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# $(G.\psi)-$Ciric-Reich-Rus contraction on metric space endowed with a graph

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**Abstract**

In this paper, we introduce the $(G,\psi)-$Ciric-Reich-Rus contraction on metric space endowed with a graph, such that $(X,d)$ is a metric space, and $V(G)$ is the vertices of $G$ coincides with $X$. We give an example to show that our results generalize some known results

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## 1. Introduction and preliminaries

One of the most attractive areas of the fixed point theory is the existence of fixed points in a metric space respect to a given graph. Recently Jachymski [? ] has given some generalizations of the Banach Contraction Principle to mappings on a metric space respect to a graph. In order to study $\psi-$Ciric-Reich-Rus type contraction, we need the following definitions. (see also [? ? ? ? ? ? ? ? ? ? ? ? ? ? ])
Let $(X,d)$ be a metric space, and $\Delta$ be the diagonal of $X\times X$. Let $G$ be a directed graph such that the set $V(G)$ of its vertices coincides with $X$, and the set $E(G)$ of its edges contains all loops, i.e., $E(G)\supseteq\Delta$. Let $G$ has no parallel edges, so one can identify $G$ with the pair $(V(G),E(G))$.
By $G^{-1}$ we denote the graph obtained from $G$ by reversing the direction of edges, and call it the reverse of graph $G$. Thus,

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$E(G^{-1}) = \{(x,y) \in X \times X | (y,x) \in E(G)\}$.
$\tilde{G}$ is the undirected graph that obtained from $G$ by remove the direction of edges. So we have,

$E(\tilde{G}) = E(G) \bigcup E(G^{-1})$.

A path from $x$ to $y$ of length $N(N \in \mathbf{N})$ is a sequence $(x_i)_{i=0}^{N}$ of $N+1$ vertices such that $x_0 = x, x_N = y$ and $(x_{n-1}, x_n) \in E(G)$ for $i = 1, ..., N$.
$G$ is weakly connected if $\tilde{G}$ is connected. $[x]_G$ is the equivalence class of relations $\Re$ defined on $V(G)$ by the rule:

$z \Re y$ if there is a path in $G$ from $z$ to $y$.
$G_x$ is called the component of $G$ which consists of all edges and vertices which are contained in some path beginning at $x$.
If $f : X \to X$ is an operator, then

$X^f := \{x \in X : (x, fx)\} \in E(G)\}$,
and the set of all fixed points of $f$ is denoted by

$F_f := \{x \in X : f(x) = x\}$.

**Definition 1.1.** *[? ] The operator $f : X \to X$ is called a $G-$Ciric-Reich-Rus operator if:*

1. *for all $x, y \in X$ if $(x,y) \in E(G)$ then $(Tx, Ty) \in E(G)$;*
2. *There exists $\alpha, \beta, \gamma \in \boldsymbol{R}^+$ with $\alpha + \beta + \gamma \in (0,1)$, such that for each $x, y \in X$ we have, $d(fx, fy) \leq \alpha d(x,y) + \beta d(x, fx) + \gamma d(y, fy)$.*

**Definition 1.2.** *[? ] The operator $f : X \to X$ is called a Picard operator $(PO)$ if:*

*(i) $f$ has a unique fixed point $x^*$;*

*(ii) For all $x \in X$, we have $\lim_{n\to\infty} T^n x = x^*$.*

**Definition 1.3.** *[? ] The operator $f : X \to X$ is called a weakly Picard operator $(WPO)$ if:*

*(i) $F_f \neq \varnothing$;*

*(ii) for all $x \in X$, we have $\lim_{n\to\infty} T^n x = x^*(x)$.*
*($x^*(x)$ is the fixed point of $f$ which depened on $x$ )*

**Definition 1.4.** *[? ] A mapping $f : X \to X$ is called orbitally continuous if for all $x, y \in X$ and any sequence $(K_n)_{n\in N}$ of positive integers,*
*$f^{k_n}x \to y$, implise $f(f^{k_n}x) \to fy$ as $n \to \infty$.*

**Definition 1.5.** *[? ] A mapping $f : X \to X$ is called orbitally $G-$ continuous if for all $x, y \in X$ and any sequence $(K_n)_{n\in N}$ of positive integers,*
*$f^{k_n}x \to y$, $(f^{k_n}x, f^{k_{n+1}}x) \in E(G)$ imply $f(f^{k_n}x) \to fy$ as $n \to \infty$ .*

