(x) = 0 \Leftrightarrow x = 0$ and let $\{b_n\}_{n=0}^{\infty}$ be a positive real sequence satisfying*

$$\sum_{n=0}^{\infty} b_n = +\infty \ \ \text{and} \ \ \lim_{n\to\infty} b_n = 0.$$

Suppose that $\{\lambda_n\}_{n=0}^{\infty}$ is a nonnegative real sequence. If there exists an integer $N_0 > 0$ satisfying

$$\lambda_{n+1}^2 < \lambda_n^2 + \sigma_n - b_n\psi(\lambda_{n+1}), \ \ \forall n \geq N_0$$

where $\{\sigma_n\}$ is a sequence of nonnegative numbers such that $\sigma_n = o(b_n)$, then $\lim_{n\to\infty} \lambda_n = 0$.

Proposition 1.9. *Let X be a real Banach space, K a nonempty closed convex subset of X, $T : K \to K$ be I- nearly weak uniformly L_1- Lipschitzian mappings and $I : K \to K$ be nearly weak uniformly L_2- Lipschitzian mappings with $F(T) \cap F(I) \neq \emptyset$. Then, there exist a point $\rho \in F(T) \cap F(I) \neq \emptyset$ and sequences $\{u_n\}_{n\geq 1}, \{v_n\}_{n\geq 1}$ with $u_n, v_n \to 0$ as $n \to \infty$ such that $\|T^n x - T^n \rho\| \leq L_1(\|I^n x - I^n \rho\| + u_n)$ and $\|I^n x - I^n \rho\| \leq L_2(\|x - \rho\| + v_n)$, for all $x \in K$, $1 < L_1 \leq L_2$ and $n \geq 1$.*

Proof . Since $T : K \to K$ be I- nearly weak uniformly L_1- Lipschitzian map with a sequence $\{u_n\}_{n\geq 1}$ and $I : K \to K$ be nearly weak uniformly L_2- Lipschitzian map with a sequence $\{v_n\}_{n\geq 1}$, there exists $\rho \in F(T) \cap F(I) \neq \emptyset$ and Lipschitz constants $L_1 > 1, L_2 > 1$ such that $\|T^n x - T^n \rho\| \leq L_1(\|I^n x - I^n \rho\| + u_n)$ and $\|I^n x - I^n \rho\| \leq L_2(\|x - \rho\| + v_n)$, for all $x \in K$ and $n \geq 1$. $\square$

2. The Main results

We prove the main result of this paper.

Theorem 2.1. *Let X be a real Banach space in which the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of X, K be a nonempty closed convex subset of X. Let $T : K \to K$ be I- nearly weak uniformly L_1- Lipschitzian map with a sequence $\{u_n\}_{n\geq 1} \subset [0, \infty)$ and $I : K \to K$ be nearly weak uniformly L_2- Lipschitzian map with a sequence $\{v_n\}_{n\geq 1} \subset [0, \infty)$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and, $\rho \in F(T) \cap F(I) \neq \emptyset$. Define a sequence $\{x_n\}$ in K by : $x_1 \in K$,*

$$x_{n+1} = (1 - a_n)x_n + a_n T^n x_n, \; n \geq 1, \tag{2.1}$$

where $\{a_n\} \in [0, 1]$ such that (i)$\sum_{n\geq 1} a_n = \infty$, (ii) $a_n \to 0$ as $n \to \infty$. Suppose there exists a strictly increasing continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^n x - T^n \rho, j(x - \rho) >\leq k_n \|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n, \tag{2.2}$$

for all $j(x - y) \in J(x - y)$ and $x \in K$, the sequence $\{x_n\}_{n\geq 1}$ defined by (2.1) converges strongly to the unique fixed point of T and I.

Proof . Observe from (2.2) that

$$\epsilon_n + \langle k_n(x - \rho) - (T^n x - \rho), j(x - \rho)\rangle \geq \Phi(\|x - \rho\|), \forall n \geq 1. \tag{2.3}$$