**Definition 1.6.** *[? ] Let us define the class $\Psi = \{\psi : \boldsymbol{R}^+ \to \boldsymbol{R}^+ \mid \psi$ is nondecreasing $\}$ which satisfies the following conditions:*

*(i)* $\psi(w) = 0$ *if and only if* $w = 0$;

*(ii) for every* $(w_n) \in \boldsymbol{R}^+$, $\psi(w_n) \to 0$ *if and only if* $w_n \to 0$;

*(iii) for every* $w_1, w_2 \in \boldsymbol{R}^+, \psi(w_1 + w_2) \leq \psi(w_1) + \psi(w_2)$.

In the next section, we state two fixed point theorems for $(G, \psi)-$Ciric-Reich-Rus type contraction.

## 2. Main results

In this section, we assume that $(X, d)$ is a metric space, and $G$ is a directed graph such that $V(G) = X, \triangle \subseteq E(G)$ and $G$ has no parallel edges.

**Definition 2.1.** *A mapping* $f : X \to X$ *is called* $(G, \psi) - Ciric - Reich - Rus$ *contraction if:*

*(i) for all* $x, y \in X$ *if* $(x, y) \in E(G)$ *then* $(Tx, Ty) \in E(G)$;

*(ii) there exists* $\alpha, \beta, \gamma \in \boldsymbol{R}^+$, *with* $\alpha + \beta + \gamma \in (0, 1)$, *such that for each* $(x, y) \in E(G)$ *implies* $\psi(d(fx, fy)) \leq \alpha\psi(d(x, y)) + \beta\psi(d(x, fx)) + \gamma\psi(d(y, fy))$.

The following Lemma is immediately.

**Lemma 2.2.** *If* $f : X \to X$ *is a* $(G, \psi) - Ciric - Reich - Rus$ *contraction then* $f$ *is both a* $(G^{-1}, \psi) - Ciric - Reich - Rus$ *contraction and a* $(\tilde{G}, \psi) - Ciric - Reich - Rus$ *contraction.*

**Lemma 2.3.** *Let* $f : X \to X$ *be a* $(G, \psi) - Ciric - Reich - Rus$ *with the constants* $\alpha, \beta, \gamma$. *Then, for given* $x \in X^f$, *there exists* $r(x) \geq 0$ *such that*

$$\psi(d(f^n x, f^{n+1}x)) \leq a^n r(x),$$

*for all* $n \in N$, *where* $a := \dfrac{\alpha + \beta}{1 - \gamma}$.

**Proof .** *Assume that* $x \in X^f$, *then by induction, we have* $(f^n x, f^{n+1}x) \in E(G)$ *for each* $n \in N$. *So* $\psi(d(f^n x, f^{n+1}x)) \leq \alpha\psi(d(f^{n-1}x, f^n x)) + \beta\psi(d(f^{n-1}x, f^n x)) + \gamma\psi(d(f^n x, f^{n+1}x))$.
*Hence* $\psi(d(f^n x, f^{n+1}x)) \leq \dfrac{\alpha + \beta}{1 - \gamma}\psi(d(f^{n-1}x, f^n x)) \leq \cdots \leq a^n\psi(d(x, fx))$. *Set* $r(x) := \psi(d(x, fx))$.
□

**Lemma 2.4.** *Assume that* $(X, d)$ *is a complete metric space and* $f : X \to X$ *is a* $(G, \psi) - Ciric - Reich - Rus$ *contraction with the constants* $\alpha, \beta, \gamma$. *Then, for each* $x \in X^f$, *there exists* $x^*(x) \in X$ *such that the sequence* $(f^n x)_{n \in N}$ *converges to* $x^*(x)$ *as* $n \to \infty$.

**Proof .** *Let* $x \in X^f$. *By Lemma 2.3,* $\psi(d(f^n x, f^{n+1}x)) \leq a^n r(x)$. *Hence* $\sum_{n=0}^{\infty} \psi(d(f^n x, f^{n+1}x)) < \infty$. *Thus* $\psi(d(f^n x, f^{n+1}x)) \to 0$ *as* $n \to \infty$.
*Then we have* $d(f^n x, f^{n+1}x) \to 0$. *So the sequence* $(f^n x)_{n \in N}$ *is a Cauchy sequence. Since the space* $X$ *is complete, there exists* $x^*(x) \in X$ *such that the sequence* $(f^n x)_{n \in N}$ *converges to* $x^*(x)$ *as* $n \to \infty$.
□

**Theorem 2.5.** *Let* $(X, d)$ *be a complete metric space endowed with a graph* $G$, *and let the triple* $(X, d, G)$ *has the following condition:*
*For any* $(x_n)_{n \in N}$ *in* $X$, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in N$, *then there is a subsequence* $(x_{k_n})_{n \in N}$ *with* $(x_{k_n}, x) \in E(G)$ *for all* $n \in N$.
*Let* $f : X \to X$ *be a* $(G, \psi) - Ciric - Reich - Rus$ *contraction and* $f$ *be orbitally* $G-$*continuous. Then the following statements hold.*

*(i)* $F_f \neq \varnothing$ *if and only if* $X^f \neq \varnothing$.

*(ii) If* $X^f \neq \varnothing$ *and* $G$ *is weakly connected, then* $f$ *is a weakly Picard operator.*

*(iii) For any* $X^f \neq \varnothing$, $f\mid_{[x]_{\tilde{G}}}$ *is a weakly Picard operator.*

**Proof .** *First we prove (iii). Let* $x \in X^f$; *by Lemma 2.4, there exists* $x^* \in X$ *such that* $\lim_{n\to\infty} f^n x = x^*$. *Since* $x \in X^f$, *then* $f^n x \in X^f$ *for every* $n \in N$. *Now assume that* $(x, fx) \in E(G)$. *By condition (P), there is a subsequence* $(f^{k_n}x)_{n\in N}$ *of* $(f^n x)_{n\in N}$ *such that* $(f^{k_n}x, x^*) \in E(G)$ *for each* $n \in N$. *Now we have a path in* $G$ *by using the points* $x, fx, \cdots, f^{k_l}x, x^*$ *and hence* $x^* \in [x]_{\tilde{G}}$. *On the other hand since* $f$ *is orbitally* $G-$*continuous, we have* $x^*$ *is a fixed point forf* $\mid_{[x]_{\tilde{G}}}$ .
*(i) is obtained using (iii), because* $F_f \neq \varnothing$ *if* $X^f \neq \varnothing$. *Now suppose that* $F_f \neq \varnothing$. *By using the assumption that* $\triangle \subseteq E(G)$, *we obtain* $X^f \neq \varnothing$.
*For proving (ii) let* $x \in X^f$. *Because* $G$ *is weakly connected, we have* $X = [x]_{\tilde{G}}$ *and (iii) complete the proof.* □

**Remark 2.6.** *Set* $\psi(w) = w$ *in Theorem 2.5, then Theorem* 2.2 *in [**?** ] obtain immediately.*

In the next we study the case that $f : X \to X$ as a $(G, \psi) - Ciric - Reich - Rus$ contraction can be a Picard operator. So we need the following definition.

**Definition 2.7.** *Let* $(X, d)$ *be a metric space endowed with a graph* $G$ *and* $f : X \to X$ *be a mapping. We say that the graph* $G$ *has a* $f-$*path property, if for any path in* $G$, $(x_i)_{i=0}^N$ *from* $x$ *to* $y$ *such that* $x_0 = x, x_N = y$ *we have* $fx_{i-1} = x_i$ *for all* $i = 1, \cdots, N$.

**Lemma 2.8.** *Let* $(X, d)$ *be a metric space endowed with a graph* $G$ *and* $f : X \to X$ *be a* $(G, \psi) - Ciric - Reich - Rus$ *contraction such that the graph* $G$ *has the* $f-$*path property. Then for any* $x \in X$ *and* $y \in [x]_{\tilde{G}}$ *two sequences* $(f^n x)_{n\in N}$ *and* $(f^n y)_{n\in N}$ *are equivalent.*
**Proof .** *Let* $x \in X$, *and let* $y \in [x]_{\tilde{G}}$; *then there exists a path* $(x_i)_{i=0}^l$ *in* $\tilde{G}$ *from* $x$ *to* $y$ *such that* $x_0 = x, x_l = y$ *with* $(x_{i-1}, x_i) \in E(G)$ *and* $fx_{i-1} = x_i$ *for all* $i = 1, \cdots, l$. *From Lemma 2.2,* $f$ *is a* $(\tilde{G}, \psi) - Ciric - Reich - Rus$. *Then for all* $n \in N$ $(f^n x_{i-1}, f^n x_i) \in E(\tilde{G})$, *so*