Firstly, we prove that there exists $x_1 \in K$ with $x_1 \neq Tx_1$ such that $r_0 = \epsilon_n + (k_n + L_1)\|x_1 - \rho\|^2 + L_1\|x_1 - \rho\|^2 + u_1 L_1 \|x_1 - \rho\| \in R(\Phi)$, where $R(\Phi)$ is the range of Φ. Infact, if $x_1 = Tx_1$, then we are done. Otherwise, there exists the smallest positive integer $n_1 \in N$ such that $x_{n_1} \neq Tx_{n_1}$. We denote $x_{n_1} = x_1$, and then we obtain that $r_0 = \epsilon_n + (k_n + L_1)\|x_1 - \rho\|^2 + L_1\|x_1 - \rho\|^2 + u_1 L_1\|x_1 - \rho\| \in R(\Phi)$. Indeed, if $\Phi(r) \to +\infty$ as $r \to \infty$, then $r_0 \in R(\Phi)$; If $\sup\{\Phi(r) : r \in [0, \infty]\} = r_1 < +\infty$ with $r_1 < r_0$, then for $\rho \in K$, there exists a sequence $\{w_n\}$ in K such that $\eta_n \to \rho$ as $n \to \infty$ with $w_n \neq \rho$. Clearly, we have that $Tw_n \to T\rho$ as $n \to \infty$ thus $\{w_n - T\eta_n\}$ is a bounded sequence. So therefore, there exists a natural number n_0 such that $\epsilon_n + (k_n + L_1)\|w_n - \rho\|^2 + L_1\|w_n - \rho\|^2 + u_1 L_1\|w_n - \rho\| < \frac{r_1}{2}$ for $n \geq n_1$, and then we redefine $x_1 = w_{n_0}$ to have $\epsilon_n + (k_n + L_1)\|x_1 - \rho\|^2 + L_1\|x_1 - \rho\|^2 + u_1 L_1\|x_1 - \rho\|^2 \in R(\Phi)$.

Secondly, we show that $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence using induction process.
Set $R = \Phi^{-1}(r_0)$, then from (2.3), we obtain that $\|x_1 - \rho\| \leq R$. Since $\{u_n\}$ and $\{v_n\}$ are bounded sequences in X, there exists $M > 0$ such that

$$\max\{\sup_{n\geq 1} u_n, \sup_{n\geq 1} v_n\} \leq M.$$

Denote $B_1=\{x\in K:\|x-\rho\|\leq R\}, B_2=\{x\in K:\|x-\rho\|\leq 2R\}$.
Now, we prove that $x_n\in B_1$. If $n=1$, then $x_1\in B_1$. Now, assume that it holds for some n, that is, $x_n\in B_1$. Suppose that, it is not the case, then $\|x_{n+1}-\rho\|>R$.
Denote $\tau_0\in R^+$ by

$$\tau_0 \quad = \quad min\{1,\frac{R}{L_1L_2(R+2M)},\frac{\Phi(R)}{12R^2},\frac{\Phi(R)}{16R(R+L_1L_2(R+2M))},\frac{\Phi(R)}{8}\}. \tag{2.4}$$

Since $\lim_{n\to\infty}a_n=0$ and $\lim_{n\to\infty}k_n=1$. Without loss of generality, let $0\leq a_n,k_n-1,\epsilon_n\leq\tau_0$, for any $n\geq 1$. Thus, we get

$$\begin{aligned}
\|x_{n+1}-\rho\| &\leq (1-a_n)\|x_n-\rho\|+a_nL_1(\|I^nx_n-I^n\rho\|+u_n)\\
&\leq (1-a_n)\|x_n-\rho\|+a_nL_1L_2(\|x_n-\rho\|+v_n+u_n)\\
&\leq R+\tau_0L_1L_2(R+2M)\\
&\leq 2R,
\end{aligned}$$

$$\begin{aligned}
\|x_{n+1}-x_n\| &\leq a_n\|T^nx_n-x_n\|\\
&\leq a_n(\|x_n-\rho\|+\|T^nx_n-T^n\rho\|)\\
&\leq a_n(\|x_n-\rho\|+L_1(\|I^nx_n-I^n\rho\|+u_n))\\
&\leq a_n(R+L_1L_2(R+2M)\\
&\leq \tau_0(R+L_1L_2(R+2M))2R
\end{aligned}$$

and

$$\begin{aligned}
\|T^nx_{n+1}-T^nx_n\| &\leq L_1(\|I^nx_{n+1}-I^nx_n\|+u_n)\\
&\leq L_1L_2(a_n\|T^nx_n-x_n\|+v_n+u_n)\\
&\leq L_1^2L_2^2(2a_n(\|x_n-\rho\|+v_n+u_n))\\
&\leq 2\tau_0L_1^2L_2^2(\|x_n-\rho\|+v_n+u_n)\\
&\leq 2\tau_0L_1^2L_2^2(R+2M)\\
&\leq \frac{\Phi(R)}{16R}.
\end{aligned} \tag{2.5}$$

Using Lemma 1.7 and the above estimates, we have

$$\begin{aligned}
\|x_{n+1}-\rho\|^2 &\leq (1-a_n)^2\|x_n-\rho\|^2+2a_n<T^nx_n-x_n,j(x_{n+1}-\rho)>\\
&\leq (1-a_n)^2\|x_n-\rho\|^2\\
&\quad +2a_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|)+\epsilon_n)\\
&\quad +2a_n\|T^nx_{n+1}-T^nx_n\|\|x_{n+1}-\rho\|\\
&\quad +2a_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\|\\
&\leq (1-a_n)^2R^2+2a_n(k_n\|x_{n+1}-\rho\|^2-\Phi(R)+\epsilon_n)\\
&\quad +\tfrac{2a_n}{16R}\Phi(R)2R+2a_n\tau_0(R+L_1L_2(R+2M))2R\\
&\leq R^2+2a_n[\tau_0+\tfrac{\tau_0}{2}]R^2-2a_n\Phi(R)+2a_n\tau_0\\
&\quad +\tfrac{a_n}{4}\Phi(R)+\tfrac{a_n}{4}\Phi(R)\\
&\leq R^2,
\end{aligned} \tag{2.6}$$

which is a contradiction. Hence $\{x_n\}_{n=1}^{\infty}$ is a bounded sequence.
Step 2. We prove that $\|x_n-\rho\|\to 0$ as $n\to\infty$.
Let

$$M_o=\sup_n\{\|x_n-\rho\|\}.$$

Observe that

$$\begin{aligned} \|x_{n+1}-x_n\| &\le a_n\|T^nx_n-x_n\| \\ &\le a_n(\|x_n-\rho\|+\|T^nx_n-T^n\rho\|) \\ &\le a_n(\|x_n-\rho\|+L_1(\|I^nx_n-I^n\rho\|+u_n)) \\ &\le a_n(\|x_n-\rho\|+L_1L_2(\|x_n-\rho\|+u_n+v_n)) \\ &\le a_n(M_o+L_1L_2(M_o+2M)). \end{aligned} \tag{2.7}$$

Employing Lemma 1.8, equations (2.5) and (2.7), we have

$$\begin{aligned} \|x_{n+1}-\rho\|^2 &\le (1-a_n)^2\|x_n-\rho\|^2+2a_n < T^nx_n-x_n, j(x_{n+1}-\rho) > \\ &\le (1-a_n)^2\|x_n-\rho\|^2 \\ &\quad +2a_n(k_n\|x_{n+1}-\rho\|^2-\Phi(\|x_{n+1}-\rho\|)+\epsilon_n) \\ &\quad +2a_n(\|T^nx_{n+1}-T^nx_n\|)\|x_{n+1}-\rho\| \\ &\quad +2a_n\|x_{n+1}-x_n\|\|x_{n+1}-\rho\| \\ &\le \|x_n-\rho\|^2 \\ &\quad +2a_n((k_n-1)M_o^2+a_n^2M_o^2-a_n\Phi(\|x_{n+1}-\rho\|)+2a_n\epsilon_n) \\ &\quad +2a_nL_1L_2(\|x_{n+1}-x_n\|+u_n+v_n)M_o \\ &\quad +a_n(M_o+L_1L_2(M_o+2M)). \\ &\le \|x_n-\rho\|^2-a_n\Phi(\|x_{n+1}-\rho\|)+\sigma_n. \end{aligned} \tag{2.8}$$

By applying Lemma 1.8 with ψ given by $\psi(t)=\Phi(\sqrt{t})$, $\sigma_n=2a_n((k_n-1)M_o^2+a_n^2M_o^2+2a_n\epsilon_n)+2a_nL_1L_2(\|x_{n+1}-x_n\|+u_n+v_n)M_o+a_n(M_o+L_1L_2(M_o+2M))$, we have that

$$\lim_{n\to\infty}\|x_n-\rho\|=0,$$

i.e., $x_n\to\rho$ as $n\to\infty$. This completes the proof. □

Remark 2.2. *Theorem 2.1 remains true for modified Mann iteration with errors [10]. The proof of such iteration with error term is unnecessary since it follows easily from the arguments presented here.*

Remark 2.3. *If X is a uniformly smooth real Banach space, the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of X. Partcular interest is the fact that, the normalized duality mapping is norm-to-norm uniformly continuous on bounded subsets of L_p, $1<p<\infty$ [4]. Thus, we have the following corollaries.*

Corollary 2.4. *Let X be uniformly smooth real Banach space, K be a nonempty closed convex subset of X. Let $T:K\to K$ be I- nearly weak uniformly L_1- Lipschitzian map with a sequence $\{u_n\}_{n\ge1}\subset[0,\infty)$ and $I:K\to K$ be nearly weak uniformly L_2- Lipschitzian map with a sequence $\{v_n\}_{n\ge1}\subset[0,\infty)$. Let $k_n\subset[1,\infty)$ and ϵ_n be sequences with $\lim_{n\to\infty}k_n=1$, $\lim_{n\to\infty}\epsilon_n=0$ and, $\rho\in F(T)\cap F(I)\neq\emptyset$. Define a sequence $\{x_n\}$ in K by : $x_1\in K$,*

$$x_{n+1}=(1-a_n)x_n+a_nT^nx_n,\ n\ge1, \tag{2.9}$$

where $\{a_n\}\in[0,1]$ such that (i)$\sum_{n\ge1}a_n=\infty$, (ii) $a_n\to0$ as $n\to\infty$. Suppose there exists a strictly increasing continuous function $\Phi:[0,\infty)\to[0,\infty)$ with $\Phi(0)=0$ such that

$$< T^nx-T^n\rho, j(x-\rho) >\le k_n\|x-\rho\|^2-\Phi(\|x-\rho\|)+\epsilon_n, \tag{2.10}$$

for all $j(x-y)\in J(x-y)$ and $x\in K$, the sequence $\{x_n\}_{n\ge1}$ defined by (2.9) converges strongly to the unique fixed point of T and I.