$$\begin{aligned}\psi(d(f^n x_{i-1}, f^n x_i)) \leq\ & \alpha\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)) + \beta\psi(d(f^{n-1}x_{i-1}, f^n x_{i-1})) + \gamma\psi(d(f^{n-1}x_i, f^n x_i)) \\ =\ & \alpha\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)) + \beta\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)) + \gamma\psi(d(f^n x_{i-1}, f^n x_i))\end{aligned}$$

*then,*

$$\psi(d(f^n x_{i-1}, f^n x_i)) \leq \frac{\alpha+\beta}{1-\gamma}\psi(d(f^{n-1}x_{i-1}, f^{n-1}x_i)).$$

*Hence, for all* $n \in N$

$$\psi(d(f^n x_{i-1}, f^n x_i)) \leq a^n \psi(d(x_{i-1}, x_i)), \tag{2.1}$$

*where* $a = \dfrac{\alpha+\beta}{1-\gamma}$. *We know that* $(f^n x_i)_{i=0}^l$ *is a path in* $\tilde{G}$ *from* $f^n x$ *to* $f^n y$. *Using the triangle inequality and (2.1),*

$$\psi(d(f^n x, f^n y)) \leq \sum_{i=1}^{l} \psi(d(f^n x_{i-1}, f^n x_i)) \leq a^n \sum_{i=1}^{l} \psi(d(x_{i-1}, x_i)).$$

*Letting* $n \to \infty$, *we get* $d(f^n x, f^n y) \to 0$. □

**Theorem 2.9.** *Let $(X,d)$ be a complete metric space endowed with a graph $G$, and $f : X \to X$ be a $(G,\psi)$-Ciric-Reich-Rus contraction such that the graph $G$ has the $f-$path property and $f$ be orbitally $G-$continuous. Let the triple $(X,d,G)$ has the following condition:*
*For any $(x_n)_{n\in N}$ in $X$, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in N$, then there is a subsequence $(x_{k_n})_{n\in N}$ with $(x_{k_n}, x) \in E(G)$ for all $n \in N$. Let there exists $z \in X$ such that $z \in X^f$, then the following statements hold:*

*(1) $f\mid_{[z]_{\tilde{G}}}$ is a Picard operator;*

*(2) if $G$ is weakly connected, then $f$ is a Picard operator.*

**Proof .** (1) *Using* $(iii)$ *Theorem 2.5, there exists $x^*(z) \in [z]_{\tilde{G}}$ such that $\lim_{n\to\infty} f^n(z) = x^*(z)$, and $x^*(z)$ is a fixed point of $f$. Now if $y \in [z]_{\tilde{G}}$ and $\lim_{n\to\infty} f^n(y) = x^*(y)$. Then by Lemma 2.8 two sequences $(f^n z)_{n\in N}$ and $(f^n y)_{n\in N}$ are equivalent. Since both are convergent sequence, then they are Cauchy sequences. Hence they are Cauchy equivalent. This means $x^*(y) = x^*(z)$.*
(2) *Since $z \in X^f$ and $G$ is weakly connected, we have $X = [z]_{\tilde{G}}$. Then we only need to apply* (1). □

**Definition 2.10.** *[? ] We say that mapping $f : X \to X$ is a $(G,\psi)-$contraction if the following hold:*

*(i) $f$ preserves edges of $G$, i.e, for all $x, y \in X$ if $(x,y) \in E(G)$ then $(fx, fy) \in E(G)$);*

*(ii) $f$ decreases the weight of $G$, that is, there exists $c \in (0,1)$ such that for all $x, y \in X$ if*

$$(x,y) \in E(G) \qquad then \quad \psi(d(fx,fy)) \le c\psi(d(x,y)).$$

In the following example we show that $(G,\psi)-$Ciric-Reich-Rus contraction is a generalization of $(G,\psi)-$contraction.