Corollary 2.5. *Let X be any L_p, $1 < p < \infty$, K be a nonempty closed convex subset of X. Let $T : K \to K$ be I- nearly weak uniformly L_1- Lipschitzian map with a sequence $\{u_n\}_{n\geq 1} \subset [0, \infty)$ and $I : K \to K$ be nearly weak uniformly L_2- Lipschitzian map with a sequence $\{v_n\}_{n\geq 1} \subset [0, \infty)$. Let $k_n \subset [1, \infty)$ and ϵ_n be sequences with $\lim_{n\to\infty} k_n = 1$, $\lim_{n\to\infty} \epsilon_n = 0$ and, $\rho \in F(T) \cap F(I) \neq \emptyset$. Define a sequence $\{x_n\}$ in K by : $x_1 \in K$,*

$$x_{n+1} = (1 - a_n)x_n + a_nT^nx_n,\ n \geq 1, \tag{2.11}$$

where $\{a_n\} \in [0, 1]$ such that (i)$\sum_{n\geq 1} a_n = \infty$, (ii) $a_n \to 0$ as $n \to \infty$. Suppose there exists a strictly increasing continuous function $\Phi : [0, \infty) \to [0, \infty)$ with $\Phi(0) = 0$ such that

$$< T^nx - T^n\rho, j(x - \rho) >\leq k_n\|x - \rho\|^2 - \Phi(\|x - \rho\|) + \epsilon_n, \tag{2.12}$$

for all $j(x - y) \in J(x - y)$ and $x \in K$, the sequence $\{x_n\}_{n\geq 1}$ defined by (2.11) converges strongly to the unique fixed point of T and I.

Now, we give an example to illustrate the validity of our result.

Example 2.6. *The mapping $Tx = \frac{2x^3}{1+2x^2}$, where $x \in [0, \infty)$ is strictly monotone increasing, i.e. $Tx \leq 2x$ for any $x \in [0, \infty)$ with $L > 1$, $k_n = 1$ for all n. Suppose we define $\Phi(x) : [0, \infty) \to [0, \infty)$ by $\Phi(x) = \frac{x^2}{1+2x^2}$. We have that $\rho = 0 \in F(T) \cap F(I)$. Then*

$$\begin{aligned} < T^nx - T^n\rho, j(x - \rho) > &= < T^nx - 0, j(x - 0) > \\ &= < \tfrac{2x^3}{1+2x^2} - 0, j(x - 0) > \end{aligned}$$

$$\begin{aligned} .k_n|x - \rho|^2 - \Phi(|x - \rho|) &= k_n|x - 0|^2 - \Phi(|x - 0|) \\ &= k_n|x - 0|^2 - \tfrac{|x-0|^2}{1+2|x-0|^2}. \end{aligned}$$

It is clear that $< T^nx - T^n\rho, j(x - \rho) >\leq k_n|x - \rho|^2 - \Phi(|x - \rho|) + \epsilon_n$ with the sequences $k_n = 1$ and $\epsilon_n = 0$.
Also,

$$\begin{aligned} |T^nx - T^n\rho| &= |T^nx - 0| \\ &\leq |Tx - 0 + (\tfrac{2}{3})^n| \\ &= |\tfrac{2x^3}{1+2x^2} - 0 + (\tfrac{2}{3})^n| \\ &\leq |2x - 0| + (\tfrac{2}{3})^n \\ &\leq 2(|x - 0|) + (\tfrac{2}{3})^n. \end{aligned}$$

Thus, T is I- nearly weak uniformly 2-Lipschitzian map with a sequence $\{(\frac{2}{3})^n\}$.

3. Applications

Fixed point theory has an abundance of applications in proving the existence of solutions for a wider class of differential and integral equations [7] and [24]. In recent times, the existence of fixed point for a nearly uniformly Lipschitzian mapping has been considered extensively by many authors, in uniformly convex Banach spaces, uniformly smooth Banach spaces and also in metric spaces [1], [2], [3], [5], [6], [9], [12], [13], [17] and [19]. Some problems in the stability of differential equations are closely related to the study of the existence of fixed points for uniformly Lipschitzian mappings. The result has been applicable in the solvability of the linear Fredholm integral equations of the second kind [24].

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