**Example 2.11.** *Let $X = [0,1]$ and $d(x,y) =\mid x - y \mid$ . Define the graph $G$ by*
$E(G) = \{(0,0),(0,1)\} \bigcup \{(x,y) \in (0,1] \times [0,1] \qquad x \geqslant y\}$.

*$f : X \to X$ and*

$$fx = \begin{cases} \frac{x}{2}, & x \in (0,1]; \\ \frac{3}{4}, & x = 0. \end{cases}$$

*$G$ is weakly connected, and $f$ is a $(G,\psi)-$Ciric-Reich-Rus contraction with constants,*
$\alpha = \frac{1}{8}, \beta = \frac{3}{4}, \gamma = \frac{1}{16}, \psi(w) = \dfrac{w}{2}$. *But $f$ is not $(G,\psi)-$contraction, because if we consider*

$$\psi(d(f(0), f(\frac{1}{2})) \le c\psi(d(0,\frac{1}{2})$$

*Then we have $\frac{1}{4} \le c\frac{1}{4}$ which is a contradiction since $c \in [0,1)$.*

**Definition 2.12.** *The mapping $f : X \to X$ is called a $(G,\psi)-$Kannan mapping if:*

*(i) for all $x, y \in X$ if $(x,y) \in E(G)$ then $(fx, fy) \in E(G)$);*

*(ii)* *there exists a constant* $a \in (0,1)$ *such that for all* $x, y \in X, (x,y) \in E(G)$ *then,*

$$\psi(d(fx, fy)) \le a[\psi(d(x, fx)) + \psi(d(y, fy))].$$

**Corollary 2.13.** *Let* $(X, d)$ *be a complete metric space endowed with a graph* $G$, *and* $f : X \to X$ *be a* $(G, \psi)-$ *contraction such that the graph* $G$ *has the* $f-$*path property and* $f$ *be orbitally* $G-$*continuous. Let the triple* $(X, d, G)$ *has the following condition:*
*For any* $(x_n)_{n\in N}$ *in* $X$, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in N$, *then there is a subsequence* $(x_{k_n})_{n\in N}$ *with* $(x_{k_n}, x) \in E(G)$ *for all* $n \in N$. *Let there exists* $z \in X$ *such that* $z \in X^f$, *then the following statements hold:*

(1) $f\mid_{[z]_{\tilde{G}}}$ *is a Picard operator;*

(2) *if* $G$ *is weakly connected, then* $f$ *is a Picard operator.*

**Proof .** *If* $f$ *is a* $(G, \psi)-$*contraction with constant* $c \in [0, 1)$, *then* $f$ *is a* $(\tilde{G}, \psi)-$*Ciric-Reich-Rus contraction with constants* $\alpha = c, \beta = \gamma = 0$. *Hence according to Theorem* **??**, $f$ *is a Picard operator.* □

**Corollary 2.14.** *Let* $(X, d)$ *be a complete metric space endowed with a graph* $G$, *and* $f : X \to X$ *be a* $(G, \psi)-$*Kannan mapping such that the graph* $G$ *has the* $f-$*path property and* $f$ *be orbitally* $G-$*continuous. Let the triple* $(X, d, G)$ *has the following condition:*
*For any* $(x_n)_{n\in N}$ *in* $X$, *if* $x_n \to x$ *and* $(x_n, x_{n+1}) \in E(G)$ *for all* $n \in N$, *then there is a subsequence* $(x_{k_n})_{n\in N}$ *with* $(x_{k_n}, x) \in E(G)$ *for all* $n \in N$. *Let there exists* $z \in X$ *such that* $z \in X^f$, *then the following statements hold:*

(1) $f\mid_{[z]_{\tilde{G}}}$ *is a Picard operator;*

(2) *if* $G$ *is weakly connected, then* $f$ *is a Picard operator.*

**Proof .** *If* $f$ *is a* $(G, \psi)-$*Kannan with constant* $a \in [0, 1)$, *then* $f$ *is a* $(\tilde{G}, \psi) - Ciric - Reich - Rus$ *contraction with constants* $\alpha = 0, \beta = \gamma = a$. *Hence according to Theorem* **??**, $f$ *is a Picard operator.* □

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